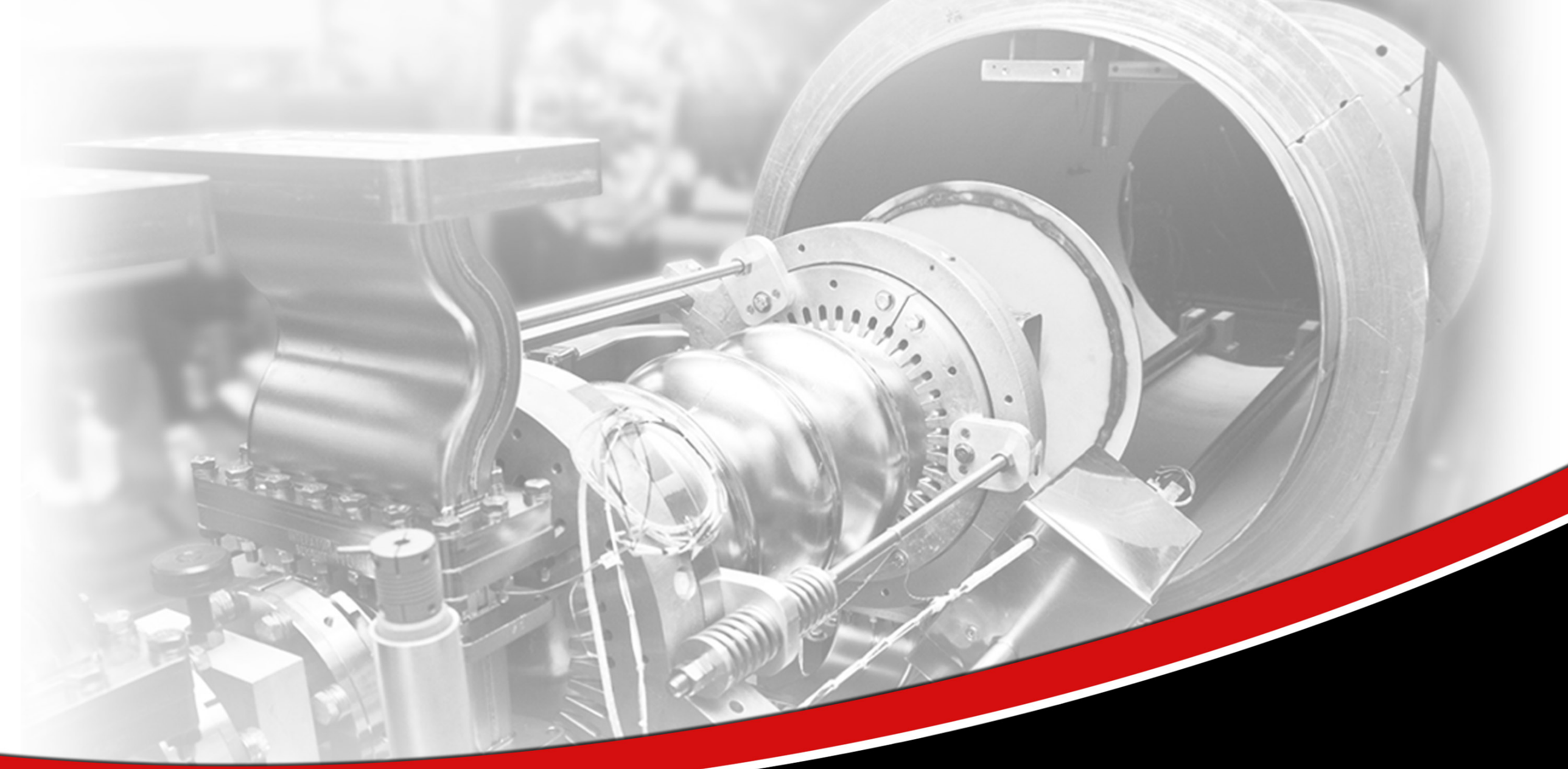

# Rapidity evolution of gluon TMD from low to moderate x

**Andrey Tarasov**

**DIS 2015**
**Dallas, April 27 - May 1, 2015**

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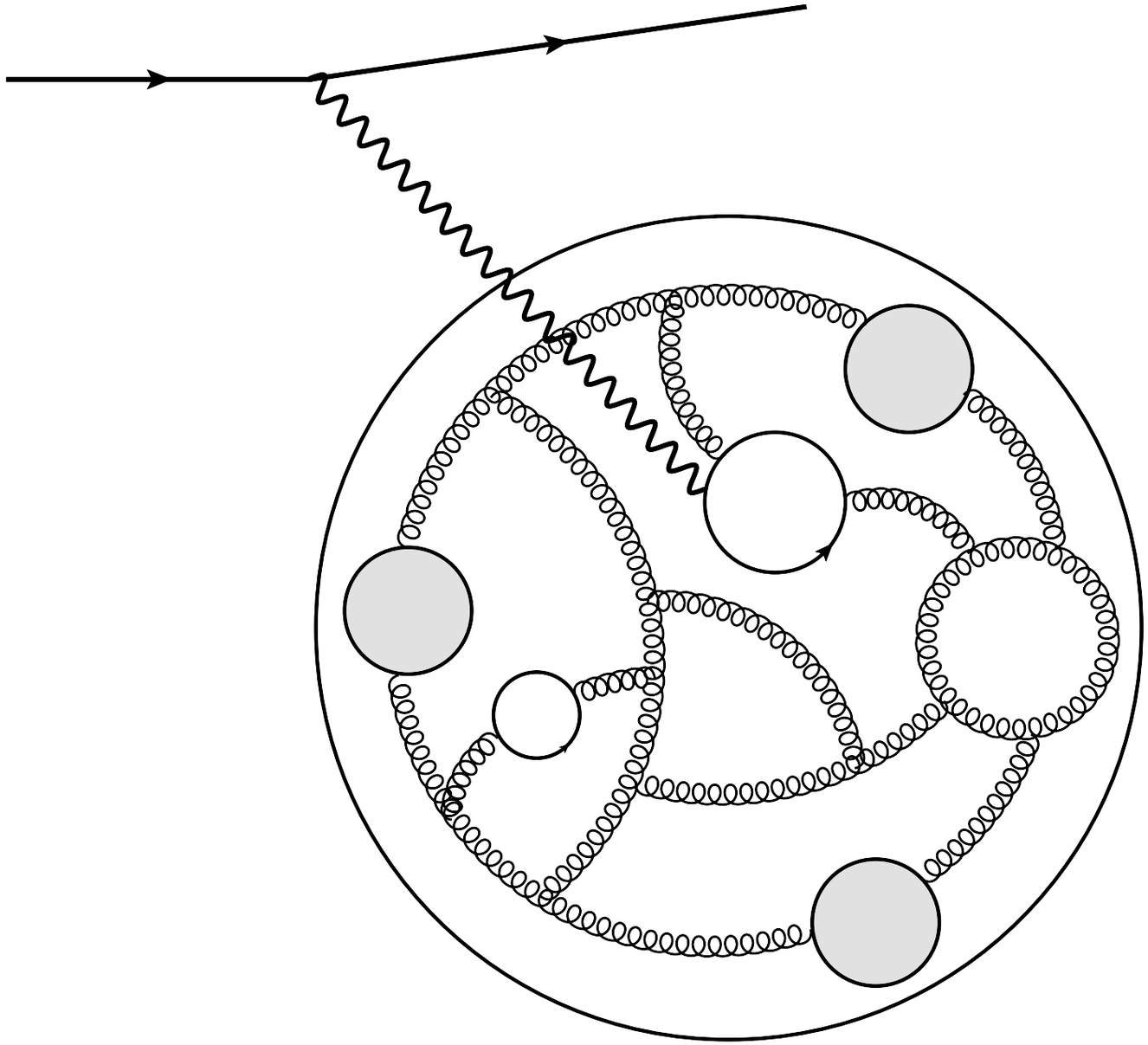
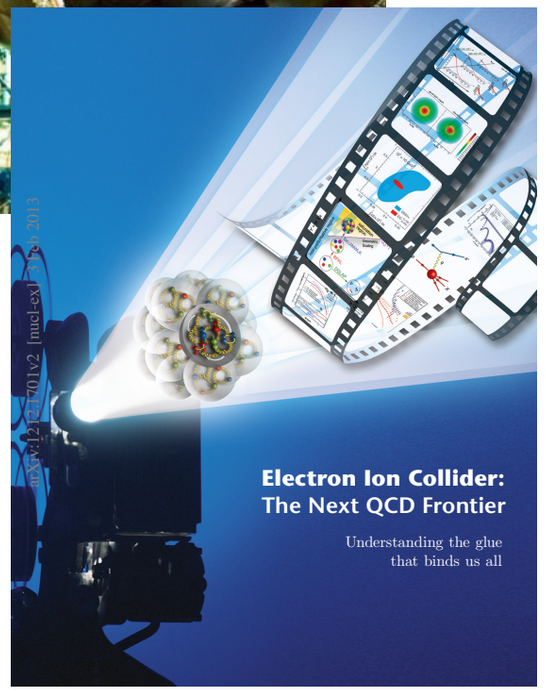
# Outline

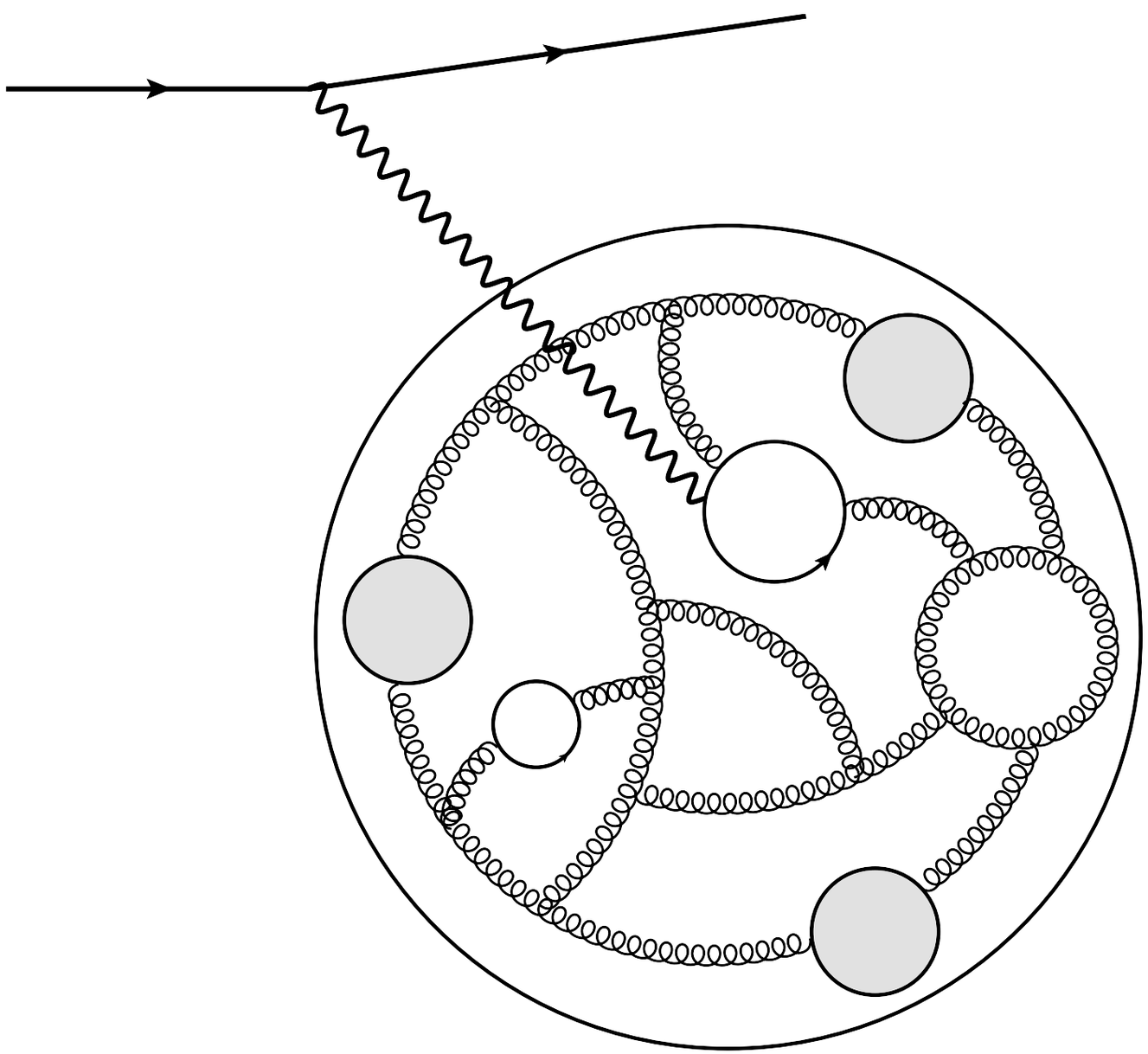
- Definition of gluon TMD
- Light-cone expansion at moderate x
- Rapidity factorization approach at small x
- Generalized evolution at moderate and small x
- BK, DGLAP and Sudakov limits
- Gluon TMD factorization function

