On 3D Minimal Massive Gravity

Mohammad M. Qaemmaqami

Institute for Researches in Fundamental Sciences (IPM), Tehran, Iran

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Motivation

3D Einstein gravity

- The action of Einstein-Hilbert gravity is:

$$I_{EH} = \frac{1}{16\pi G_3} \int \sqrt{-g}(R - 2\Lambda) d^3x$$

- The Equation of Motion is $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \Lambda g_{\mu\nu}$
- No gravitons
  Because the number of diffeomorphisms and Einstein equation constraints is enough to kill all the dynamical DOF.
- In AdS vacuum of radius L, central charges $c_{\pm}$, of (putative) dual 2D CFT are [Brown Henneaux- 1986]

$$c_{\pm} = \frac{3L}{2G_3}$$

Motivation

Topologically Massive Gravity

TMG (parity violating) action is $\frac{1}{\mu} I_{CS}[g] + \sigma I_{EH}$ where μ is a mass parameter and σ is a sign (Jakiw, Deser and templeton- 1982).

$$I_{CS}[g] = \frac{1}{2\mu} \int tr[\Gamma d\Gamma + \frac{1}{3}\Gamma^3] d^3x$$

Chern-Simons term is third-order in derivatives of g. Variation with respect to metric from CS term gives Cotton tensor (curl of Schouten tensor):

$$C^{\mu\nu} = \epsilon_\mu{}^{\rho\sigma} \nabla_\rho S_{\sigma\nu}, \qquad S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}.$$

So TMG equation is:

$$\frac{1}{\mu} C_{\mu\nu} + \sigma G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

Propagates one spin-2 mode of mass μ, but positive energy requires wrong-sign EH term: $\sigma = -1$

Motivation

Bulk-Boundary clash for TMG:

- TMG has AdS vacuum ($\Lambda = -1/L^2$) and central charges of dual CFT are(Li, Song Strominger- 2008):

$$c_{\pm} = \frac{3L}{2G_3}(\sigma \pm \frac{1}{\mu L})$$

- Positive c_+ and c_- requires $\sigma = 1$ but positive energy graviton (no ghost) requires $\sigma = -1$, this is the Bulk-Boundary Clash(B.B.C)
- [$c_- = 0$ is special case, which is null or logarithmic, we know that resulting chiral gravity (Li, Song Strominger- 2008) has non-unitary dual "log CFT" therefore TMG is not a viable unitary theory (D. Grumiller and N. Johansson- 2010).]

Minimal Massive Gravity(MMG)[E. Bergshoeff, et.al]

Minimal Massive Gravity (MMG)

This proposal is a three dimensional massive gravity which has the same minimal local structure as TMG. One may define the action of the MMG model from that of TMG by adding an extra term.

$$L_{\mathrm{TMG}} = -\sigma e \cdot R + \frac{\Lambda_0}{6} e \cdot e \times e + h \cdot T(\omega) + \frac{1}{2\mu}\left(\omega \cdot d\omega + \frac{1}{3}\omega \cdot \omega \times \omega\right),$$

$$L_{\mathrm{MMG}} = L_{\mathrm{TMG}} + \frac{\alpha}{2} e \cdot h \times h.$$

$$c_{\pm} = \frac{3l}{2G}\left(\sigma \pm \frac{1}{\mu l} + \frac{\alpha - \alpha^2 \Lambda_0 l^2}{2\mu^2 l^2 (1+\sigma\alpha)^2}\right).$$

Then, it was shown that in the specific range of the parameters of the model, it is possible to get positive central charges for the dual conformal field theory and, at the same time to have a graviton with positive energy.

Minimal Massive Gravity

The equations of motion derived from the action of MMG are:

$$T(\omega) - \alpha e \times h = 0, \quad (vs,\ h)$$

$$R(\omega) + \mu e \times h + \sigma\mu T(\omega) = 0, \quad (vs,\ \omega)$$

$$-\sigma R(\omega) + \frac{\Lambda_0}{2} e \times e + D(\omega) h + \frac{\alpha}{2} h \times h = 0. \quad (vs,\ e)$$

Defining $\Omega = \omega + \alpha h$ one can solve the first and second equations to find Ω and h in terms of e. Then plugging the results into the equation third equation one can find an equation for the metric as follows

$$\bar{\sigma} G_{\mu\nu} + \bar{\Lambda}_0 g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} + \frac{\gamma}{\mu^2} J_{\mu\nu} = 0,$$

