Anomalous Transport in Second Order Hydrodynamics

Eugenio Megías*

In collaboration with: Prof. Manuel Valle (UPV/EHU, Spain)

***Max-Planck-Institut für Physik (Werner-Heisenberg-Institut), Munich, Germany.**

*Supported by the European Union under a Marie Curie Intra European-Fellowship, PIEF-GA-2013-623006.

4th International Conference on
New Frontiers in Physics (ICNFP2015)
27 August, 2015, Crete, Greece.

Based on EM, M. Valle, JHEP 1411 (2014) 005.
Related works: K. Landsteiner, EM, F. Peña-Benítez, PRL 107 (2011) 021601, EM, F. Peña-Benítez, JHEP 1305 (2013) 115.

Issues

Issues

1. Introduction: Anomalous Transport
   - The Chiral Magnetic Effect
   - Hydrodynamics of Relativistic Fluids
2. Equilibrium Partition Function Formalism to Hydrodymamics
   - Equilibrium Partition Function
3. Second Order Transport
   - Free theory of Dirac fermions
   - Partition function at second order
   - Non-dissipative constitutive relations

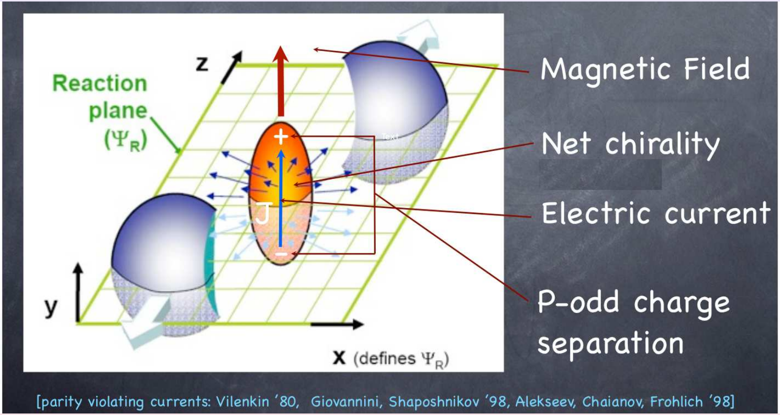
Issues

1. Introduction: Anomalous Transport
   - The Chiral Magnetic Effect
   - Hydrodynamics of Relativistic Fluids
2. Equilibrium Partition Function Formalism to Hydrodymamics
   - Equilibrium Partition Function
3. Second Order Transport
   - Free theory of Dirac fermions
   - Partition function at second order
   - Non-dissipative constitutive relations

Issues

The Chiral Magnetic Effect

[Kharzeev, McLerran, Warringa '07]



Strong Magnetic field induces a $\mathcal{P}$-odd charge separation $\Longrightarrow$

$\Longrightarrow$ Electric current: $\vec{J} = \sigma^{\mathcal{B}} \vec{B}$.

Issues

Hydrodynamics of Relativistic Fluids

[Son,Surowka], [Eling,Neiman,Oz], [Erdmenger et al.], [Banerjee et al.], [Loganayagam], [Kharzeev, Yee], [Sadovyev et al.]

$$\langle T^{\mu\nu}\rangle = \underbrace{(\varepsilon+\mathcal{P})u^\mu u^\nu + \mathcal{P}g^{\mu\nu}}_{\text{Ideal Hydro}} + \underbrace{\langle T^{\mu\nu}\rangle_{\text{diss \& anom}}}_{\text{Dissipative \& Anomalous}} ,$$

$$\langle J^{\mu}\rangle = \underbrace{\rho u^\mu}_{\text{Ideal Hydro}} + \underbrace{\langle J^{\mu}\rangle_{\text{diss \& anom}}}_{\text{Dissipative \& Anomalous}} .$$

- Landau frame: $\langle T^{0i}\rangle \sim u^i$

$$\langle T^{\mu\nu}\rangle_{\text{diss \& anom}} = -\eta P^{\mu\alpha}P^{\nu\beta}\left(D_\alpha u_\beta + D_\beta u_\alpha - \frac{2}{3}g_{\alpha\beta}D^\lambda u_\lambda\right) - \zeta P^{\mu\nu}D^\alpha u_\alpha + \cdots$$

$$\langle J^{\mu}\rangle_{\text{diss \& anom}} = -\sigma T P^{\mu\nu}D_\nu\left(\frac{\mu}{T}\right) + \sigma E^\mu + \sigma^{\mathcal{B}}B^\mu + \sigma^{\mathcal{V}}\omega^\mu + \cdots$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, and vorticity: $\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}u_\nu D_\rho u_\lambda$.