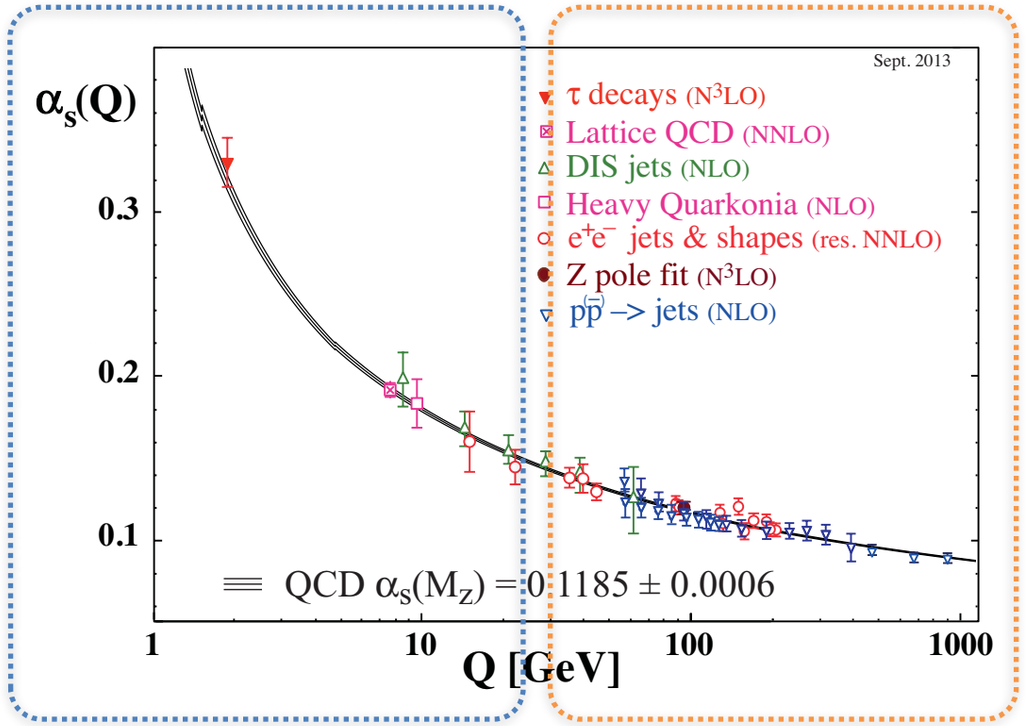
# Transverse momentum dependent (TMD) distribution





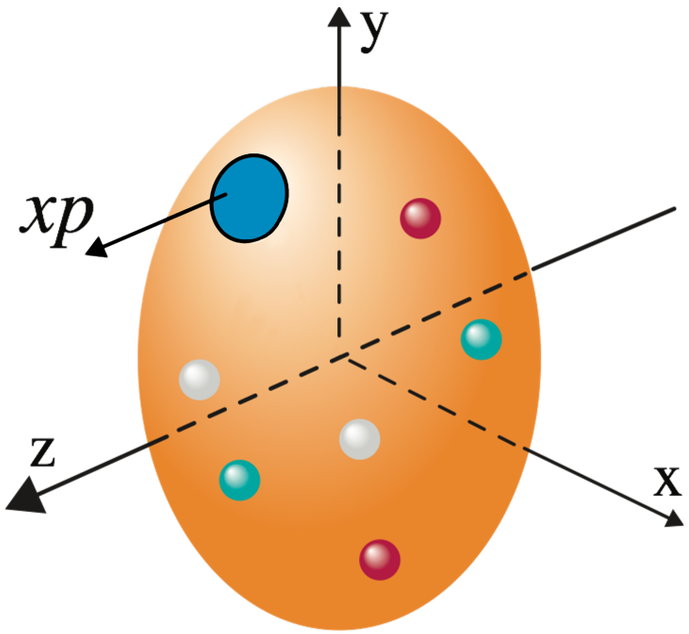
# Transverse momentum dependent (TMD) distribution



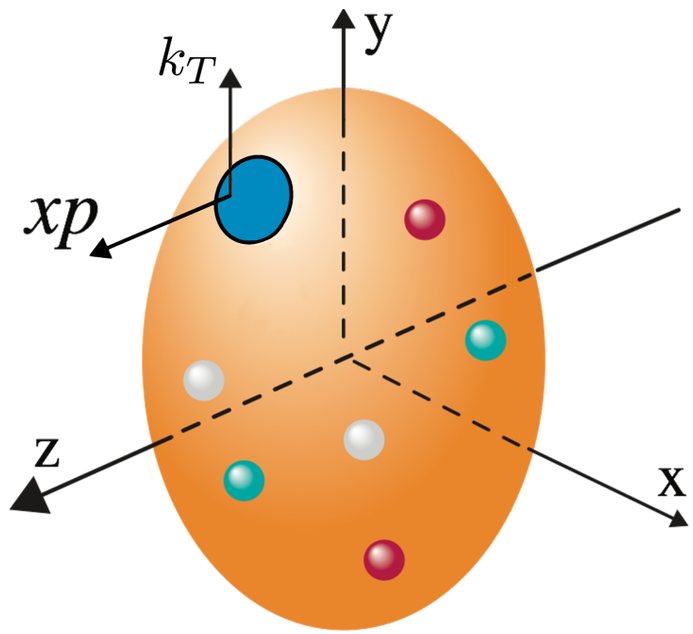


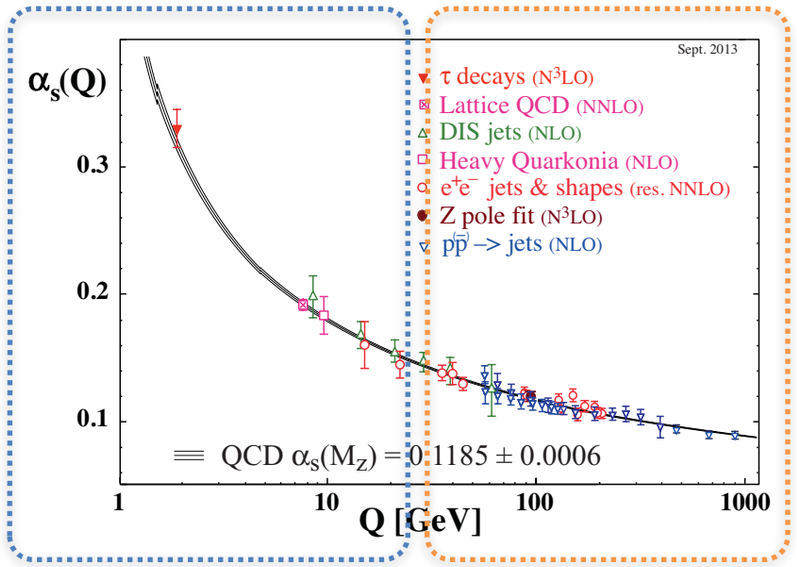
# Transverse momentum dependent (TMD) distribution

Collinear distribution:



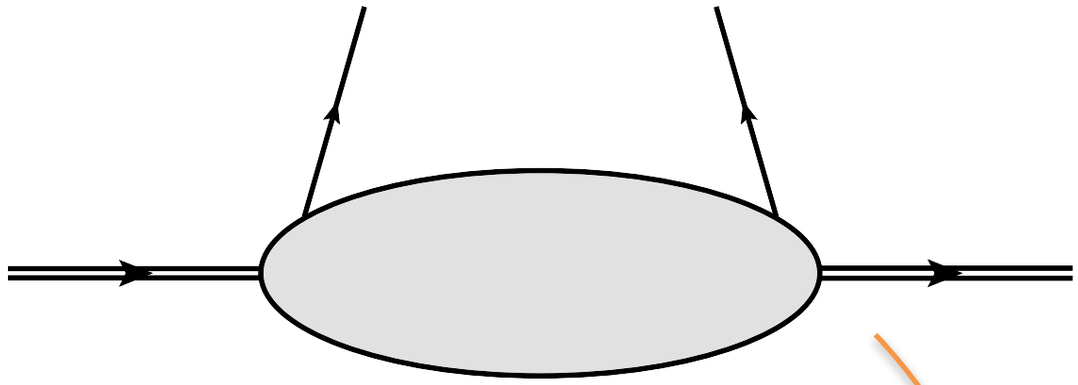

TMD distribution:



# Current status

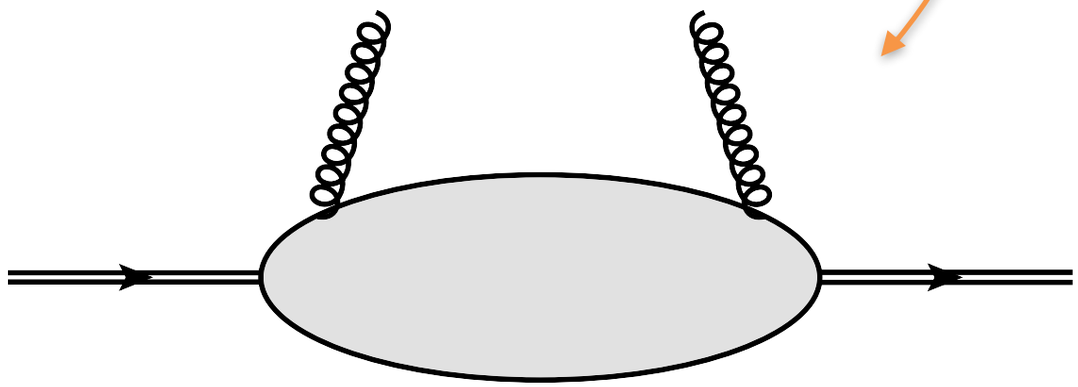
Quark TMD at moderate x:



J.C. Collins, D. E. Soper and G. Sterman, Nucl. Phys. B250, 199 (1985)
X. Ji, Jian-Ping Ma, and F. Yuan, Phys. Rev. D71, 034005 (2005)
J. C. Collins, Foundations of Perturbative QCD (CUP, 2011)

moderate x

Gluon TMD:



