False vacuum as a quantum unstable state

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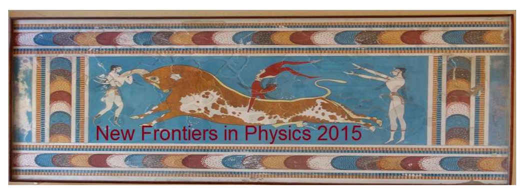

1 Introduction

The problem of false vacuum decay became famous after the publication of pioneer papers by Coleman and his colleagues [1, 2, 3]. The instability of a physical system in a state which is not an absolute minimum of its energy density, and which is separated from the minimum by an effective potential barrier was discussed there. It was shown, in those papers, that even if the state of the early Universe is too cold to activate a "*thermal*" transition (via thermal fluctuations) to the lowest energy (i.e. *"true vacuum"*) state, a quantum decay from the false vacuum to the true vacuum may still be possible through a barrier penetration via macroscopic quantum tunneling. Not long ago, the decay of the false vacuum state in a cosmological context has attracted interest, especially in view of its possible relevance in the process of tunneling among the many vacuum states of the string landscape (a set of vacua in the low energy approximation of string theory). In many models the scalar field potential driving inflation has a multiple, low–energy minima or *"false vacuua"*. Then the absolute minimum of the energy density is the *"true vacuum"*.
Recently the problem of the instability the false vacuum state triggered much discussion in the context of the discovery of the Higgs–like resonance at 125 — 126 GeV (see, eg., [4, 5, 6, 7]).
In the recent analysis [5] assuming the validity of the Standard Model up to Planckian energies it was shown that a Higgs mass $m_h < 126$ GeV implies that the electroweak vacuum is a metastable state. This means that a discussion of Higgs vacuum stability must be considered in a cosmological framework, especially when analyzing inflationary processes or the process of tunneling among the many vacuum states of the string landscape.
Krauss and Dent analyzing a false vacuum decay [8, 9] pointed out that in eternal inflation, even though regions of false vacua by assumption should decay exponentially, gravitational effects force space in a region that has not decayed yet to grow exponentially fast. This effect causes that many false vacuum regions can survive up to the times much later than times when the exponential decay law holds. In the mentioned paper by Krauss and Dent the attention was focused on the possible behavior of the unstable false vacuum at very late times, where deviations from the exponential decay law become to be dominat.
The aim of this presentation is to analyze properties of the false vacuum state as an unstable state, the form of the decay law from the canonical decay times t up to asymptotically late times and to discuss the late time behavior of the energy of the false vacuum states.

2 Unstable states in short

If $|M\rangle$ is an initial unstable state then the survival probability, $\mathcal{P}(t)$, equals $\mathcal{P}(t) = |a(t)|^2$, where $a(t)$ is the survival amplitude, $a(t) = \langle M|M;t\rangle$, and $a(0) = 1$, and, $|M;t\rangle = \exp[-it\mathfrak{H}]\,|M\rangle$, $\mathfrak{H}$ is the total Hamiltonian of the system under considerations. (The units $\hbar = c = 1$ are used in this presentation). The spectrum, $\sigma(\mathfrak{H})$, of $\mathfrak{H}$ is assumed to be bounded from below, $\sigma(\mathfrak{H}) = [E_{min}, \infty)$ and $E_{min} > -\infty$.
From basic principles of quantum theory it is known that the amplitude $a(t)$, and thus the decay law $\mathcal{P}(t)$ of the unstable state $|M\rangle$, are completely determined by the density of the energy distribution function $\omega(\mathcal{E})$ for the system in this state