Parity and Time Reversal Properties (I)

- Hydrodynamics at 1st order in derivative expansion:

$$\underbrace{\langle \vec{J} \rangle_1}_{\mathcal{P}-odd,\mathcal{T}-odd} = \underbrace{\sigma^{\mathcal{B}}}_{\mathcal{P}-odd,\mathcal{T}-even} \cdot \underbrace{\vec{B}}_{\mathcal{P}-even,\mathcal{T}-odd}$$

$$\overbrace{\langle \vec{J} \rangle_1} = \overbrace{\sigma^{\mathcal{V}}} \cdot \overbrace{\vec{\omega}}$$

$\mathcal{T} - even \Longrightarrow \sigma^{\mathcal{B}}$ and $\sigma^{\mathcal{V}}$ are non dissipative, i.e. they cannot contribute to entropy production:

$$\frac{\partial}{\partial t} s > 0 \qquad \text{(Only } \mathcal{T}\text{-odd contributions in } s\text{)}$$

Electric conductivity is dissipative: $\langle \vec{J} \rangle_1 = \sigma \vec{E} \Longrightarrow \partial_t s > 0$.

Issues

Equilibrium Partition Function

[Banerjee et al '12], [Jensen et al '13], [Bhattacharyya '14], [EM, Valle '14]

- Relativistic Invariant Quantum Field Theory on the manifold

$$ds^2 = -e^{2\sigma(\vec{x})}(dt + a_i(\vec{x})dx^i)^2 + g_{ij}(x)dx^i dx^j$$

and time independent background $U(1)$ gauge connection:

$$\mathcal{A} = \mathcal{A}_0(\vec{x})dx^0 + \mathcal{A}_i(\vec{x})dx^i \,.$$

- Partition function of the system:

$$Z = \mathrm{Tr}\, e^{-\frac{H-\mu_0 Q}{T_0}}$$

→ Dependence of Z on σ, g_{ij} and a_i?

- Most general partition function consistent with:
  - 3-dim diffeomorphism invariance.
  - Kaluza-Klein invariance: $t \to t + \phi(\vec{x})\,,\ \vec{x} \to \vec{x}$.
  - $U(1)$ *time-independent* gauge invariance (up to an anomaly).

Equilibrium Partition Function

- Stress Tensor and $U(1)$ current ➞ under t-indep variations

$$\delta \log Z = \frac{1}{T_0}\int d^3x\sqrt{-g_3}\left(-\frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu} + J^\mu \delta \mathcal{A}_\mu\right)$$

➞ $T_{\mu\nu} = -2T_0 \frac{\delta \log Z}{\delta g^{\mu\nu}}, \qquad J^\mu = T_0 \frac{\delta \log Z}{\delta \mathcal{A}_\mu}.$

- In particular, for $\log Z = \mathcal{W}(e^\sigma, A_0, a_i, A_i, g^{ij}, T_0, \mu_0)$ one gets

$$\langle J^i\rangle = \frac{T_0}{\sqrt{-G}}\frac{\delta\mathcal{W}}{\delta A_i}, \qquad \langle J_0\rangle = -\frac{T_0 e^{2\sigma}}{\sqrt{-G}}\frac{\delta\mathcal{W}}{\delta A_0},$$

$$\langle T_0{}^i\rangle = \frac{T_0}{\sqrt{-G}}\left(\frac{\delta\mathcal{W}}{\delta a_i} - A_0\frac{\delta\mathcal{W}}{\delta A_i}\right), \quad \langle T_{00}\rangle = -\frac{T_0 e^{2\sigma}}{\sqrt{-G}}\frac{\delta\mathcal{W}}{\delta\sigma}.$$

- ➞ $\mathcal{W}$ is a generating functional for the hydrodynamic constitutive relations.