F. Dominguez, A.H. Mueller, S. Munier, and Bo-Wen Xiao, Phys. Lett. B705, 106 (2011)
P. J. Mulders and J. Rodrigues, Phys. Rev. D63, 094021 (2001)
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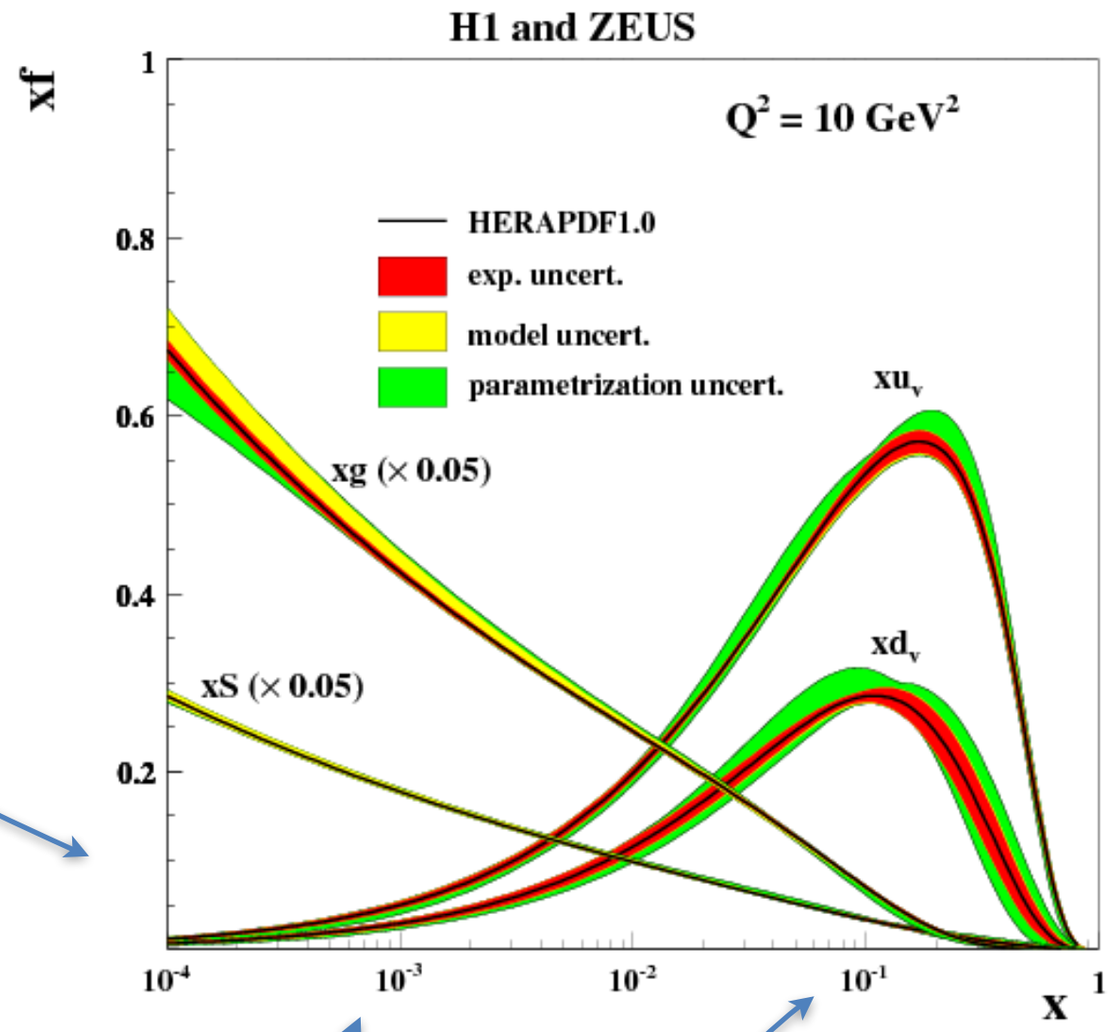
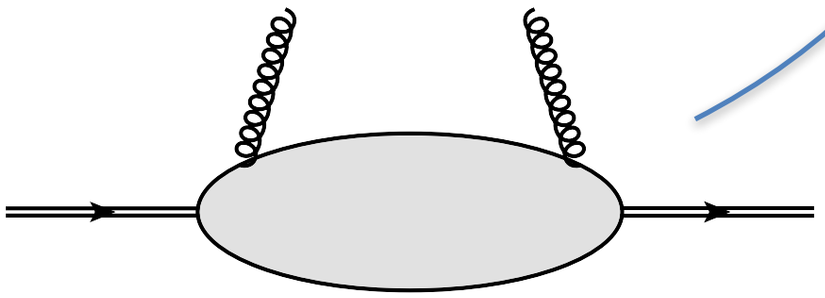
Y. V. Kovchegov, M.D. Sievert, Phys. Rev. D86, 034028 (2012)
G. A. Chirilli, Bo-Wen Xiao, and F. Yuan, Phys. Rev. D86, 054005 (2012)
M. G. Echevarria, T. Kasemets, P. J. Mulders, and C. Pisano, hep-ph/1502.0535

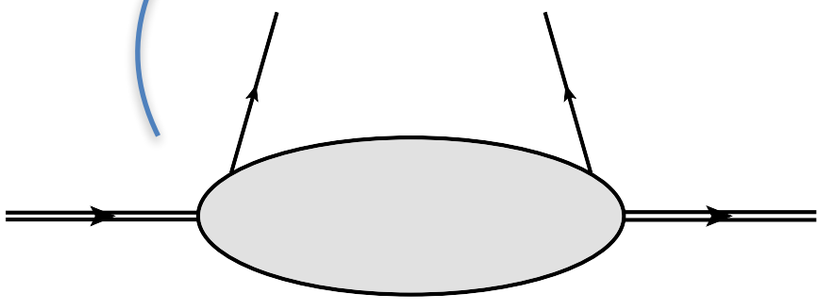
# From moderate to small-x

I. Balitsky, Nucl. Phys. B463, 99 (1996)
Yu.V. Kovchegov, Phys. Rev. D60, 034008 (1999)

Non-linear effects at small x





# Two definitions of gluon TMD

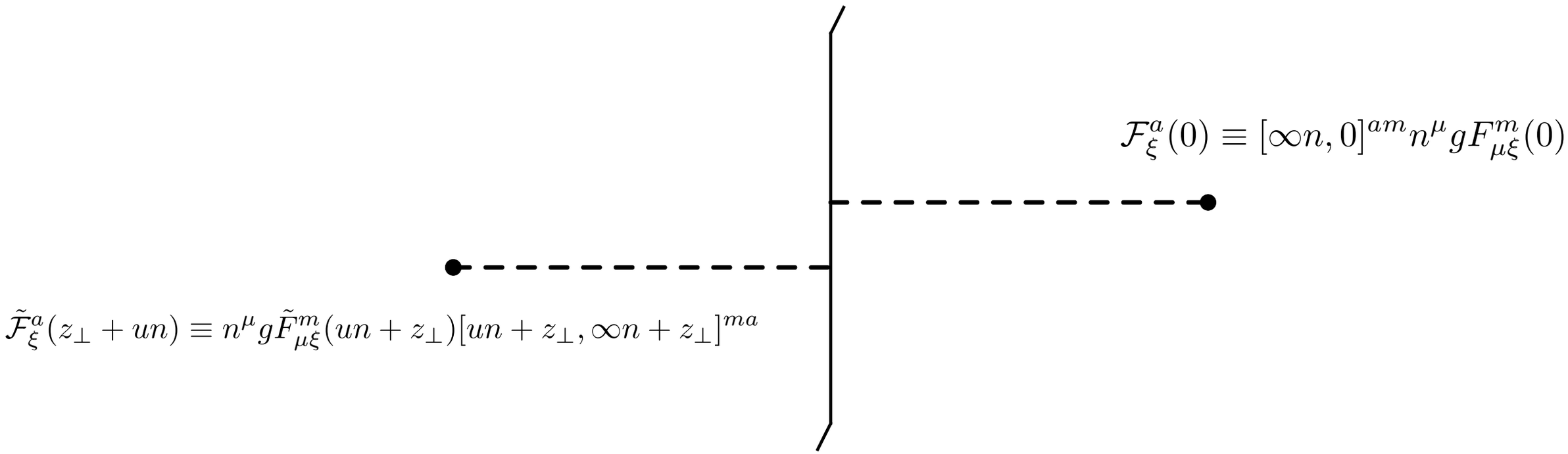
Moderate x:

$$\alpha_s \mathcal{D}(x, z_\perp) \equiv \frac{2\pi^2 \alpha_s}{N_c} G(x, z_\perp) = -\frac{1}{8\pi^2 (P \cdot n) x} \int du e^{-ixu(P\cdot n)} \langle P | \tilde{\mathcal{F}}^a_\xi (z_\perp + un) \mathcal{F}^{a\xi}(0) | P \rangle$$

there is no exponential factor

Small x (Weiszacker-Williams):

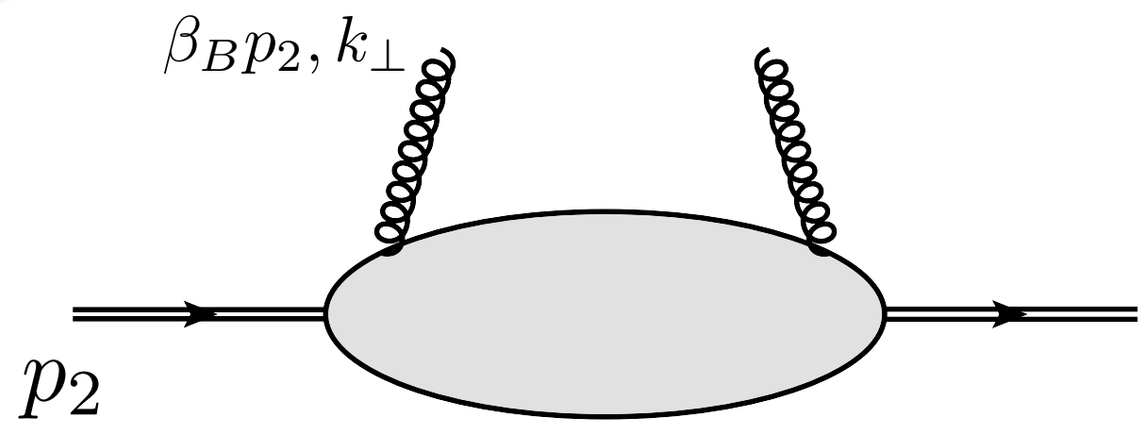
$$\alpha_s \mathcal{D}(x, z_\perp) \equiv \frac{2\pi^2 \alpha_s}{N_c} G(x, z_\perp) = -\frac{1}{8\pi^2 (P \cdot n) x} \int du \langle P | \tilde{\mathcal{F}}^a_\xi (z_\perp + un) \mathcal{F}^{a\xi}(0) | P \rangle$$



# Definition of gluon TMD

$$\beta_B \mathcal{D}(\beta_B, k_\perp) = -\frac{4}{\alpha_s \langle P|P\rangle} \int d^2x_\perp d^2y_\perp e^{ik_\perp(x_\perp - y_\perp)} \langle P|\tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\mathcal{F}^{ai}(\beta_B, y_\perp)|P\rangle$$

$$\tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\mathcal{F}_j^a(\beta_B, y_\perp)$$

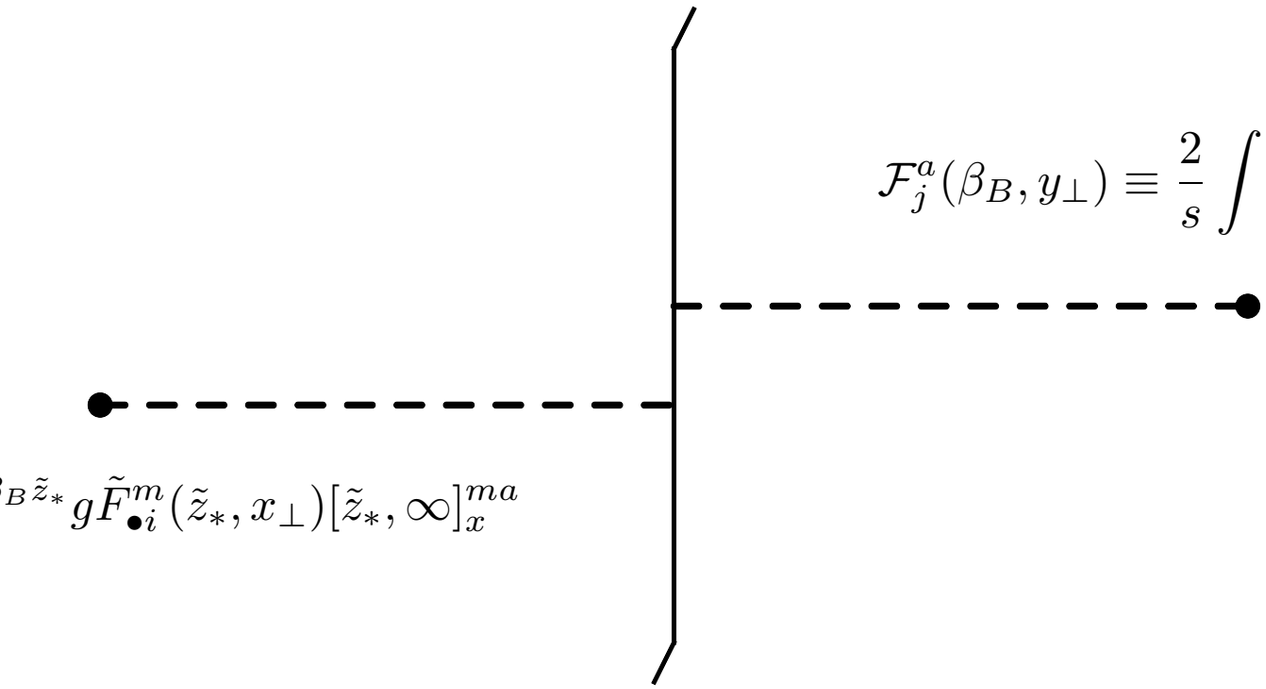



- We consider evolution of this operator
- Distribution and fragmentation function

# Definitions of gluon TMD

$$\tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\mathcal{F}_j^a(\beta_B, y_\perp)$$