$$a(t) = \int_{Spec.(\mathfrak{H})} \omega(E)\, e^{-i\,E\,t}\, dE, \tag{1}$$

where $\omega(E) \geq 0$ for $E \geq E_{min}$ and $\omega(E) = 0$ for $E < E_{min}$. From this last condition and from the Paley–Wiener Theorem it follows that there must be [10] $|a(t)| \geq A\exp[-b\,t^q]$ for $|t| \to \infty$. Here $A > 0$, $b > 0$ and $0 < q < 1$. This means that the decay law $\mathcal{P}(t)$ of unstable states decaying in the vacuum can not be described by an exponential function of time t if time t is suitably long, $t \to \infty$, and that for these lengths of time $\mathcal{P}(t)$ tends to zero as $t \to \infty$ more slowly than any exponential function of t. The analysis of the models of the decay processes shows that $\mathcal{P}(t) \simeq \exp[-\varGamma_M^0 t]$, (where $\varGamma_M^0$ is the decay rate of the state $|M\rangle$), to an very high accuracy at the canonical decay times t: From t suitably later than the initial instant t_0 up to $t \gg \tau = 1/\varGamma_M^0$, ($\tau$ is a lifetime), and smaller than $t = T$, where T is the crossover time and it denotes the time t for which the non–exponential deviations of $a(t)$ begin to dominate.
In general, in the case of quasi–stationary (metastable) states it is convenient to express $a(t)$ in the following form: $a(t) = a_c(t) + a_{non}(t)$, where $a_c(t)$ is the exponential (canonical) part of $a(t)$, that is $a_c(t) = N\,\exp[-it(E_M^0 - \frac{i}{2}\varGamma_M^0)]$, ($E_M^0$ is the energy of the system in the state $|M\rangle$ measured at the canonical decay times, N is the normalization constant), and $a_{non}(t)$ is the non–exponential part of $a(t)$). For times $t \sim \tau$: $|a_c(t)| \gg |a_{non}(t)|$. The crossover time T can be found by solving the following equation,

$$|a_c(t)|^2 = |a_{non}(t)|^2. \tag{2}$$

The amplitude $a_{non}(t)$ exhibits inverse power–law behavior at the late time region: $t \gg T$. Indeed, the integral representation (1) of $a(t)$ means that $a(t)$ is the Fourier transform of the energy distribution function $\omega(E)$. Using this fact we can find asymptotic form of $a(t)$ for $t \to \infty$. Results are rigorous. If to assume that

$$\omega(E) = (E - E_{min})^{\lambda}\, \eta(E) \;\in\; L_1(-\infty, \infty), \tag{3}$$

(where $0 \leq \lambda < 1$), and $\eta(E_{min}) \stackrel{\text{def}}{=} \eta_0 > 0$, and $\eta^{(k)}(E) = \frac{d}{dE}\eta(E)$, $(k = 0, 1, \ldots, n)$, exist and they are continuous in $[E_{min}, \infty)$, and limits $\lim_{E \to E_{min}+} \eta^{(k)}(E) \stackrel{\text{def}}{=} \eta_0^{(k)}$ exist, and $\lim_{E\to\infty} (E - E_{min})^{\lambda}\,\eta^{(k)}(E) = 0$ for all above mentioned k, then one finds that

$$\begin{aligned} a(t) \underset{t\to\infty}{\sim} & (-1)\, e^{-iE_{min}t} \Big[\Big(-\frac{i}{t}\Big)^{\lambda+1} \Gamma(\lambda+1)\, \eta_0 \\ & + \lambda \Big(-\frac{i}{t}\Big)^{\lambda+2} \Gamma(\lambda+2)\, \eta_0^{(1)} + \ldots \Big] = a_{non}(t). \end{aligned} \tag{4}$$

From (4) it is seen that asymptotically late time behavior of the survival amplitude $a(t)$ depends rather weakly on a specific form of the energy density $\omega(E)$. The same concerns a decay curves $\mathcal{P}(t) = |a(t)|^2$. A typical form of a decay curve, that is the dependence on time t of $\mathcal{P}(t)$ when t varies from $t = t_0 = 0$ up to $t > 30\tau$ is presented in panels A od Figs (1) — (3). Results presented in these Figures were obtained for the Breit–Wigner energy distribution function, $\omega(E) \equiv \omega_{BW} = \frac{N}{2\pi}\,\Theta(E - E_{min})\,\frac{\varGamma_M^0}{(E - E_M^0)^2 + (\varGamma_M^0/2)^2}$, which corresponds with $\lambda = 0$ in (3).