Equilibrium Partition Function at 0th order

- Most general partition function up to 0th order in derivatives:

$$\log Z = \mathcal{W}_0 = \int d^3x \sqrt{g_3} \frac{e^\sigma}{T_0} \underbrace{P(T_0 e^{-\sigma}, e^{-\sigma} A_0)}_{\text{Arbitrary function of 2 variables}} , \quad A_0 = \mathcal{A}_0 + \mu_0 .$$

- Constitutive relations ($a \equiv e^{-\sigma} T_0 \,,\; b \equiv e^{-\sigma} A_0$):

$$\langle T^{ij} \rangle = P g^{ij} , \qquad \langle T_{00} \rangle = e^{2\sigma} (P - a\partial_a P - b\partial_b P) , \qquad \langle T^i_0 \rangle = 0 ,$$
$$\langle J^0 \rangle = e^{-\sigma} \partial_b P , \qquad \langle J^i \rangle = 0 .$$

- By comparison with the hydrodynamic constitutive relations:

$$\langle T^{\mu\nu} \rangle = (\varepsilon + \mathcal{P}) u^\mu u^\nu + \mathcal{P} g^{\mu\nu} , \qquad \langle J^\mu \rangle = \rho u^\mu ,$$

one gets

$$u^\mu = e^{-\sigma} (1, 0, \cdots, 0) ,$$
$$\mathcal{P} = P , \qquad \varepsilon = -P + a\partial_a P + b\partial_b P , \qquad \rho = \partial_b P .$$

→ ε, $\mathcal{P}$ and ρ are not independent functions, but are determined in terms of a single *master* function.

Equilibrium Partition Function at first order

- Most general partition function at 1st order in derivative expansion **[Banerjee et al '12]**:

$$\mathcal{W}_1=\int d^3x\sqrt{g_3}\Big[\alpha_1(\sigma,A_0)\epsilon^{ijk}A_iF_{jk}+\alpha_2(\sigma,A_0)\epsilon^{ijk}A_if_{jk}+\alpha_3(\sigma,A_0)\epsilon^{ijk}a_if_{jk}\Big]$$

where $F_{ij}=\partial_iA_j-\partial_jA_i\,,\ f_{ij}=\partial_ia_j-\partial_ja_i$,

- Ideal gas of Dirac fermions ⟶ from a computation of $\langle T^i_0\rangle$ and $\langle J^i\rangle$, one gets

$$\alpha_1(\sigma,A_0)=\frac{C}{6T_0}A_0\,,\quad \alpha_2(\sigma,A_0)=\frac{1}{2}\left(\frac{C}{6T_0}A_0^2+C_2T_0\right)\,,\quad \alpha_3(\sigma,A_0)=0\,.$$

where: $\begin{cases} C=-\frac{1}{4\pi^2} \\ C_2=\frac{1}{24}\end{cases}$

(chiral anomaly): **[Son, Surowka '09], [Erdmenger et al '09], . . .**

(gauge-gravitational anomaly): **[Landsteiner, EM, Pena-Benitez '11]**

- Coefficients related to chiral magnetic $\sigma^{\mathcal{B}}$ and chiral vortical $\sigma^{\mathcal{V}}$ conductivities.

Equilibrium Partition Function at second order

- Most general partition function at 2nd order **[Bhattacharyya '14]**:

$$\mathcal{W}_2 = \int d^3x \sqrt{g} \Big[M_1 g^{ij} \partial_i T \partial_j T + M_2 g^{ij} \partial_i \nu \partial_j \nu + M_3 g^{ij} \partial_i \nu \partial_j T + T_0^2 M_4 f_{ij} f^{ij} + M_5 F_{ij} F^{ij} + T_0 M_6 f_{ij} F^{ij} + M_7 R \Big] ,$$

where $M_i = M_i(T, \nu)$ with

$$T = T_0 \, e^{-\sigma} , \qquad \nu = \frac{A_0}{T_0} .$$

- From the relations $\langle J_0 \rangle \propto \frac{\partial \mathcal{W}}{\delta A_0}$, $\langle T_{00} \rangle \propto \frac{\partial \mathcal{W}}{\partial \sigma}$ → Determine M_i.