We consider evolution of this operator



$$\mathcal{F}_j^a(\beta_B, y_\perp) \equiv \frac{2}{s}\int dz_* e^{i\beta_B z_*}[\infty, z_*]_y^{am} g F_{\bullet j}^m(z_*, y_\perp)$$

$$\tilde{\mathcal{F}}_i^a(\beta_B, x_\perp) \equiv \frac{2}{s}\int d\tilde{z}_* e^{-i\beta_B \tilde{z}_*} g\tilde{F}_{\bullet i}^m(\tilde{z}_*, x_\perp)[\tilde{z}_*, \infty]_x^{ma}$$

$$p = \alpha p_1 + \beta p_2 + p_\perp$$

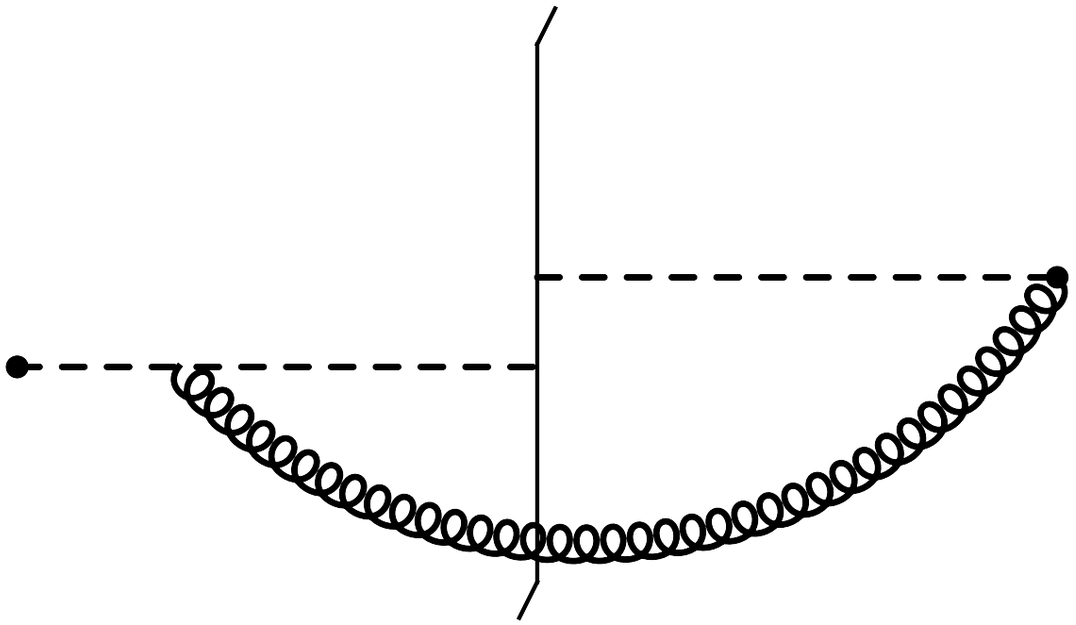
$$z_* = z_\mu p_2^\mu = \sqrt{\frac{s}{2}} z^-$$

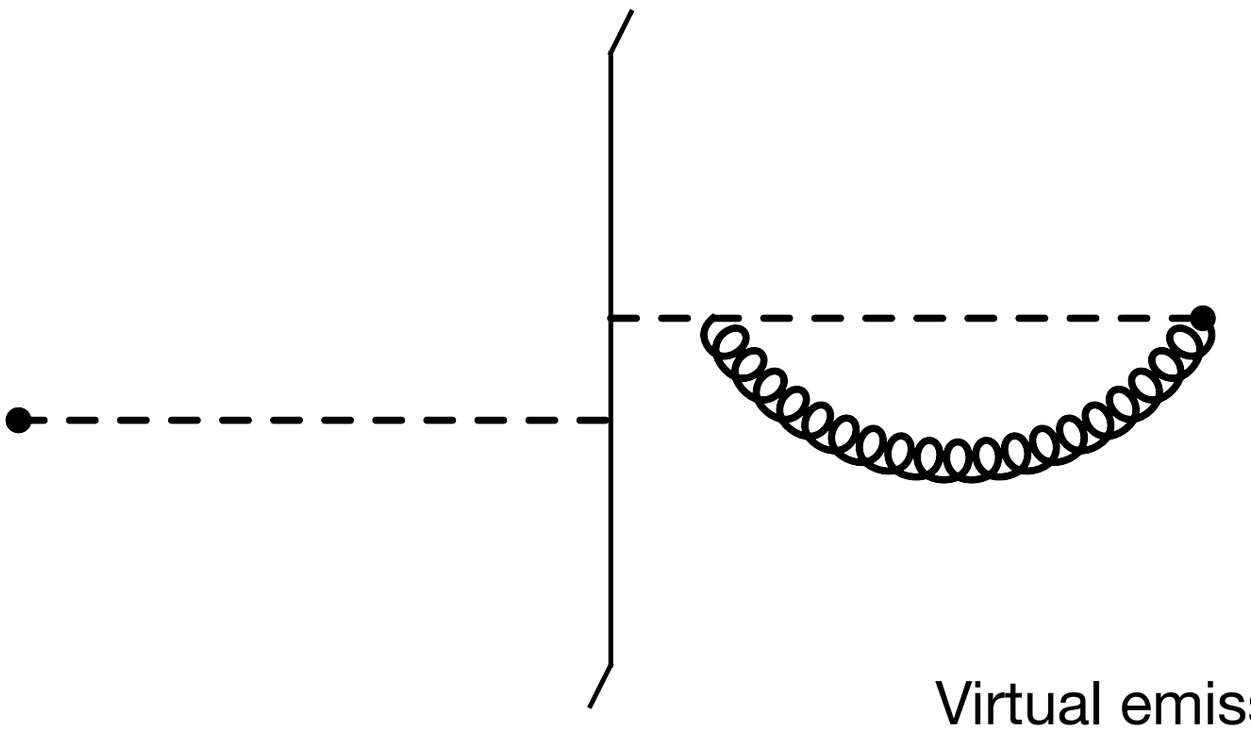
$$z_\bullet = z_\mu p_1^\mu = \sqrt{\frac{s}{2}} z^+$$

# Evolution of gluon TMD

Note: there are several diagrams of this kind

Real emission

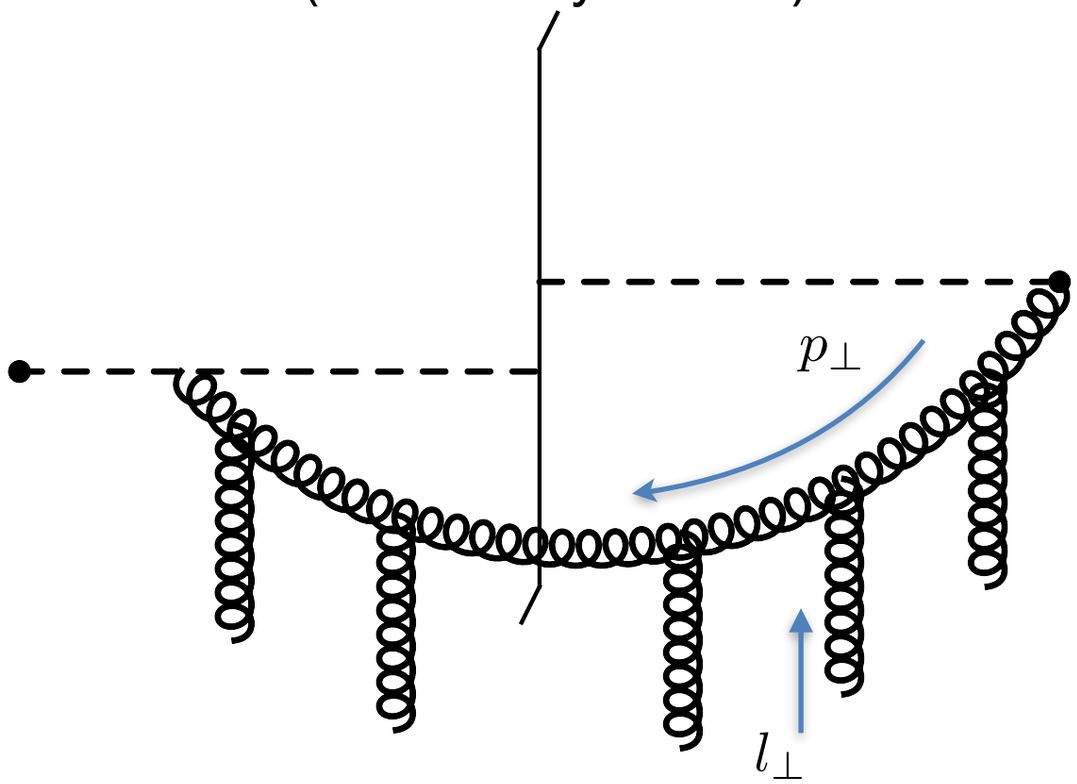




Virtual emission

# Evolution at small and moderate x

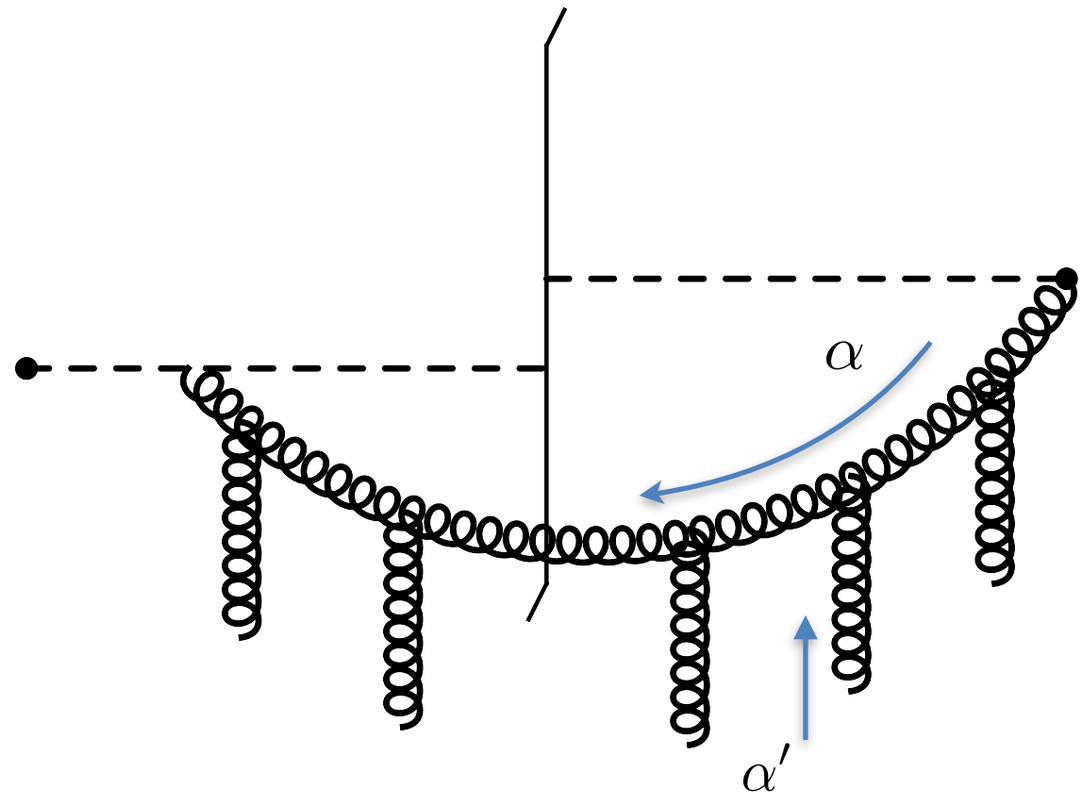
Moderate x (DGLAP dynamics)




- Ordering in transverse momentum
  $p_\perp > l_\perp$
- No ordering in rapidity
  $\alpha \sim \alpha'$

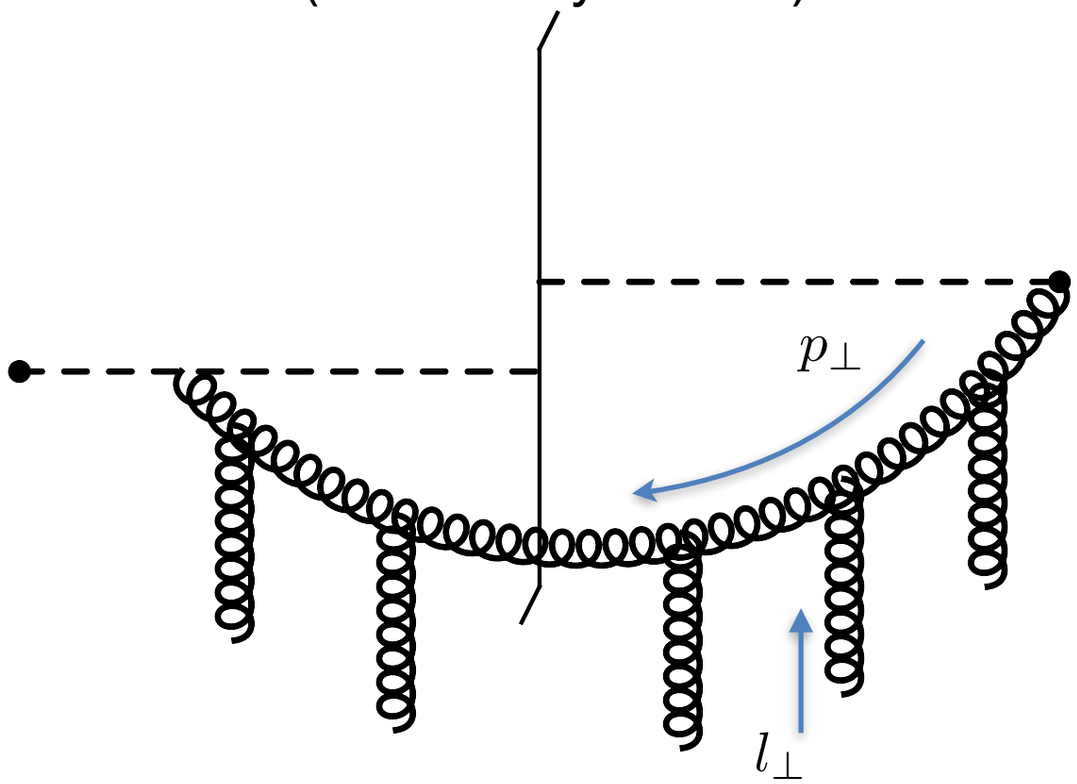
Small x

- Ordering in rapidity
  $\alpha > \alpha'$
- No ordering in transverse momentum
  $p_\perp \sim l_\perp$