3 Instantaneous energy and instantaneous decay rate

The amplitude $a(t)$ contains information about the decay law $\mathcal{P}(t)$ of the state $|M\rangle$, that is about the decay rate $\varGamma_M^0$ of this state, as well as the energy E_M^0 of the system in this state. This information can be extracted from $a(t)$. Indeed if $|M\rangle$ is an unstable (a quasi–stationary) state then $a(t) \cong \exp[-i(E_M^0 - \frac{i}{2}\varGamma_M^0)\,t] \equiv a_c(t)$ for $t \sim \tau$. So, there is

$$E_M^0 - \frac{i}{2}\varGamma_M^0 \equiv i\,\frac{\partial a_c(t)}{\partial t}\,\frac{1}{a_c(t)}, \quad (t \sim \tau), \tag{5}$$

in the case of quasi–stationary states.
The standard interpretation and understanding of the quantum theory and the related construction of our measuring devices are such that detecting the energy E_M^0 and decay rate $\varGamma_M^0$ one is sure that the amplitude $a(t)$ has the canonical form $a_c(t)$ and thus that the relation (5) occurs. Taking the above into account one can define the "effective Hamiltonian", h_M, for the one–dimensional subspace of states $\mathcal{H}_{||}$ spanned by the normalized vector $|M\rangle$ as follows [11, 12]

$$h_M \stackrel{\text{def}}{=} i\,\frac{\partial a(t)}{\partial t}\,\frac{1}{a(t)} \stackrel{\text{def}}{=} \mathcal{E}_M(t) - \frac{i}{2}\,\gamma_M(t). \tag{6}$$

In general, h_M can depend on time t, $h_M \equiv h_M(t)$. One meets this effective Hamiltonian when one starts with the Schrödinger Equation for the total state space $\mathcal{H}$ and looks for the rigorous evolution equation for the distinguished subspace of states $\mathcal{H}_{||} \subset \mathcal{H}$ [11, 12]. Thus, one finds the following expressions for the energy and the decay rate of the system in the state $|M\rangle$ under considerations, to be more precise for the instantaneous energy $\mathcal{E}_M(t)$ and the instantaneous decay rate, $\gamma_M(t)$ [11],

$$\mathcal{E}_M \equiv \mathcal{E}_M(t) = \Re\,(h_M(t)), \qquad \gamma_M \equiv \gamma_M(t) = -\,2\,\Im\,(h_M(t)), \tag{7}$$

where $\Re\,(z)$ and $\Im\,(z)$ denote the real and imaginary parts of z respectively.

Starting from the asymptotic expression (4) for $a(t)$ and using (6) after some algebra one finds for times $t \gg T$ that

$$h_M(t)\big|_{t\to\infty} \simeq E_{min} + (-\frac{i}{t})\,c_1 + (-\frac{i}{t})^2\,c_2 + \ldots, \tag{8}$$

where $c_i = c_i^*$, $i = 1, 2, \ldots$; (coefficients c_i depend on $\omega(E)$). This last relation means that

$$\mathcal{E}_M(t) \simeq E_{min} - \frac{c_2}{t^2}\ldots, \qquad \gamma_M(t) \simeq 2\,\frac{c_1}{t} + \ldots, \quad (\text{for } t \gg T). \tag{9}$$

These properties take place for all unstable states which survived up to times $t \gg T$.
Note that from (9) it follows that $\lim_{t\to\infty} \mathcal{E}_M(t) = E_{min}$ and $\lim_{t\to\infty} \gamma_M(t) = 0$.
For the density $\omega(E)$ of the form (3) (i. e. for $a(t)$ having the asymptotic form given by (4)) we have

$$c_1 = \lambda + 1, \quad c_2 = (\lambda + 1)\,\frac{\eta^{(1)}(E_{min})}{\eta(E_{min})}. \tag{10}$$

For the most general form (3) of the density $\omega(E)$ (i. e. for $a(t)$ having the asymptotic form given by (4)) we have

$$c_1 = \lambda + 1, \quad c_2 = (\lambda + 1)\,\frac{\eta^{(1)}(E_{min})}{\eta(E_{min})}. \tag{11}$$

The energy densities $\omega(E)$ considered in quantum mechanics and in quantum field theory can be described by $\omega(E)$ of the form (3), eg. quantum field theory models correspond with $\lambda = \frac{1}{2}$.
A general form of

$$\kappa(t) = \frac{\mathcal{E}_M(t) - E_{min}}{E_M^0 - E_{min}}, \tag{12}$$

as a function of time t varying from $t = t_0 = 0$ up to $t > T$ is presented in panels B of Figs (1) — (3). These results were obtained for $\omega(E) = \omega_{BW}(E)$. The crossover time T, that is the time region where fluctuations of $\mathcal{P}_M(t)$ and $\mathcal{E}_M(t)$ take place depend on the value of the parameter $s_0 = (E_M^0 - E_{min})/\varGamma_M^0$ in the model considered : The smaller s_0 the shorter T.