Issues

Free theory of Dirac fermions

- Free theory of Dirac fermions:

$$S = \int d^4x \sqrt{-G}\mathcal{L} \,, \qquad \text{where} \qquad \mathcal{L} = -i\bar{\Psi}\underline{\gamma}^\mu \nabla_\mu \Psi + im\bar{\Psi}\Psi \,.$$

- $U(1)$ current and energy-momentum tensor:

$$J^\mu = -\bar{\Psi}\underline{\gamma}^\mu\Psi \,, \qquad T_{\mu\nu} = \frac{i}{4}\bar{\Psi}\left[\underline{\gamma}_\mu \overrightarrow{\nabla}_\nu - \overleftarrow{\nabla}_\nu \underline{\gamma}_\mu + (\mu \leftrightarrow \nu)\right]\Psi \,,$$

- Explicit currents and energy-momentum tensor (left part):

$$J_0 = -e^{-\sigma}\psi^\dagger\psi \,, \qquad J^i = -\psi^\dagger\sigma_i\psi \,,$$

$$T_{00} = \frac{i}{2}e^\sigma\left(\psi^\dagger\partial_t\psi - \partial_t\psi^\dagger\psi\right) + e^\sigma A_0\psi^\dagger\psi - \frac{1}{4}e^{3\sigma}\epsilon^{ijk}\partial_j a_k \psi^\dagger\sigma_i\psi \,,$$

where $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ and $\psi \equiv \psi_L$.

Free theory of Dirac fermions

- Expectation values of J_μ and $T^{\mu\nu}$ at equilibrium ⟶ Thermal Green function

$$\langle T\psi(-i\tau, x)\psi^\dagger(0, x')\rangle_\beta = T_0 \sum_n e^{-i\omega_n\tau}\mathcal{G}(x, x', \omega_n)\,, \quad \omega_n = \frac{2\pi}{\beta}\left(n + \frac{1}{2}\right)$$

then one gets

$$\langle J_0\rangle = T_0 \sum_n \mathrm{tr}\left[-e^\sigma \mathrm{tr}\,\mathcal{G}(x, x, \omega_n)\right]\,,$$

$$\langle J^i\rangle = -T_0 \sum_n \mathrm{tr}\left[\sigma_i\, \mathcal{G}(x, x, \omega_n)\right]\,,$$

$$\langle T_{00}\rangle = T_0 \sum_n \left[e^\sigma(i\omega_n + A_0)\,\mathrm{tr}\,\mathcal{G}(x, x, \omega_n) - \frac{1}{4}e^{3\sigma}\epsilon^{ijk}\partial_j a_k\,\mathrm{tr}\left[\sigma_i\,\mathcal{G}(x, x, \omega_n)\right]\right]\,.$$

- From the relations $\langle J_0\rangle \propto \frac{\partial \mathcal{W}}{\delta A_0}\,, \cdots$ ⟶ We can determine $\mathcal{W}$.

The Green function

- Action:

$$S = -\int d^4x \sqrt{-G}\, \bar{\Psi} \underline{\gamma}^0 \left[i\partial_t - \mathcal{H} \right] \Psi \,,$$

with the Hamiltonian

$$\mathcal{H} = -i \left(\frac{1}{4} \omega_0{}^{ab} \gamma_{ab} - iA_0 \right) - \frac{i}{g^{00}} \underline{\gamma}^0 \left(\underline{\gamma}^k \nabla_k - m \right) \,.$$

- Rotation to imaginary time $t \to -i\tau$ ➞ Green function obeys:

$$-\sqrt{-G}\, \gamma^0 \underline{\gamma}^0 (i\omega_n - \mathcal{H}) \mathcal{G}(\vec{x}, \vec{x}', \omega_n) = \delta^{(3)}(\vec{x} - \vec{x}') \,.$$

- Computation of $\mathcal{G}$ in a derivative expansion ($\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1 + \mathcal{G}_2 \cdots$):

$$(i\omega_n - H_0(\vec{x}))\mathcal{G}_0(\vec{x}, \vec{x}') = \delta^{(3)}(\vec{x} - \vec{x}') \,,$$
$$(i\omega_n - H_0(\vec{x}))\mathcal{G}_1(\vec{x}, \vec{x}') = \mathcal{F}(\mathcal{G}_0, H_1) \,,$$
$$(i\omega_n - H_0(\vec{x}))\mathcal{G}_2(\vec{x}, \vec{x}') = \mathcal{F}(\mathcal{G}_0, \mathcal{G}_1, H_1, H_2) \,, \cdots$$