# Evolution at small and moderate x

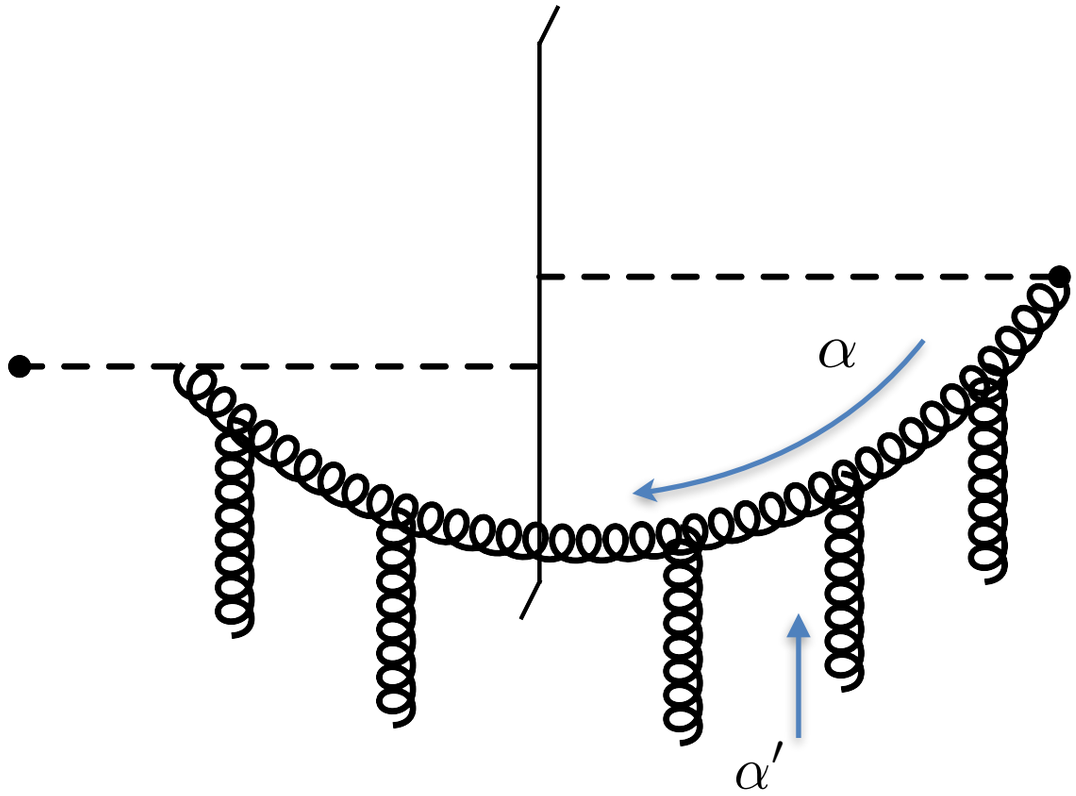
Moderate x (DGLAP dynamics)




light-cone expansion

Small x

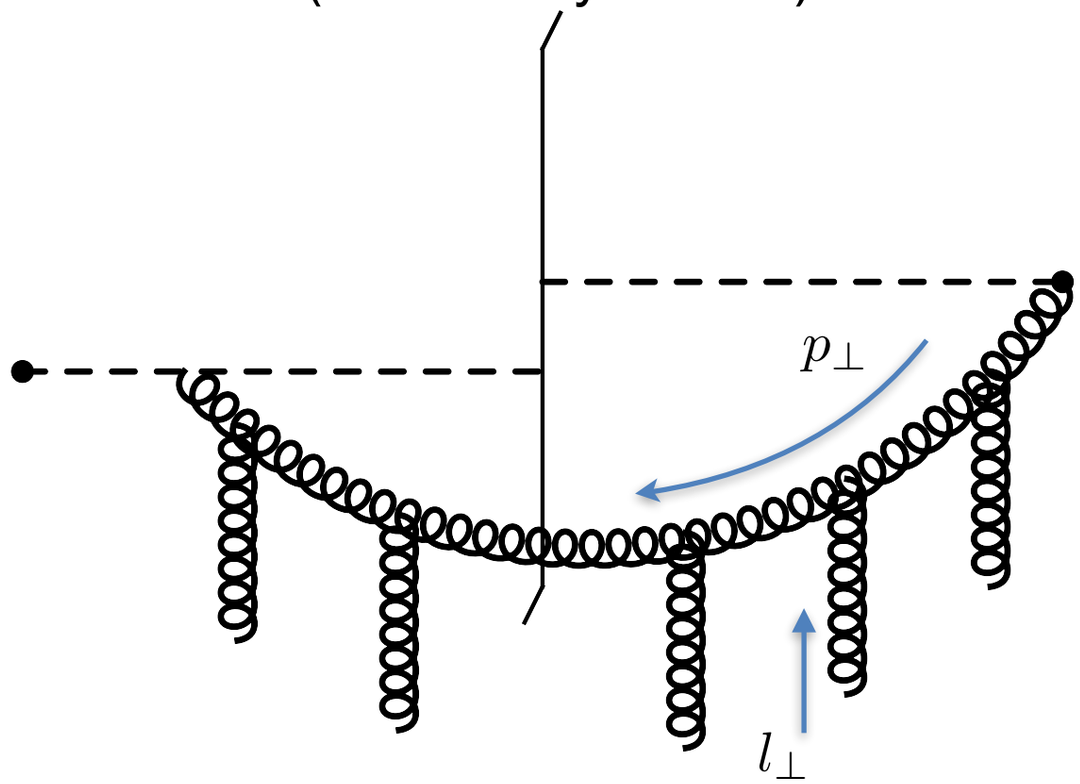
rapidity factorization approach




We consider emission in both limits and combine results

# Light-cone expansion (moderate x)

Moderate x (DGLAP dynamics)




- Parameter of expansion $l_\perp/p_\perp \ll 1$
- In coordinate representation is an expansion in powers of deviation from the light-cone
- Gluon propagator is linear in F

$$\lim_{k^2\to 0} k^2\langle A^a_\mu(k)A^b_\nu(y)\rangle = -ie^{i\frac{k_\perp^2}{\alpha s}y_*}e^{-ik_\perp y_\perp}\mathcal{G}^{ab}_{\mu\nu}(\infty,y_*;k_\perp)|_{y_\perp}$$

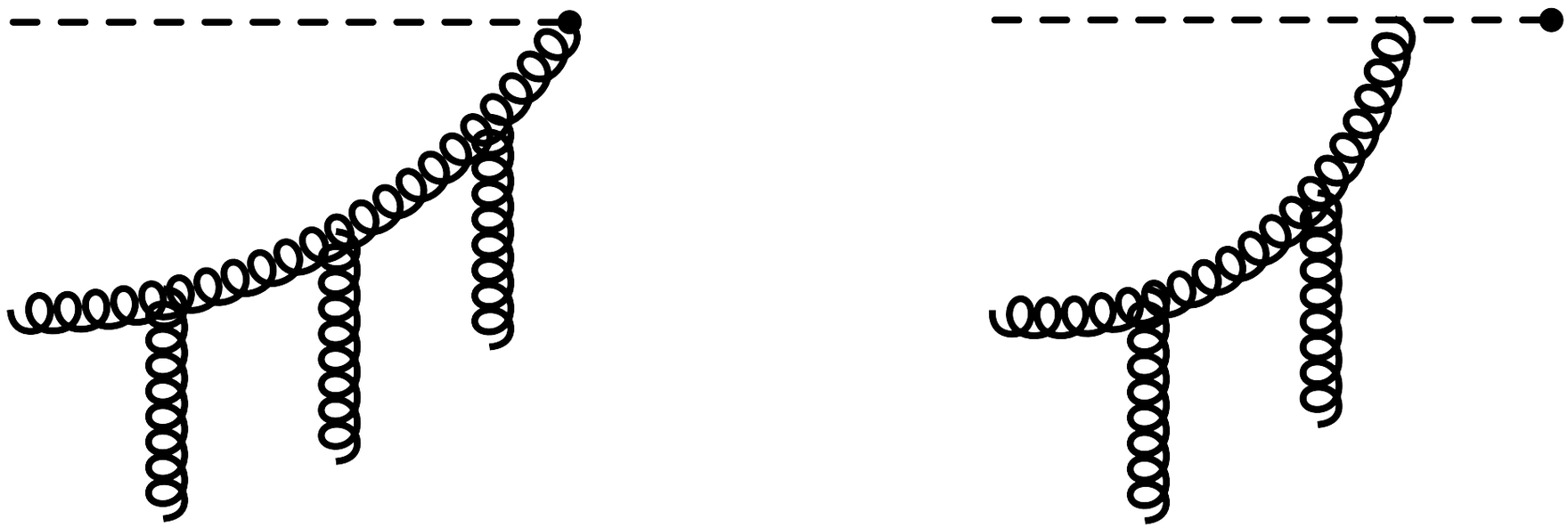
$$\mathcal{G}_{\mu\nu}(\infty,y_*,k_\perp)|_{y_\perp} = g_{\mu\nu}[\infty,y_*]_{y_\perp} + \int_{y_*}^{\infty} dz_*\Big(-\frac{4i}{\alpha s^2}k^j(z-y)_* g_{\mu\nu}[\infty,z_*]_{y_\perp}F_{\bullet j}[z_*,y_*]_{y_\perp} + \frac{4}{\alpha s^2}(p_{2\nu}\delta^j_\mu - p_{2\mu}\delta^j_\nu)[\infty,z_*]_{y_\perp}F_{\bullet j}[z_*,y_*]_{y_\perp}\Big)$$

The structure at small x (rapidity factorization) is different

# Light-cone expansion. Real emission

Lipatov vertex

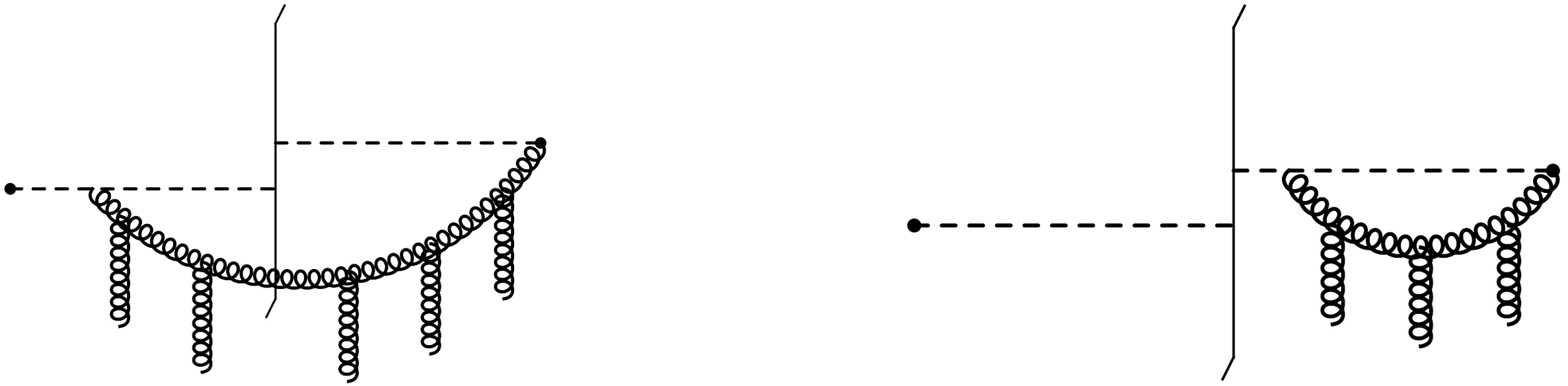
$$L^{ab}_{\mu i}(k, y_\perp, \beta_B) = i \lim_{k^2 \to 0} k^2 \langle A^a_\mu(k) \mathcal{F}^b_i(\beta_B, y_\perp) \rangle$$



$$\langle \tilde{\mathcal{F}}^a_i(\beta_B, x_\perp) \mathcal{F}^a_j(\beta_B, y_\perp) \rangle = - \int \frac{d\alpha}{2\alpha} d^2 k_\perp \tilde{L}^\mu_i(k, x_\perp, \beta_B) L_{\mu j}(k, y_\perp, \beta_B)$$