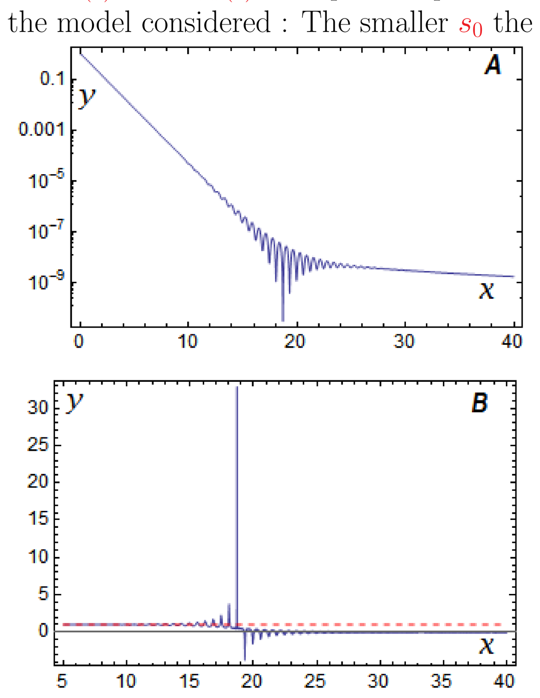

Figure 1: *The case $s_0 = 10$. Panel A — Axes: $y = \mathcal{P}(t)$, The logarithmic scale), $x = t/\tau$. Panel B, Axes: $y = \kappa(t)$, $x = t/\tau$. The horizontal red dashed line denotes $y = \kappa(t) = 1$.*

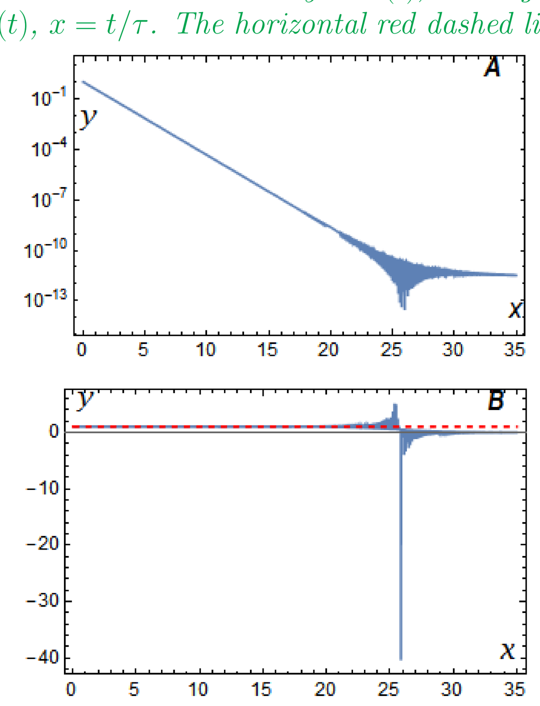

Figure 2: *The case $s_0 = 50$. Panel A — Axes: $y = \mathcal{P}(t)$, The logarithmic scale), $x = t/\tau$. Panel B, Axes: $y = \kappa(t)$, $x = t/\tau$. The horizontal red dashed line denotes $y = \kappa(t) = 1$, that is $\mathcal{E}_M(t) = E_M^0$.*

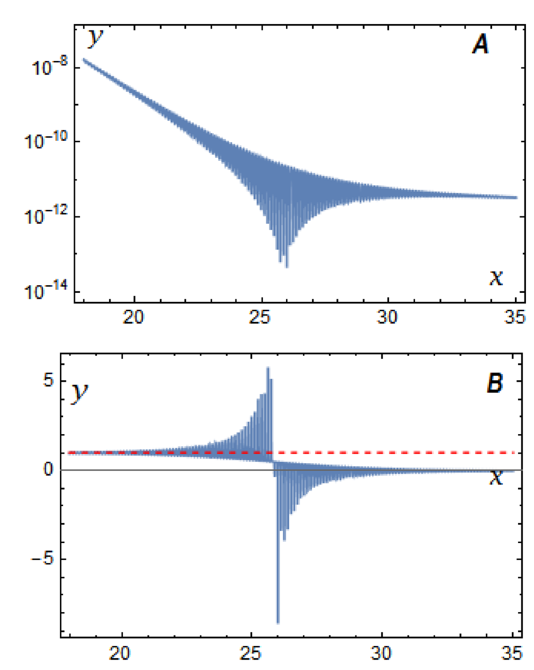

Figure 3: *The case $s_0 = 50$. The enlarged part of Fig. 2. Panel A — Axes: $y = \mathcal{P}(t)$, The logarithmic scale), $x = t/\tau$. Panel B, Axes: $y = \kappa(t)$, $x = t/\tau$.*