Where the Einstein tensor $G_{\mu\nu}$, and the Cotton tensor $C_{\mu\nu}$ are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \qquad C_{\mu\nu} = \epsilon_{\mu}{}^{\alpha\beta}\nabla_{\alpha}(R_{\beta\nu} - \frac{1}{4}g_{\beta\nu}R)$$

while the curvature-squared symmetric tensor $J_{\mu\nu}$ is defined by,

$$J_{\mu\nu} = R_{\mu}{}^{\alpha}R_{\alpha\nu} - \frac{3}{4}RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(R^{\alpha\beta}R_{\alpha\beta} - \frac{5}{8}R^2\right).$$

Here the parameters $\gamma, \bar{\sigma}$ and $\bar{\Lambda}_0$ are defined in terms of the parameters of the action as follows

$$\gamma = \frac{\alpha}{(1+\sigma\alpha)^2}, \quad \bar{\sigma} = -\left(\sigma + \alpha + \frac{\alpha^2\Lambda_0}{2\mu^2(1+\sigma\alpha)^2}\right),$$

$$\bar{\Lambda}_0 = -\Lambda_0\left(1 + \sigma\alpha - \frac{\alpha^3\Lambda_0}{4\mu^2(1+\sigma\alpha)^2}\right), \quad \sigma\alpha \neq -1.$$

Linearized analysis in metric formalism

Using the metric formulation of MMG, we study linear excitations of metric around an AdS_3 vacuum. Denoting the metric by $g_{\mu\nu}$, one sets

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu},$$

where $h_{\mu\nu}$ is the small perturbation and $\bar{g}_{\mu\nu}$ satisfies the following equation

$$\bar{R}_{\mu\nu} = -\frac{2}{l^2}\bar{g}_{\mu\nu}, \qquad \text{with} \qquad l^2 = -\frac{1}{2\mu}\frac{1}{\bar{\Lambda}_0}(\mu\bar{\sigma} \pm \sqrt{\mu^2\bar{\sigma}^2 - \bar{\Lambda}_0\gamma}).$$

At the linearized level, the equation of motion reduces to

$$\bar{\sigma}(R^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(1)} + \frac{3}{l^2}h_{\mu\nu}) + \bar{\Lambda}_0 h_{\mu\nu}$$

$$+ (1/\mu)\epsilon_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha\left(R^{(1)}_{\beta\nu} - \tfrac{1}{4}\bar{g}_{\beta\nu}R^{(1)} + \tfrac{2}{l^2}h_{\beta\nu}\right) = -\tfrac{\gamma}{2\mu^2 l^2}J^{(1)}_{\mu\nu},$$

Linearized analysis in metric formalism

where

$$J^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(1)} + \frac{5}{2l^2}h_{\mu\nu}.$$

with $h \equiv \bar{g}^{\mu\nu}h_{\mu\nu}$. Using these expressions and in the gauge $\bar{\nabla}^{\mu}h_{\mu\nu} = \bar{\nabla}_{\nu}h$ the equation reads

$$\left(\bar{\nabla}^2 + \frac{2}{l^2}\right)\left(\frac{1}{\tilde{\mu}}\epsilon_{\mu}{}^{\alpha\beta}\,\bar{\nabla}_{\alpha}h_{\beta\nu} + h_{\mu\nu}\right) = 0,.$$

with,

$$\tilde{\mu} = \frac{\gamma + 2\mu^2 l^2\bar{\sigma}}{2\mu l^2}$$

Therfore MMG has same local DOF with TMG at linear order about AdS_3 vacuum.

Linearized analysis in metric formalism

This equation is exactly the same as TMG around an AdS vacuum with the replacement of μ with $\tilde{\mu}$, Indeed, One may rewrite the linearized equation in terms of the quadratic Casimir operators of the $SL(2,R)_L \times SL(2,R)_R$

$$\left[2(L^2+\bar{L}^2)+3+\tilde{\mu}^2 l^2\right]\left[L^2+\bar{L}^2+2\right]h_{\mu\nu}=0.$$