# Light-cone expansion. Real and virtual emission

$$\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp)|p\rangle \;=\; \frac{g^2N_c}{\pi}\int đ^2k_\perp\,\Big\{e^{i(k,x-y)_\perp}\langle p|\tilde{\mathcal{F}}_k^a\big(\beta_B+\frac{k_\perp^2}{\sigma s},x_\perp\big)\mathcal{F}_l^a\big(\beta_B+\frac{k_\perp^2}{\sigma s},y_\perp\big)|p\rangle\Big[\frac{\delta_i^k\delta_j^l}{k_\perp^2}-\frac{2\delta_i^k\delta_j^l}{\sigma\beta_Bs+k_\perp^2}$$
$$+\frac{k_\perp^2\delta_i^k\delta_j^l+\delta_j^k k_ik^l+\delta_i^l k_jk^k-\delta_j^l k_ik^k-\delta_i^k k_jk^l-g^{kl}k_ik_j-g_{ij}k^kk^l}{(\sigma\beta_Bs+k_\perp^2)^2}+k_\perp^2\frac{2g_{ij}k^kk^l+\delta_i^k k_jk^l+\delta_j^l k_ik^k-\delta_j^k k_ik^l-\delta_i^l k_jk^k}{(\sigma\beta_Bs+k_\perp^2)^3}$$
$$-\frac{k_\perp^4g_{ij}k^kk^l}{(\sigma\beta_Bs+k_\perp^2)^4}\Big]\theta\big(1-\beta_B-\frac{k_\perp^2}{\sigma s}\big)-\frac{\sigma\beta_Bs}{k_\perp^2(\sigma\beta_Bs+k_\perp^2)}\langle p|\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp)|p\rangle\Big\}$$



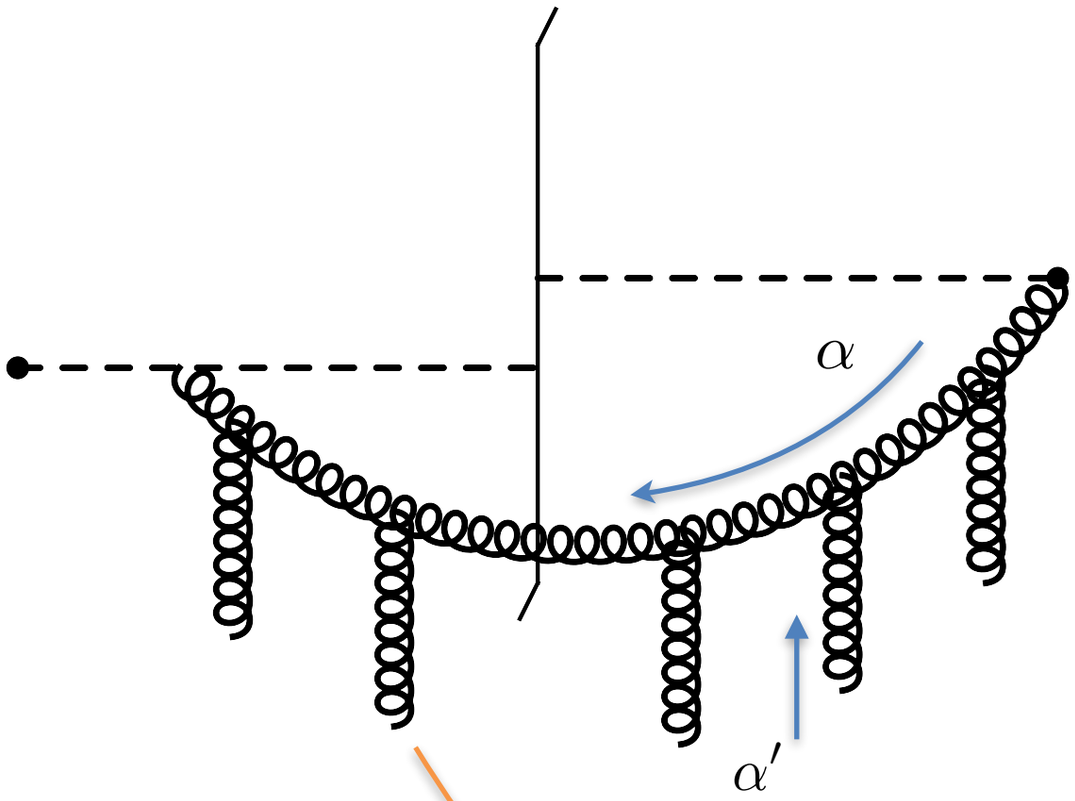
Collinear limit $x_\perp = y_\perp$

$$\mu^2\frac{d}{d\mu^2}\langle\tilde{\mathcal{F}}^{ni}(\beta_B,x_\perp)\mathcal{F}_i^n(\beta_B,x_\perp)\rangle=\frac{\alpha_s(\mu)}{\pi}N_c\int_{\beta_B}^1dz\Big[\frac{1}{z(1-z)_+}-2+z(1-z)\Big]\tilde{\mathcal{F}}^{ni}(\frac{\beta_B}{z},x_\perp)\mathcal{F}_i^n(\frac{\beta_B}{z},x_\perp)$$

DGLAP equation

$(k_\perp^2<\mu^2)$

# Rapidity factorization approach (small x)

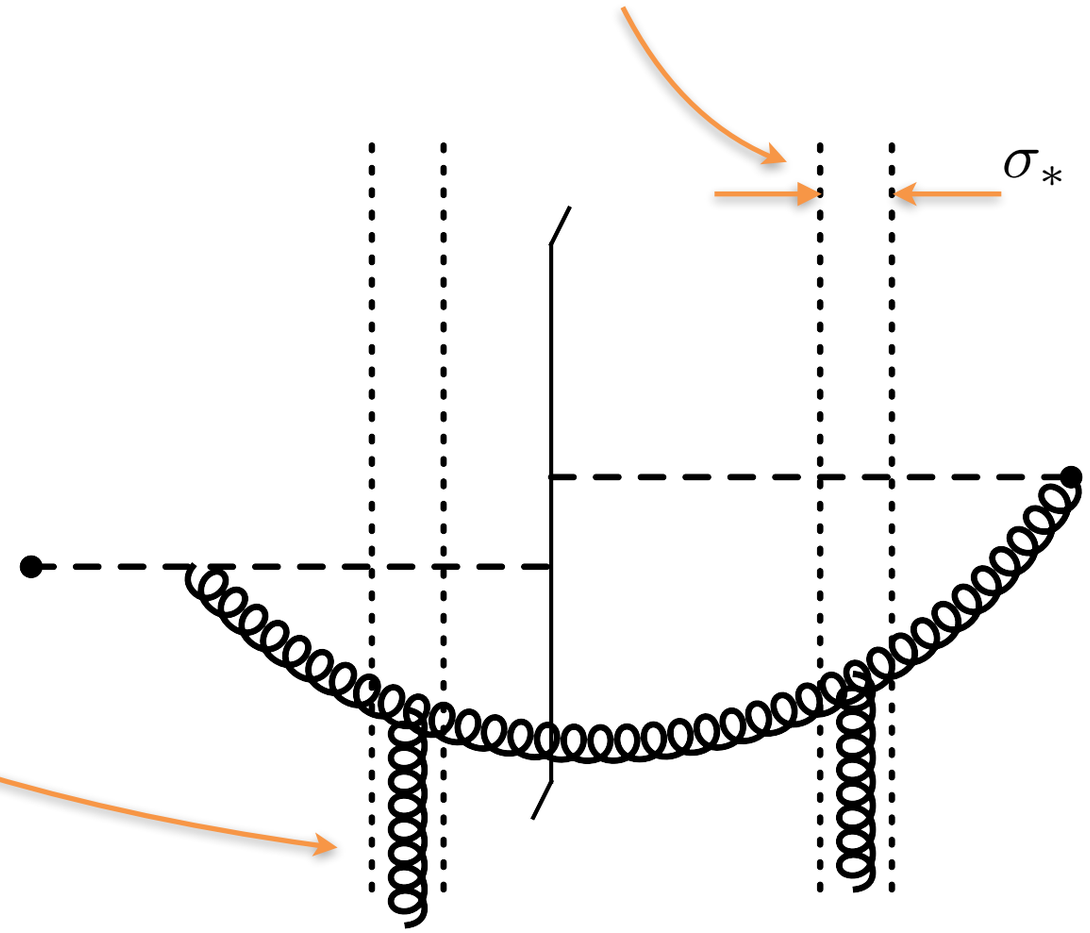



Separate slow $\alpha > \sigma'$

and fast fields $\alpha' < \sigma'$

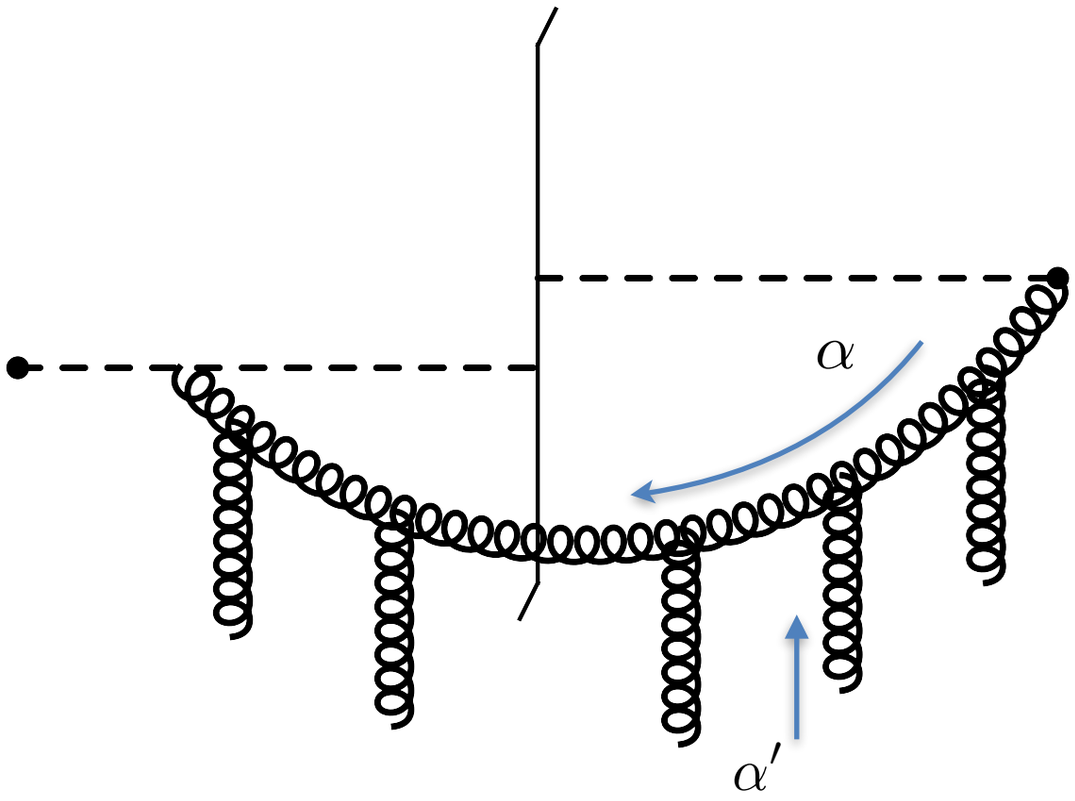
Fast fields form the shock-wave

To construct transition to moderate x we keep a finite width of the shock-wave




Interaction with external field is inside very thin shock-wave

# Rapidity factorization approach. Gluon propagator




Parameter of expansion $\frac{p_\perp^2}{\alpha s}\sigma_* \ll 1$

No ordering in transverse momentum $p_\perp \sim l_\perp$

The gluon propagator has tortuous structure: nonlinear in F terms and quark contribution

Quark contribution

$$\lim_{k^2\to 0} k^2\langle A_\mu^a(k)A_\nu^b(y)\rangle = -ie^{i\frac{k_\perp^2}{\alpha s}y_*}e^{-ik_\perp y_\perp}\mathcal{O}_{\mu\nu}^{ab}(\infty,y_*;k_\perp)|_{y_\perp}$$