4 Cosmological applications

Krauss and Dent in their paper [8] mentioned earlier made a hypothesis that some false vacuum regions do survive well up to the time T or later. Let $|M\rangle = |0\rangle^{false}$, be a false, $|0\rangle^{true}$ – a true, vacuum states and E_0^{false} be the energy of a state corresponding to the false vacuum measured at the canonical decay time and E_0^{true} be the energy of true vacuum (i.e. the true ground state of the system). As it is seen from the results presented in previous Section, the problem is that the energy of those false vacuum regions which survived up to T and much later differs from E_0^{false} [13].
Now, if one assumes that $E_0^{true} \equiv E_{min}$ and $E_0^{false} = E_M^0$. and takes into account results of the previous Section (including those in Panels B of Figs (1) — (3) then one can conclude that the energy of the system in the false vacuum state has the following general properties:

$$\mathcal{E}_0^{false}(t) = E_0^{true} + \Delta E\;\cdot\;\kappa(t), \tag{13}$$

where $\Delta E = E_0^{false} - E_0^{true}$ and $\kappa(t) \simeq 1$ for $t \sim \tau_0^{false} < T$. $\kappa(t)$ is a fluctuating function of t at $t \sim T$ and $\kappa(t) \propto \frac{1}{t^2}$ for $t \gg T$.
At asymptotically late times, $t \gg T$, one finds that

$$\mathcal{E}_0^{false}(t) \simeq E_0^{true} - \frac{c_2}{t^2}\ldots \;\neq\; E_0^{false}, \tag{14}$$

where $c_2 = c_2^*$ and it can be positive or negative depending on the model considered. Similarly $\gamma_0^{false}(t) \simeq +2\,c_1/t\;\ldots$ for $t \gg T$. Two last properties of the false vacuum states mean that

$$\mathcal{E}_0^{false}(t) \to E_0^{true} \;\text{ and }\; \gamma_0^{false}(t) \to 0 \;\text{ as }\; t \to \infty. \tag{15}$$

Going from quantum mechanics to quantum field theory one should take into account among others a volume factors so that survival probabilities per unit volume per unit time should be considered. The standard false vacuum decay calculations shows that the same volume factors should appear in both early and late time decay rate estimations (see Krauss and Dent [8, 14]). This means that the calculations of cross–over time T can be applied to survival probabilities per unit volume. For the same reasons within the quantum field theory the quantity $\mathcal{E}_M(t)$ can be replaced by the energy per unit volume $\rho_M(t) = \mathcal{E}_M(t)/V$ because these volume factors V appear in the numerator and denominator of the formula (6) for $h_M(t)$. This conclusion seems to hold when considering the energy $\mathcal{E}_0^{false}(t)$ of the system in false vacuum state $|0\rangle^{false}$ because Universe is assumed to be homogeneous and isotropic at suitably large scales. So at such scales to a sufficiently good accuracy we can extract properties of the energy density $\rho_0^{false} = E_M^0/V = E_0^{false}/V$ of the system in the false vacuum state $|0\rangle^{false}$ from properties of the energy $\mathcal{E}_0^{false}(t)$ of the system in this state defining $\rho_0^{false}(t)$ as $\rho_0^{false}(t) = \mathcal{E}_0^{false}(t)/V$. This means that in the case of a meta–stable (unstable or decaying, false) vacuum the following important property of $\kappa(t)$ holds:

$$\kappa(t) \;\equiv\; \frac{\rho_0^{false}(t) - \rho_{bare}}{\rho_0^{false} - \rho_{bare}},$$

where $\rho_{bare} = E_{min}/V$ is the energy density of the true (bare) vacuum. From the last equations the following relation follows