Consider primary states $|\psi_{\mu\nu}\rangle$ with weight $(h,\bar{h})$, so that

$$L_0|\psi_{\mu\nu}\rangle = h|\psi_{\mu\nu}\rangle, \qquad L_1|\psi_{\mu\nu}\rangle = 0,$$

$$L_0 = i\partial_u, \qquad L_{-1} = ie^{-iu}\left[\frac{cosh2\rho}{sinh2\rho}\partial_u - \frac{1}{sinh2\rho}\partial_v + \frac{i}{2}\partial_\rho\right],$$

$$L_1 = ie^{iu}\left[\frac{cosh2\rho}{sinh2\rho}\partial_u - \frac{1}{sinh2\rho}\partial_v - \frac{i}{2}\partial_\rho\right],$$

$$L^2 = \frac{1}{2}(L_1L_{-1}+L_{-1}L_1)-L_0^2.$$

Linearized analysis in metric formalism

Thus, in the present case the primary weights $(h, \bar{h})$ satisfy the following equation

$$\left[2h(h-1)+2\bar{h}(\bar{h}-1)-3-\tilde{\mu}^2 l^2\right]\left[h(h-1)+\bar{h}(\bar{h}-1)-2\right]=0.$$

Here we have used the fact that

$$L^2 \mid \psi_{\mu\nu}\rangle = -h(h-1) \mid \psi_{\mu\nu}\rangle, \quad \bar{L}^2 \mid \psi_{\mu\nu}\rangle = -\bar{h}(\bar{h}-1) \mid \psi_{\mu\nu}\rangle,$$

The above equation has two branches of solutions where either of its factors is zero

$$(h^{(0)}, \bar{h}^{(0)}) = \begin{cases} (\frac{3}{2} \pm \frac{1}{2}, -\frac{1}{2} \pm \frac{1}{2}), & h^{(0)} - \bar{h}^{(0)} = 2, \\ (-\frac{1}{2} \pm \frac{1}{2}, \frac{3}{2} \pm \frac{1}{2}), & h^{(0)} - \bar{h}^{(0)} = -2, \end{cases}$$

$$(\mathrm{h}^{(m)}, \bar{h}^{(m)}) = \begin{cases} (\frac{3}{2} \pm \frac{1}{2}\tilde{\mu} l, -\frac{1}{2} \pm \frac{1}{2}\tilde{\mu} l), & h^{(m)} - \bar{h}^{(m)} = 2, \\ (-\frac{1}{2} \pm \frac{1}{2}\tilde{\mu} l, \frac{3}{2} \pm \frac{1}{2}\tilde{\mu} l), & h^{(m)} - \bar{h}^{(m)} = -2. \end{cases}$$

Linearized analysis in metric formalism

one could still have critical points $\tilde{\mu} l = \pm 1$ at which the massive gravitons degenerate with either of left or right massless gravitons. In terms of the original parameters in the metric formulation the critical points are given by

$$\gamma = -2\mu l(\mu l \bar{\sigma} \mp 1),$$

while in terms of parameters of Chern-Simons-like formalism, they occur at

$$\alpha = -2(\sigma \pm \frac{1}{\mu l}).$$

One may wonder if the model reduces to a three dimensional conformal gravity in the limit of $\tilde{\mu} \to 0$. One observes that corresponds to $\gamma = -2\mu^2 l^2 \bar{\sigma}$, in this limit the equation of motion reduces to:

$$\frac{1}{\mu}\epsilon_{\mu}{}^{\alpha\beta}\bar{\nabla}_{\alpha}\Big(-\frac{1}{2}\bar{\nabla}^2 h_{\beta\nu} + \frac{1}{2}\bar{\nabla}_{\nu}\bar{\nabla}_{\beta} h + \frac{1}{2l^2}\bar{g}_{\beta\nu} h - \frac{1}{l^2} h_{\beta\nu}\Big) = 0,$$

In particular one may study wave solutions in this model . To proceed we consider an ansatz for a wave solution as follows

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + F k_\mu k_\nu,$$

with k_μ being a null vector with respect to the background AdS metric $\bar{g}_{\mu\nu}$ which could be parametrized as

$$ds^2 = \frac{l^2}{r^2}(dr^2 - 2du dv),$$

With this notation, our ansatz for the wave solution takes the following form

$$ds^2 = \frac{l^2}{r^2}\left(dr^2 - 2du dv - F(r,u) du^2\right).$$

Wave solutions

By substituting the ansatz into the equations of motion, one finds

$$\frac{1}{4\mu^2 l^2 r}\left[-2\mu l r^2\frac{\partial^3 F}{\partial r^3}+(2\mu^2 l^2\bar{\sigma}+\gamma)\left(r\frac{\partial^2 F}{\partial r^2}-\frac{\partial F}{\partial r}\right)\right]=0.$$