Energy and charge density up to second order

- Then one can compute the constitutive relations up to p-th order:

$$\langle J_0 \rangle_p = T_0 \sum_n \mathrm{tr}\left[-e^{\sigma} \mathrm{tr}\, \mathcal{G}_p(x,x,\omega_n) \right] ,$$

$$\langle T_{00} \rangle_p = T_0 \sum_n \left[e^{\sigma}(i\omega_n + A_0)\, \mathrm{tr}\, \mathcal{G}_p(x,x,\omega_n) - \frac{1}{4} e^{3\sigma} \epsilon^{ijk} \partial_j a_k\, \mathrm{tr}\left[\sigma_i \mathcal{G}_p(x,x,\omega_n) \right] \right]$$

- To get $\mathcal{W}_2$, it is enough to compute $\langle J_0 \rangle_2$ and $\langle T_{00} \rangle_2$ including only bilinear terms $\sim \partial_i X \partial_j Y$ ➞ Pauli-Villars regularization.

$$\langle J_0 \rangle_2 = \frac{1}{24\pi^2}\left(-\nabla^i A_0 \nabla_i \sigma + \frac{1}{2} e^{2\sigma} f_{ij} F^{ij} + \frac{1}{2} A_0 e^{2\sigma} f_{ij} f^{ij} \right) \mathcal{N}_\Lambda(\sigma, A_0)$$
$$+ \frac{1}{48\pi^2}\left(\nabla^i A_0 \nabla_i A_0 + \frac{e^{2\sigma}}{2} A_0^2 f_{ij} f^{ij} + \frac{e^{2\sigma}}{2} F_{ij} F^{ij} + e^{2\sigma} A_0 f_{ij} F^{ij} \right) \frac{\partial \mathcal{N}_\Lambda}{\partial A_0}$$
$$- \frac{1}{24\pi^2} A_0 \nabla^i \sigma \nabla_i \sigma + \frac{7}{96\pi^2} \nabla^i A_0 \nabla_i \sigma + \frac{5}{192\pi^2} e^{2\sigma} f_{ij} F^{ij}$$
$$+ \frac{3}{64\pi^2} e^{2\sigma} A_0 f_{ij} f^{ij} + \frac{A_0}{48\pi^2} R .$$

Issues

Partition function at second order

$$M_1(T,\nu) = -\frac{1}{144}\frac{1}{T} - \frac{1}{48\pi^2}\frac{\nu^2}{T},$$

$$M_2(T,\nu) = \frac{1}{48\pi^2} T \left(\ln\frac{\bar{M}^2}{T^2} + Q(\nu) - \frac{1}{4} - \frac{3}{4} \right),$$

$$M_3(T,\nu) = -\frac{1}{12\pi^2}\nu,$$

$$M_4(T,\nu) = -\frac{1}{96\pi^2}\frac{\nu^2}{T}\left(\ln\frac{\bar{M}^2}{T^2} + Q(\nu) + \frac{11}{4} + 6\pi^2 C + \frac{1}{4} \right) + \frac{1}{288}\frac{1}{T} - \frac{C_2}{8T} + \frac{1}{384\pi^2}\frac{1}{T^3} M^2 \ln 2,$$

$$M_5(T,\nu) = -\frac{1}{96\pi^2}\frac{1}{T}\left(\ln\frac{\bar{M}^2}{T^2} + Q(\nu) - \frac{1}{4} + \frac{1}{4} \right),$$

$$M_6(T,\nu) = -\frac{1}{48\pi^2}\frac{\nu}{T}\left(\ln\frac{\bar{M}^2}{T^2} + Q(\nu) + \frac{7}{4} + 6\pi^2 C + \frac{1}{4} \right),$$

$$M_7(T,\nu) = -\frac{1}{288} T - \frac{1}{96\pi^2} T \nu^2 + \frac{1}{96\pi^2}\frac{1}{T} M^2 \ln 2.$$

Renormalization and Trace Anomaly

- The action can be renormalized by adding the counterterm

$$\mathcal{W}_2^{\rm ct} = -\frac{M^2 \ln 2}{96\pi^2}\int d^4x\sqrt{-G}\,\tilde{R}\,,$$

so that $\mathcal{W}_2^{\rm ren} = \mathcal{W}_2 + \mathcal{W}_2^{\rm ct}$.