$$\rho_0^{false}(t) - \rho_{bare} = (\rho_0^{false} - \rho_{bare})\,\kappa(t).$$

Thus, because for $t < T$ there is $\kappa(t) = 1$, one finds that

$$\rho_0^{false}(t) = \rho_0^{false}, \;\text{ for }\; t < T,$$

whereas for $t \gg T$ we have

$$\rho_0^{false}(t) - \rho_{bare} = (\rho_0^{false} - \rho_{bare})\,\kappa(t) \simeq \pm\,d_2\,\frac{\hbar^2}{t^2}, \;\; (t \gg T), \tag{16}$$

where $d_2 = d_2^*$. The units $\hbar = 1 = c$ will be used in the next formulas. Analogous relations (with the same $\kappa(t)$) take place for $\Lambda(t) = \frac{8\pi G}{c^2}\,\rho(t)$, or $\Lambda(t) = 8\pi G\,\rho(t)$ in $\hbar = c = 1$ units:

$$\Lambda(t) - \Lambda_{bare} = (\Lambda_0 - \Lambda_{bare})\,\kappa(t), \tag{17}$$

or,

$$\Lambda(t) = \Lambda_{bare} + (\Lambda_0 - \Lambda_{bare})\,\kappa(t). \tag{18}$$

One may expect that Λ_0 equals to the cosmological constant calculated within quantum field theory. From (18) it is sen that for $t < T$,

$$\Lambda(t) \simeq \Lambda_0, \quad \text{for} \quad (t < T), \tag{19}$$

because $\kappa(t < T) \simeq 1$. Now if to assume that Λ_0 corresponds to the value of the cosmological "constant" Λ calculated within the quantum field theory, than one should expect that [15]

$$\frac{\Lambda_o}{\Lambda_{bare}} \geq 10^{120}, \tag{20}$$

which allows one to write down Eq. (18) as follows

$$\Lambda(t) \simeq \Lambda_{bare} + \Lambda_0\,\kappa(t). \tag{21}$$

Note that for $t \gg T$ there should be (see (16))

$$\Lambda_0\,\kappa(t) \simeq 8\pi G\,\frac{d_2}{t^2} \equiv \pm\frac{\alpha^2}{t^2}, \;\; \text{for} \;\; (t \gg T). \tag{22}$$

5 Final Remarks

Parametrization following from quantum theoretical treatment of metastable vacuum states can explain why the cosmologies with the time–dependent cosmological constant $\Lambda(t)$ are worth considering and may help to explain the cosmological constant problem [16, 17]. The time dependence of Λ of the type $\Lambda(t) = \Lambda_{bare} + \frac{\alpha^2}{t^2}$ was assumed eg. in [18] but there was no any explanation what physics suggests such a choice of the form of Λ. Earlier analogous form of Λ was obtained in [19], where the invariance under scale transformations of the generalized Einstein equations was studied. Such a time dependence of Λ was postulated also in [20] as the result of the analysis of the large numbers hypothesis. The cosmological model with time dependent Λ of the above postulated form was studied also in [21] and in much more recent papers.
The nice feature and maybe even the advantage of the formalism presented in Sect. 4 is that in the case of the universe with metastable (false) vacuum if one realized that the decay of this unstable vacuum state is the quantum decay process then it emerges automatically that there have to exist the true ground state of the system that is the true (or bare) vacuum with the minimal energy, $E_{min} > -\infty$, of the system corresponding to him and equivalently, $\rho_{bare} = E_{min}/V$, or Λ_{bare}. What is more, in this case such $\Lambda \equiv \Lambda(t)$ emerges that at suitable late times it has the form described by relations (21), (22). In such a case the function $\kappa(t)$ given by the relation (12) describes time dependence for all times t of the energy density $\rho_M(t)$ or the cosmological "constant" $\Lambda_M(t)$ and it general form is presented in panels B in Figs (1) — (3). Note that results presented in Sections 2 — 4 are rigorous. The formalism mentioned was applied in [14, 15], where cosmological models with $\Lambda(t) = \Lambda_{bare} \pm \frac{\alpha^2}{t^2}$ were studied: The most promising result is reported in [15] where using the parametrization following from the mentioned quantum theoretical analysis of the decay process of the unstable vacuum state an attempt was made to explain the small today's value of the cosmological constant Λ. This shows that formalism and the approach described in this paper and in [14, 15] is promising and can help to solve the cosmological constant and other cosmological problems and it needs further studies, especially it to take into account the LHC result concerning the mass of the Higgs boson and cosmological consequences of this result.
Acknowledgments: The work was supported in part by the NCN grant No DEC-2013/09/B/ST2/03455.

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