Leaky quantum wires. On relation between geometry and spectrum.

Sylwia Kondej

Institute of Physics, University of Zielona Góra

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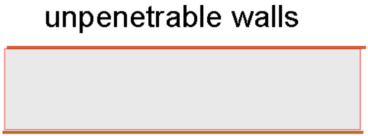


$\Omega = R \times [0, \pi]$, unpenetrable walls- Dirichlets boundary conditions

Hamiltonian: $H = -\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)$, $\frac{\hbar^2}{2m} = 1$

"transversal energies" $E_k = \{k^2\}_{k=1}^{\infty}$, ef. $\chi_k(y) = \sin ky$

"longitudinal energies" $[0, \infty)$, generalized eigenfunctions e^{ipx}

Energies of a particle in a straight waveguide, cont.

"Eigenfunctions:" $\psi_k = e^{ipx} \sin ky$,

$$H\psi_k = -(p^2 + k^2)\psi_k \,, \quad p \in R, \quad k \in N$$

Consequently, the spectrum of H is continuous and takes the form

$$\sigma_{ess} = [1 \,, \infty]$$

A curved, asymptotically straight wavequide, P.Duclos, P.Exner, at al.

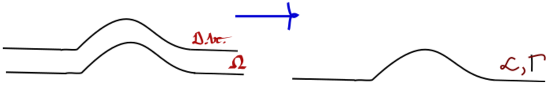
Deformation of the straight wavequide

- Stability of the essential spectrum,
- Bending acts as an attractive potential

Spectrum of Hamiltonian $H = -\Delta_{\Omega}$

$$\sigma_{ess}(H) = [1, \infty)\,, \quad \sigma_d(H) \neq \emptyset$$

Particle moving in delta potential



- Two dimensional system $L^2(R^2)$, potential supported by a curve Γ.

$$"H = -\Delta - \alpha\delta_\Gamma" \qquad \alpha > 0\,.$$

Serious definition of Hamiltonian

$$\mathcal{E}[f] = \int_{R^2} |\nabla f|^2 - \alpha \int_\Gamma |f|^2\,, \quad f \in W^{1,2}(\Omega)\,.$$

and

$$(Hf, f) = \mathcal{E}[f]\,.$$

Straight line interaction

$$"H = -\Delta - \alpha\delta_\Sigma"$$

Transversal component: $-\Delta - \alpha\delta(x)$

$$-\frac{\alpha^2}{4} \cup [0, \infty)\,,$$

Spectrum of H

$$\sigma(H) = [-\frac{\alpha^2}{4}\,, \infty)$$

Curved wire. 2D: Exner, Ichinose; 3D: Exner, SK.

Bending acts as an attractive potential for quantum wires

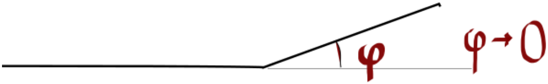
Γ- infinite asymptotically straight, C^2 piecewise, (no intersections, etc.) ‘left’ and ‘right’ tangential vectors are well-defined.

Spectrum

$$\sigma_{ess}(H) = [-\frac{\alpha^2}{4}, \infty), \quad \sigma_d(H) \neq \emptyset$$

Formulation of the problem

Γ- angle.



What is an asymptotics of the discrete spectrum is we approach to a straight line?

Some inspirations...

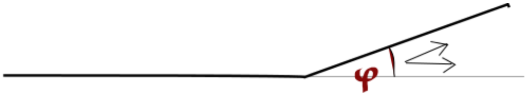
Curvature κ as an effective potential:

$$-\frac{d^2}{ds^2} - \frac{\kappa^2}{4} \,.$$

for $\alpha \to \infty$.
If $\kappa \to \beta\kappa$ then the energies $E = \mathcal{O}(\beta^4)$.

Total curvature

$\int_s^{s'} \kappa(t)$ measures the angle between tangential vectors $t(s)$ and $t(s')$.



I class of deformations and the result

Theorem [P.Exner, SK '15].

For φ small enough there is a unique discrete eigenvalue which admits the asymptotics

$$\lambda = -\frac{\alpha^2}{4} - A\varphi^4 + o(\varphi^4)\,, \quad A > 0\,.$$

Parameterization of Γ_φ: $s \mapsto \gamma_\varphi(s)$

A is defined by the first perturbation term of

$$K_0(|\gamma_\varphi(s) - \gamma_\varphi(s')|) - K_0(|s - s'|)$$

Generalization

Curvature of Γ_β is defined by $\beta\kappa$.

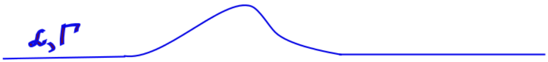
$$\Gamma_\beta(s) = (\int_0^s (\cos \int_0^u \beta\kappa du')du, \int_0^s (\sin \int_0^u \beta\kappa du')du)$$

Then

$$\lambda = -\frac{\alpha^2}{4} + B_\Gamma \beta^4 + o(\beta^4)$$

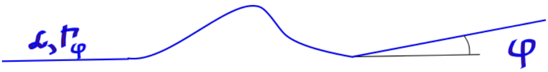
II class of deformations

Γ_{φ}- infinite, asymptotically straight - with the same straight line at infinity.



Hamiltonian has the discrete eigenvalues: λ_k, $k \in N$

Introduce deformation



Theorem [P.Exner, SK '15].

For φ small enough eigenvalues of H_{Γ_φ} admit the asymptotics

$$\lambda_k + A_k \varphi + o(\varphi)$$

Some ideas of the proof

Birman–Schwinger argument.
Analysis of the resolvent, $H = -\Delta + V$

$$(H-z)^{-1} = R(z) - R(z)V^{1/2}[\mathbf{I} + |\mathbf{V}|^{\mathbf{1/2}}\mathbf{R}(\mathbf{z})\mathbf{V}^{\mathbf{1/2}}]^{-1}|V|^{1/2}R(z)\,,$$

where $R(z) = (-\Delta - z)^{-1}$

Analysis of "poles" of $I + |V|^{1/2}R(z)V^{1/2}$

i.e. z such that

$$\ker(I + |V|^{1/2}R(z)V^{1/2}) \neq \emptyset$$

Thank you for your attention
Dziękuję za uwagę