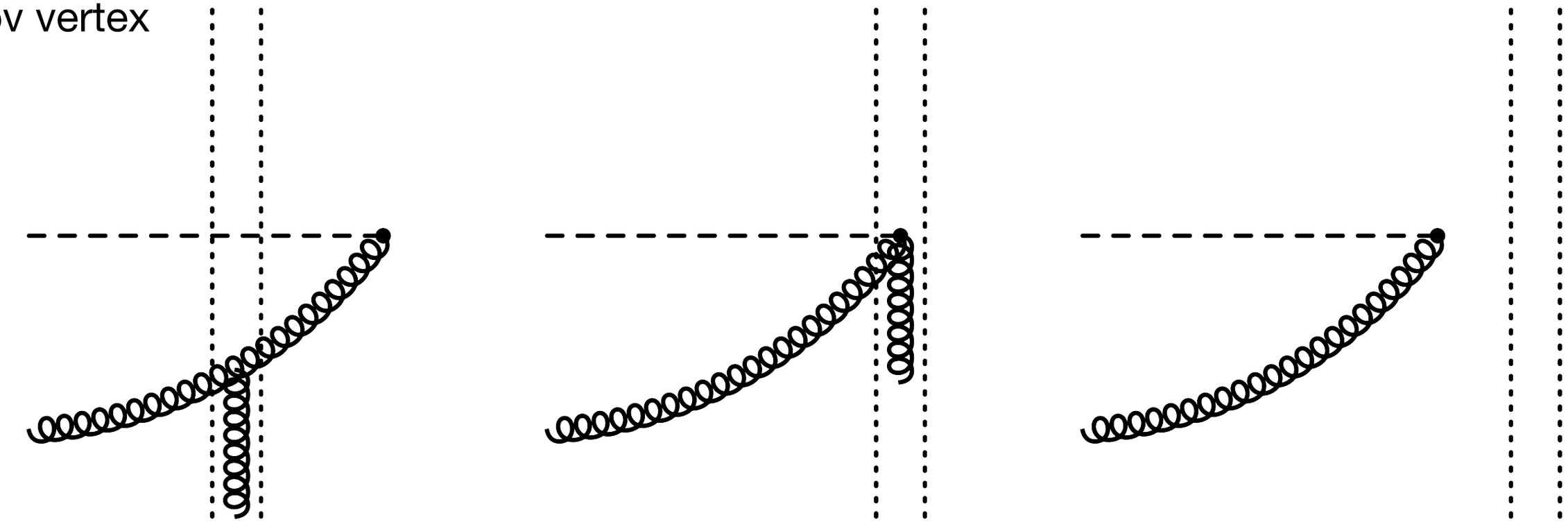
$$\mathcal{O}_{\mu\nu}^{ab}(\infty,y_*;k_\perp) = \mathcal{G}_{\mu\nu}^{ab}(\infty,y_*;k_\perp) + \mathcal{J}_{\mu\nu}^{ab}(\infty,y_*;k_\perp) + \bar{\mathcal{J}}_{\mu\nu}^{ab}(\infty,y_*;-k_\perp)$$

$$\mathcal{G}_{\mu\nu}(\infty,y_*,p_\perp) = g_{\mu\nu}[\infty,y_*] + \int_{y_*}^{\infty} dz_*\Big(-\frac{2i}{\alpha s^2}(z-y)_* g_{\mu\nu}\{2p^j[\infty,z_*]F_{\bullet j} - i[\infty,z_*]D^jF_{\bullet j}\} + \frac{4}{\alpha s^2}(p_{2\nu}\delta_\mu^j - p_{2\mu}\delta_\nu^j)[\infty,z_*]F_{\bullet j}\Big)[z_*,y_*]$$
$$+\frac{8}{\alpha s^3}\int_{y_*}^{\infty} dz_*\int_{y_*}^{z_*} dz_*'[ig_{\mu\nu}(z'-y)_* - \frac{2}{\alpha s}p_{2\mu}p_{2\nu}][\infty,z_*]F_{\bullet j}[z_*,z_*']F_\bullet{}^j[z_*',y_*]$$

Nonlinear terms

# Rapidity factorization approach. Real emission

Lipatov vertex



We take into account position of the shock-wave $U \equiv [\infty, -\infty]$

$$L^{ab}_{\mu i}(k, y_\perp, \beta_B) = i \lim_{k^2 \to 0} k^2 \langle A^a_\mu(k) \mathcal{F}^b_i(\beta_B, y_\perp) \rangle = 2 e^{-i k_\perp y_\perp} \left( \frac{p_{2\mu}}{\alpha s} - \frac{p_{1\mu} \alpha}{k^2_\perp} \right) (\mathcal{F}_i(\beta_B, y_\perp) - i \partial_i U_{y_\perp} U^\dagger_{y_\perp})^{ab}$$
$$+ (k_\perp | g_{\mu i} \left\{ \frac{\alpha \beta_B s}{\alpha \beta_B s + p^2_\perp} - U \frac{\alpha \beta_B s}{\alpha \beta_B s + p^2_\perp} U^\dagger \right\} + 2 \alpha p_{1\mu} \left\{ \frac{p_i}{\alpha \beta_B s + p^2_\perp} - U \frac{p_i}{\alpha \beta_B s + p^2_\perp} U^\dagger \right\} - \frac{2 i p_{1\mu} \alpha}{p^2_\perp} \partial_i U_{y_\perp} U^\dagger_{y_\perp}$$
$$+ \left\{ \frac{2i}{\alpha s} p_{2\mu} \partial_i U - \frac{2i}{\alpha \beta_B s} \partial^\perp_\mu U p_i + \frac{2 p_{2\mu}}{\beta_B \alpha^2 s^2} \partial^2_\perp U p_i \right\} \frac{\alpha \beta_B s}{\alpha \beta_B s + p^2_\perp} U^\dagger | y_\perp)^{ab}$$

Takes into account dependence on $\beta_B \neq 0$

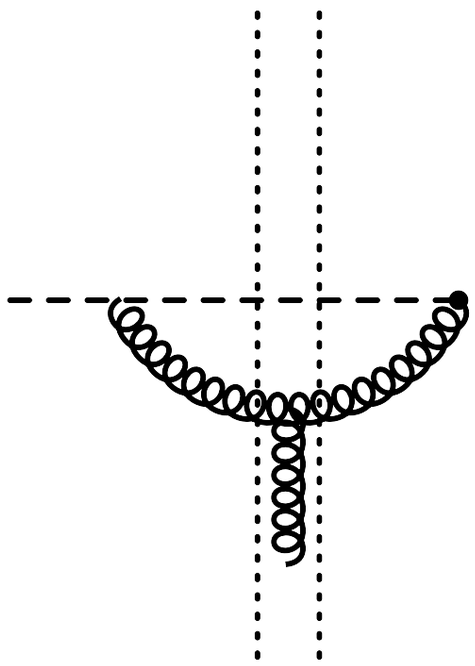
We apply the same strategy to calculation of the virtual correction

# One loop correction for the whole region of transverse momentum

One can unify result obtained for $p_\perp \gg l_\perp$ (light-cone expansion) and $p_\perp \sim l_\perp$ (rapidity factorization approach)

$$L^{ab}_{\mu i}(k, y_\perp, \beta_B) = i \lim_{k^2\to 0} k^2 \langle A^a_\mu(k) \mathcal{F}^b_i(\beta_B, y_\perp)\rangle =$$
$$\frac{2e^{-ik_\perp y_\perp}}{\alpha\beta_B s + k_\perp^2}\Big\{\frac{\alpha\beta_B s}{k_\perp^2}\Big(\frac{k_\perp^2}{\alpha s}p_{2\mu} - \alpha p_{1\mu}\Big)\delta^j_i - k_i\delta^j_\mu + \frac{\alpha\beta_B s g_{\mu i}k^j}{\alpha\beta_B s + k_\perp^2} + \frac{2\alpha k_i k^j}{\alpha\beta_B s + k_\perp^2}p_{1\mu}\Big\}\Big(\mathcal{F}^{ab}_j(\beta_B + \frac{k_\perp^2}{\alpha s}, y_\perp) - i\partial_j U U^{\dagger ab}\Big)$$
$$+(k_\perp|\frac{\alpha\beta_B s g_{\mu i} + 2\alpha p_{1\mu}k_i}{\alpha\beta_B s + k_\perp^2}(2ik^j\partial_j U - \partial_\perp^2 U)\frac{1}{\alpha\beta_B s + p_\perp^2}U^\dagger + 2i\alpha p_{1\mu}\partial_i U\frac{1}{\alpha\beta_B s + p_\perp^2}U^\dagger - \frac{2ip_{1\mu}\alpha}{p_\perp^2}\partial_i U_{y_\perp}U^\dagger_{y_\perp}$$
$$+\Big\{\frac{2i}{\alpha s}p_{2\mu}\partial_i U - \frac{2i}{\alpha\beta_B s}\partial^\perp_\mu U p_i + \frac{2p_{2\mu}}{\beta_B\alpha^2 s^2}\partial^2_\perp U p_i\Big\}\frac{\alpha\beta_B s}{\alpha\beta_B s + p_\perp^2}U^\dagger|y_\perp)^{ab}$$

Virtual emission:



$$\mathcal{F}^n_i(\beta_B, y_\perp) = \frac{2}{s}\int_{-\infty}^{\infty} dy_* e^{i\beta_B y_*}\langle[\infty, y_*]^{nm}_y g F^m_{\bullet i}(y_*, y_\perp)\rangle$$
$$= -if^{nkl}\int_{\sigma'}^{\sigma}\frac{d\alpha}{\alpha}\Big\{(y_\perp|\frac{1}{p_\perp^2}\Big(\alpha\beta_B s i\partial_i U + \partial^2_\perp U p_i\Big)\frac{1}{\alpha\beta_B s + p_\perp^2}U^\dagger|y_\perp)^{kl}$$
$$+(y_\perp|\frac{\alpha\beta_B s}{p_\perp^2(\alpha\beta_B s + p_\perp^2)}|y_\perp)\big(\mathcal{F}_i(\beta_B, y_\perp) - i\partial_i U_{y_\perp}U^\dagger_{y_\perp}\big)^{kl}\Big\}$$

- Valid in a full range of $p_\perp$
- Takes into account dependence on $\beta_B \neq 0$