- Under a Weyl rescaling:

$$g_{ij} \to e^{2\omega} g_{ij}\,, \qquad \sigma \to \sigma + \omega\,,$$

terms $\propto \ln \frac{\bar{M}^2}{T^2}$ are not invariant ➞ Trace Anomaly:

$$\mathcal{W}_{\rm anom} = \frac{1}{24\pi^2}\int d^3x\sqrt{g}\frac{1}{T}\ln\frac{\bar{M}}{T} \times \left(e^{-2\sigma}g^{ij}\partial_i A_0 \partial_j A_0 - \frac{1}{2}A_0^2 f_{ij}f^{ij} - \frac{1}{2}F_{ij}F^{ij} - A_0 f_{ij}F^{ij}\right)$$

$$= -\frac{1}{48\pi^2}\int d^4x\sqrt{-G}\ln\frac{\bar{M}}{T}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \;\rightarrow\; \langle T^\mu_\mu\rangle = -\frac{1}{48\pi^2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\,.$$

Issues

Constitutive relations

$$T^{\mu\nu} = (\varepsilon + \mathcal{P})u^\mu u^\nu + \mathcal{P}G^{\mu\nu} + T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)} + \dots ,$$
$$J^\mu = \rho u^\mu + J^\mu_{(1)} + J^\mu_{(2)} + \dots .$$

- Most general non-dissipative form of constitutive relations at 2nd order **[Bhattacharyya, David, Thakur '14]**:

$$T_{(2)\,\mu\nu} = \Delta P\,(G_{\mu\nu} + u_\mu u_\nu) + T\left(\kappa_1 \tilde{R}_{\langle\mu\nu\rangle} + \kappa_2 u^\alpha u^\beta \tilde{R}_{\langle\mu\alpha\nu\rangle\beta} + \kappa_3 \nabla_{\langle\mu}\nabla_{\nu\rangle}\nu\right) + \cdots$$
$$J_{(2)\,\mu} = \lambda_1 P_{\mu\alpha} u_\nu \tilde{R}^{\nu\alpha} + \lambda_2 P_{\mu\alpha}\nabla_\nu \mathcal{F}^{\nu\alpha} + \cdots .$$

- Determination of κ_i and λ_i by comparison with $\mathcal{W}_2$:

$$T_{00}\big|_{\rm eq} = -\frac{T_0^2}{T\sqrt{g}}\frac{\delta\mathcal{W}_2}{\delta\sigma} = T_0^2\left(-2M_1\nabla^2 T - M_3\nabla^2\nu + \frac{\partial M_7^{\rm ren}}{\partial T}R\right) + \cdots ,$$
$$J_0\big|_{\rm eq} = -\frac{T_0^2}{T\sqrt{g}}\frac{\delta\mathcal{W}_2}{\delta A_0} = \frac{T_0}{T}\left(M_3\nabla^2 T + 2M_2\nabla^2\nu - \frac{\partial M_7^{\rm ren}}{\partial\nu}R\right) + \cdots .$$

Constitutive relations

- General result for any theory:

$$T_{(2)\,\mu\nu} = T\left(\kappa_1 \tilde{R}_{\langle\mu\nu\rangle} + \kappa_2 u^\alpha u^\beta \tilde{R}_{\langle\mu\alpha\nu\rangle\beta} + \kappa_3 \nabla_{\langle\mu}\nabla_{\nu\rangle}\nu\right) + \cdots,$$
$$J_{(2)\,\mu} = \lambda_1 P_{\mu\alpha} u_\nu \tilde{R}^{\nu\alpha} + \lambda_2 P_{\mu\alpha}\nabla_\nu \mathcal{F}^{\nu\alpha} + \cdots.$$

$$\kappa_1 = -2M_7^{\text{ren}},$$
$$\kappa_2 = -2M_7^{\text{ren}} - 2T\frac{\partial M_7^{\text{ren}}}{\partial T},$$
$$\kappa_3 = 2\frac{\partial M_7^{\text{ren}}}{\partial \nu},$$

$$\lambda_1 = 4T^2\left(2\nu M_5 - M_6\right) - \frac{8\rho}{\varepsilon + P}T^3\left(M_4^{\text{ren}} + \nu^2 M_5 - \nu M_6\right),$$
$$\lambda_2 = -4TM_5 + \frac{2\rho}{\varepsilon + P}T^2\left(2\nu M_5 - M_6\right).$$