Using the definition of $\tilde{\mu}$, the above differential equation may be recast into

$$-r^2\frac{\partial^3 F}{\partial r^3}+\tilde{\mu}l\left(r\frac{\partial^2 F}{\partial r^2}-\frac{\partial F}{\partial r}\right)=0,$$

which can be solved to give a generic AdS-wave solution with

$$F(r,u)=F_0(u)+F_2(u)r^2+F_{\tilde{\mu}}(u)\ r^{1+\tilde{\mu}l}.$$

It is then obvious that the solution degenerates at the critical points $\tilde{\mu}l=\pm 1$, as expected. In fact at these points one gets

$$\tilde{\mu}l=-1\longrightarrow F(r,u)=\tilde{F}_0(u)\log(r)+F_0(u)+F_2(u)r^2$$

$$\mu l=1\longrightarrow F(r,u)=F_0(u)+F_2(u)r^2+\tilde{F}_2(u)r^2\log(r).$$

Logarithmic CFT and new anomaly

When the central charge is zero, the two point function of the corresponding energy momentum tensor should vanish as well. We note, however, that in a LCFT one has the logarithmic partner of stress-tensor which could have non-zero two point function whose expression is fixed by a new parameter known as "new anomaly".

The asymptotic symmetry algebra of AdS geometry in MMG model consists of two copies of the Virasoro algebra with central charges

$$c_{\pm} = \frac{3l}{2G}\left(\sigma \pm \frac{1}{\mu l} + \frac{\alpha - \alpha^2 \Lambda_0 l^2}{2\mu^2 l^2 (1+\sigma\alpha)^2}\right).$$

At the critical point $\tilde{\mu} l = -1$ where the corresponding geometry could provide a holographic description for a LCFT

$$\tilde{\mu} l = -1 \quad : [\text{or } \alpha = -2(\sigma - \frac{1}{\mu l})] \rightarrow c_+ = \frac{3}{\mu G}, \quad c_- = 0,$$

Logarithmic CFT and new anomaly

The new anomaly can be evaluated as follows[D. Grumiller, N. Johansson and T. Zojer]:

$$b_L \equiv b_- = \lim_{\alpha \to \alpha_c} \frac{c_-}{h^{(0)} - h^{(m)}}.$$

where $\alpha_c = -2(\sigma - \frac{1}{\mu l})$. Note that in this limit both the denominator and the numerator vanish, though their ratio remains finite and nonzero. More precisely,

$$h^{(0)} - h^{(m)} = \frac{1}{2\alpha}\left(\alpha - \mu l(1+\sigma\alpha)^2 + \sqrt{\alpha^2 + \mu^2 l^2 (1+\sigma\alpha)^2}\right).$$

By performing the limit $\alpha \to \alpha_c$ one arrives at

$$b_- = -\frac{3l}{G}\frac{\mu l}{(\mu l \sigma - 2)^2}.$$

Logarithmic CFT and new anomaly

Clearly in the TMG limit where $\sigma = 1, \quad \mu l = 1$ one has

$$c_+ = \frac{3l}{G}, \qquad b_- = -\frac{3l}{G}, \qquad c_- = 0,$$

in agreement with previous results[K. Skenderis, M. Taylor and B. C. van Rees] Therefore denoting by $(T(z), \bar{T}(\bar{z}))$ and $(t(z), \bar{t}(\bar{z}))$ the energy momentum tensor and its logarithmic partner, respectively, it is natural to expect that the three dimensional MMG gravity at the critical point would provide a holographic dual for a LCFT such that

$$\langle T(z)T(0)\rangle = \frac{c_-}{2z^4} = 0, \qquad \langle \bar{T}(\bar{z})\bar{T}(0)\rangle = \frac{3l/G}{2\bar{z}^4},$$

$$\langle T(z)t(0)\rangle = \frac{\mu l}{(\mu l\sigma - 2)^2}\frac{-3l/G}{2z^4}, \qquad \langle t(z)t(0)\rangle = \frac{3l}{G}\frac{\mu l}{(\mu l\sigma - 2)^2}\frac{\ln(m^2 z^2)}{z^4},$$

Conclusion

- We have found AdS wave solutions of the model and shown that at the critical points the model exhibit logarithmic solutions. Therefore at the critical point the model could provide a holographic dual for a LCFT. We have also calculated the new anomaly of the theory.
- To further explore the properties of MMG it would be interesting to study holographic renormalization of the model. We note, however, that since in this context the action plays an essential role one will have to work with the Chern-Simons like formulation of the model.

Thank you for your attention