# Evolution equation of gluon TMD

$$
\begin{aligned}
&\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}_i^a(x_\perp,\beta_B)\mathcal{F}_j^a(y_\perp,\beta_B)|p\rangle \\
&= \ 2\alpha_s\langle p|\mathrm{Tr}\Big\{\big(x_\perp\big|-\frac{1}{2}\Big[\tilde{U}\frac{p_\perp^2\delta_i^k+2p_ip^k}{\sigma\beta_Bs+p_\perp^2}\tilde{U}^\dagger-(\tilde{U}\to 1)\Big]\Theta\Big[U\frac{p_\perp^2g_{jk}+2p_jp_k}{\sigma\beta_Bs+p_\perp^2}U^\dagger-(U\to 1)\Big] \\
&+\breve{\breve{\mathcal{F}}}_k\Theta\big(\beta_B+\frac{p_\perp^2}{\sigma s}\big)\Big[\frac{g^{kl}p_i+\delta_i^kp^l-\delta_i^lp^k}{\sigma\beta_Bs+p_\perp^2}+\frac{p_\perp^2\delta_i^lp^k+2p_ip^kp^l}{(\sigma\beta_Bs+p_\perp^2)^2}\Big]\Big[U\frac{p_\perp^2g_{jl}+2p_jp_l}{\sigma\beta_Bs+p_\perp^2}U^\dagger-(U\to 1)\Big] \\
&+\Big[\tilde{U}\frac{p_\perp^2\delta_i^k+2p_ip^k}{\sigma\beta_Bs+p_\perp^2}\tilde{U}^\dagger-(\tilde{U}\to 1)\Big]\Big[\frac{g_{kl}p_j+g_{jl}p_k-g_{jk}p_l}{\sigma\beta_Bs+p_\perp^2}+\frac{p_\perp^2g_{jk}p_l+2p_jp_lp_k}{(\sigma\beta_Bs+p_\perp^2)^2}\Big]\Theta\breve{\mathcal{F}}^l\big(\beta_B+\frac{p_\perp^2}{\sigma s}\big) \\
&+2\breve{\breve{\mathcal{F}}}_k\big(\beta_B+\frac{p_\perp^2}{\sigma s}\big)\Big[\frac{p_\perp^2\delta_i^k\delta_j^l-g^{kl}p_ip_j+\delta_j^kp_ip^l+\delta_i^lp_jp^k-g_{ij}p^kp^l}{(\sigma\beta_Bs+p_\perp^2)^2} \\
&+p_\perp^2\frac{\delta_i^kp_jp^l+\delta_j^lp_ip^k-\delta_i^lp_jp^k-\delta_j^kp_ip^l+2g_{ij}p^kp^l}{(\sigma\beta_Bs+p_\perp^2)^3}-\frac{\delta_j^ip^lp^kp_\perp^4}{(\sigma\beta_Bs+p_\perp^2)^4}\Big]\Theta\breve{\mathcal{F}}_l\big(\beta_B+\frac{p_\perp^2}{\sigma s}\big) \\
&-\tilde{\mathcal{F}}_i\big(\beta_B+\frac{p_\perp^2}{\sigma s}\big)\Theta\frac{p^k}{p_\perp^2}\Big[U\frac{p_\perp^2g_{jk}+2p_jp_k}{\sigma\beta_Bs+p_\perp^2}U^\dagger-(U\to 1)\Big] \\
&-\Big[\tilde{U}\frac{p_\perp^2\delta_i^k+2p_ip^k}{\sigma\beta_Bs+p_\perp^2}\tilde{U}^\dagger-(\tilde{U}\to 1)\Big]\frac{p_k}{p_\perp^2}\Theta\mathcal{F}_j\big(\beta_B+\frac{p_\perp^2}{\sigma s}\big) \\
&-2\tilde{\mathcal{F}}_i\big(\beta_B+\frac{p_\perp^2}{\sigma s}\big)\Theta\Big[\frac{\delta_j^k}{\sigma\beta_Bs+p_\perp^2}+\frac{p_jp^k}{(\sigma\beta_Bs+p_\perp^2)^2}\Big]\breve{\mathcal{F}}_k\big(\beta_B+\frac{p_\perp^2}{\sigma s}\big) \\
&-2\breve{\breve{\mathcal{F}}}_i\big(\beta_B+\frac{p_\perp^2}{\sigma s}\big)\Big[\frac{\delta_j^k}{\sigma\beta_Bs+p_\perp^2}+\frac{p_jp^k}{(\sigma\beta_Bs+p_\perp^2)^2}\Big]\Theta\mathcal{F}_k\big(\beta_B+\frac{p_\perp^2}{\sigma s}\big)\big|y_\perp\big) \\
&-(x_\perp|\tilde{U}\frac{1}{\sigma\beta_Bs+p_\perp^2}(2\partial_i\partial_k\tilde{U}^\dagger+\partial_\perp^2\tilde{U}^\dagger)\frac{p^k}{p_\perp^2}|x_\perp)\mathcal{F}_j(y_\perp,\beta_B) \\
&-\tilde{\mathcal{F}}_i(x_\perp,\beta_B)(y_\perp|\frac{p^k}{p_\perp^2}(2\partial_j\partial_kU+g_{jk}\partial_\perp^2U)\frac{1}{\sigma\beta s+p_\perp^2}U^\dagger|y_\perp) \\
&+2\int\frac{\mathrm{d}^2p_\perp}{p_\perp^2}\Big[e^{i(p,x-y)_\perp}\theta\big(1-\beta_B-\frac{p_\perp^2}{\sigma s}\big)\tilde{\mathcal{F}}_i\big(\beta_B+\frac{p_\perp^2}{\sigma s},x_\perp\big)\mathcal{F}_j\big(\beta_B+\frac{p_\perp^2}{\sigma s},y_\perp\big) \\
&-\frac{\sigma\beta_Bs}{\sigma\beta_Bs+p_\perp^2}\tilde{\mathcal{F}}_i(\beta_B,x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\Big]\Big\}|p\rangle
\end{aligned}
$$

Nonlinear equation

- Valid for the whole range of $\sigma$ and $\beta_B$
- No infrared divergency
- Real emission part contains kinematical constraint $p_\perp^2<\sigma(1-\beta_B)s$

# DGLAP limit

$$\int d^2 z_\perp \; e^{i(k,z)_\perp} \langle p|\tilde{\mathcal{F}}_i^{a\eta}(z_\perp,\beta_B)\mathcal{F}_j^{a\eta}(0_\perp,\beta_B)|p+\xi p_2\rangle$$
$$= \; 2\pi^2\delta(\xi)\beta_B g^2\Big[-g_{ij}\mathcal{D}(\beta_B,k_\perp,\eta)+\big(\frac{2k_ik_j}{m^2}+g_{ij}\frac{k_\perp^2}{m^2}\big)\mathcal{H}(\beta_B,k_\perp,\eta)\Big]$$

$$\beta_B = x_B \sim 1$$

$$k_\perp^2 \sim (x-y)_\perp^{-2} \sim s$$

$$\sigma \lesssim 1$$

P. J. Mulders and J. Rodrigues, Phys. Rev. D63, 094021 (2001)

$$\frac{d}{d\eta}\alpha_s\mathcal{D}(\beta_B,z_\perp,\eta)$$
$$= \; \frac{\alpha_s N_c}{\pi}\int_{\beta_B}^1 \frac{dz'}{z'}\Big\{J_0\Big(|z_\perp|\sqrt{\sigma s\beta_B\frac{1-z'}{z'}}\Big)\Big[\big(\frac{1}{1-z'}\big)_+ + \frac{1}{z'} - 2 + z'(1-z')\Big]\alpha_s\mathcal{D}\big(\frac{\beta_B}{z'},z_\perp,\eta\big)$$
$$+ \; \frac{4}{m^2}(1-z')z'z_\perp^2 J_2\Big(|z_\perp|\sqrt{\sigma s\beta_B\frac{1-z'}{z'}}\Big)\alpha_s\mathcal{H}''\big(\frac{\beta_B}{z'},z_\perp,\eta\big)\Big\},$$

- Evolution equation is linear
- In the collinear case reproduce DGLAP

# Small x limit

$$\beta_B = x_B \sim \frac{(x-y)_\perp^{-2}}{s}$$

$$k_\perp^2 \sim (x-y)_\perp^{-2} \ll s$$

$$\frac{(x-y)_\perp^{-2}}{s} \ll \sigma \ll 1$$

$$\frac{d}{d\ln\sigma}\langle \tilde{U}_i^a(x_\perp) U_j^a(y_\perp)\rangle = -4\alpha_s Tr\Big\{(x_\perp|\tilde{U}p_i\tilde{U}^\dagger\Big(\tilde{U}\frac{p^k}{p_\perp^2}\tilde{U}^\dagger - \frac{p^k}{p_\perp^2}\Big)\Big(U\frac{p_k}{p_\perp^2}U^\dagger - \frac{p_k}{p_\perp^2}\Big)Up_jU^\dagger|y_\perp)$$
$$-\tilde{U}_i(x_\perp)\Big[(y_\perp|\frac{p^k}{p_\perp^2}U\frac{p_kp_j}{p_\perp^2}U^\dagger|y_\perp) - \frac{1}{2}(y_\perp|\frac{1}{p_\perp^2}|y_\perp)U_j(y_\perp)\Big] - \Big[(x_\perp|\tilde{U}\frac{p_ip^k}{p_\perp^2}\tilde{U}^\dagger\frac{p_k}{p_\perp^2}|x_\perp) - \frac{1}{2}(x_\perp|\frac{1}{p_\perp^2}|x_\perp)\tilde{U}_k(x_\perp)\Big]U_j(y_\perp)\Big\}$$

- Nonlinear equation

$$\frac{d}{d\ln\sigma}\langle \tilde{U}_i^a(x_\perp) U_j^a(y_\perp)\rangle = -\frac{g^2}{8\pi^3}\int d^2z_\perp Tr\Big\{(-i\overrightarrow{\partial_i^x} + \tilde{U}_i^x)\Big(\tilde{U}_x\tilde{U}_z^\dagger - 1\Big)\frac{(x-y)_\perp^2}{(x-z)_\perp^2(z-y)_\perp^2}\Big(U_zU_y^\dagger - 1\Big)(i\overleftarrow{\partial_j^y} + U_j^y)\Big\}$$

I.Balitsky, A.T. (2014)

$$U_i \equiv i\partial_i UU^\dagger$$

# Sudakov limit

$$\beta_B = x_B \sim 1$$

$$k_\perp^2 \sim (x-y)_\perp^{-2} \sim few\ GeV^2$$

$$\sigma \lesssim 1$$

No restriction for the virtual emission

$$\frac{d}{d\ln\sigma}\tilde{\mathcal{F}}_i^a(x_\perp,\beta_B)\mathcal{F}_j^a(y_\perp,\beta_B) = -\frac{g^2N_c}{\pi}\int\frac{d^2p_\perp}{p_\perp^2}\left[1-e^{i(p,x-y)_\perp}\right]\tilde{\mathcal{F}}_i^a(x_\perp,\beta_B)\mathcal{F}_j^a(y_\perp,\beta_B)$$

Real emission is restricted to the small $p_\perp$ region

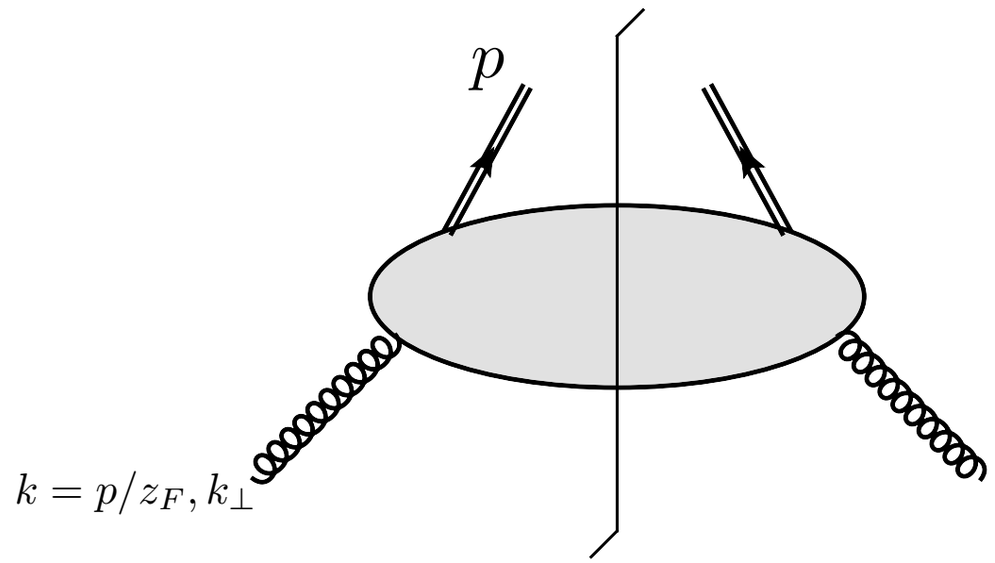
$$\mathcal{D}(\beta_B,k_\perp,\ln\sigma) \sim \exp\Big\{-\frac{\alpha_s N_c}{2\pi}\ln^2\frac{\sigma s}{k_\perp^2}\Big\}\mathcal{D}(\beta_B,k_\perp,\ln\frac{k_\perp^2}{s})$$

# Fragmentation function

$$\alpha_s \mathcal{D}_f(x_F, z_\perp) = -\frac{z_F}{8\pi^2 (p\cdot n)} \int du e^{i\frac{1}{z_F}(p\cdot n)u} \sum_X \langle 0|\tilde{\mathcal{F}}_i^a(z_\perp + un)|X+p\rangle\langle X+p|\mathcal{F}^{ai}(0)|0\rangle$$

$$\langle 0|U|p+X\rangle = 0$$

For fragmentation function non-linearity is lost




$$\frac{d}{d\eta}\alpha_s \mathcal{D}_{\rm f}(-\beta_F, z_\perp, \eta)$$

$$= \frac{\alpha_s N_c}{\pi}\int_{z_F}^1 \frac{dz'}{z'^2}\Big\{ J_0\Big(|z_\perp|\sqrt{\frac{\sigma s}{z_F}(1-z')}\Big)\Big[\big(\frac{1}{1-z'}\big)_+ + \frac{1}{z'} - 2 + z'(1-z')\Big]\alpha_s \mathcal{D}_{\rm f}\big(\frac{z'}{z_F}, z_\perp, \eta\big)$$

$$+ \frac{4}{m^2}\frac{1-z'}{z'} z_\perp^2 J_2\Big(|z_\perp|\sqrt{\frac{\sigma s}{z_F}(1-z')}\Big)\alpha_s \mathcal{H}''_{\rm f}\big(\frac{z'}{z_F}, z_\perp, \eta\big)\Big\},$$

collinear case

$$\frac{d}{d\ln\mu^2}\alpha_s(\mu) d_g^{\rm f}(z_F, \ln\mu^2) =$$

$$= \frac{\alpha_s(\mu)}{\pi} N_c \int_{z_F}^1 \frac{dz'}{z'}\Big(\big[\frac{1}{1-z'}\big]_+ + \frac{1}{z'} + z'(1-z') - 2\Big)\alpha_s(\mu) d_g^{\rm f}\big(\frac{z_F}{z'}, \ln\mu^2\big)$$

# Conclusion

- There is a difference between small and moderate x even at the level of definitions
- We've use light-cone expansion at moderate x and rapidity factorization at small x
- We've constructed evolution equation which is valid in both regions
- The equation reproduces different limits (BK, DGLAP and Sudakov)
- The equation is linear at moderate and non-linear at small x
- However evolution of the gluon fragmentation function is linear