Constitutive relations: free theory of Weyl fermions

- Free theory of Weyl fermions:

$$\kappa_1 = \frac{T}{144} + \frac{1}{48\pi^2}\frac{\mu^2}{T}\,, \qquad \kappa_2 = 2\kappa_1\,, \qquad \kappa_3 = -\frac{\mu}{24\pi^2}\,,$$

$$\lambda_1 = \frac{1}{2}\left(C + \frac{1}{3\pi^2}\right)\mu + \frac{\rho}{\varepsilon + P}\left[-\frac{1}{2}\left(C + \frac{1}{6\pi^2}\right)\mu^2 + \left(C_2 - \frac{1}{36}\right)T^2\right]$$

$$\lambda_2 = \frac{1}{24\pi^2}\left(\ln\frac{\bar{M}^2}{T^2} + Q\left(\frac{\mu}{T}\right)\right) + \frac{\rho}{\varepsilon + P}\frac{1}{4}\left(C + \frac{1}{3\pi^2}\right)\mu\,,$$

Remarks:

- λ_2 is sensitive to renormalization scale.
- No mixture of trace and chiral anomalies in constitutive relations.
- κ_1 and κ_2 in agreement with **[Moore, Sohrabi '12]** after $\mu = 0$.
- κ_3 and λ_2 computed in a holographic model in 5 dimensions in **[Erdmenger et al. '09], [Banerjee et al. '11], [EM, Pena-Benitez '13].**

$$[\kappa_3]_{\text{weak coupling}} \propto [\kappa_3]_{\text{strong coupling}}\,, \qquad [\lambda_2]_{\text{weak c.}} \sim c(T) + \frac{5}{112\pi^4}\frac{\mu^2}{T^2} \propto [\lambda_2]_{\text{strong c.}}$$

Parity and Time Reversal Properties (II)

- Equilibrium partition function can only account for non-dissipative effects $\partial_\mu s^\mu = 0$ ⟶ transport coefficients multiplying quantities that survive in equilibrium.
- Hydrodynamics at 1st order: non-dissipative coefficients are $\mathcal{P}$-odd and $\mathcal{T}$-even ⟶ $\sigma^{\mathcal{B}}$ and $\sigma^{\mathcal{V}}$.
- Hydrodynamics at 2nd order:
  - coefficients $\mathcal{P}$-odd and $\mathcal{T}$-even vanish

$$\epsilon^{ijk}\nabla_i \sigma f_{jk}\,, \qquad \epsilon^{ijk}\nabla_i \sigma F_{jk}\,, \qquad \epsilon^{ijk}\nabla_i A_0 f_{jk}\,, \qquad \epsilon^{ijk}\nabla_i A_0 F_{jk}$$

    ⟶ we have explicitly checked it with the partition function formalism.
  - non-dissipative coefficients calculated are $\mathcal{P}$-even and $\mathcal{T}$-even.

Conclusions

- We have studied non-dissipative transport effects up to 2nd order in the hydrodynamic expansion.
- Effects are induced by *external magnetic fields*, *vortices* and curvature in a relativistic fluid $\Longrightarrow$ Anomalous Transport.
- Equilibrium partition function method can only account for non-dissipative effects: *time reversal properties*.
- Dissipative effects (shear viscosity, electric conductivity, ...) ➞ other methods: Kubo formulae, Fluid/gravity correspondence, ...
- Results at 2nd order for free fermions consistent with strong coupling results.
- We have studied Renormalization and Conformal Anomaly effects ➞ chiral and conformal anomaly mix in the partition function, but not in the constitutive relations.

Conclusions

Future directions:

- Derivation of Kubo formulae for 2nd order hydrodynamics

$$\langle J^x J^y \rangle = -i\sigma^{\mathcal{B}} k_z + \tilde{\xi}_5 \omega k_z + \cdots \Longrightarrow \quad \langle \vec{J} \rangle = \sigma^{\mathcal{B}} \vec{B} + \tilde{\xi}_5 \vec{\nabla} \times \vec{E} + \cdots$$

  $\tilde{\xi}_5$ is $\mathcal{P}$-odd and $\mathcal{T}$-odd $\Longrightarrow$ Entropy production?

- Computation of entropy current up to 2nd order in holography:

$$\partial_\mu s^\mu > 0 \,.$$

- Applications, not only to the QGP, but also to condensed matter systems with triangle anomalies → Weyl semi-metals **[Basar, Kharzeev, Yee '14], [Landsteiner '14].**

Thank You!