Absence of Bound States for δ'-interaction Supported by Non-closed Curve in $\mathbb{R}^2$

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Orthodox Academy of Crete, 26.8.2015

Outline

- Introduction
- Definition of the operator
- Essential spectrum
- Discrete spectrum
- Conformal mapping
- Examples
- Summary

Motivation

- The operator can be formally written as

$$H = -\Delta - \omega\delta'(\cdot - \Lambda)(\delta'(\cdot - \Lambda), \tilde{\cdot})$$

- Related to leaky quantum graphs
- Attractive δ' coupling $\omega(x) > 0$
- Curve Λ is non-closed Lipschitz C^1 curve
- We are interested in the existence and absence of the discrete spectrum of this operator

δ' interaction on a line

- The operator can be formally written as

$$H = -\Delta - \beta\delta'(0)(\delta'(0), \cdot)$$

- Suitable approximation for following operators

$$H = -\Delta + W^{a,\beta}_{\epsilon,0}(\cdot)$$

- Potential can be written as

$$W^{a,\beta}_{\epsilon,0} = \frac{-\beta}{\epsilon a(\epsilon)^2} V_0\left(\frac{x}{\epsilon}\right) - \left(\frac{2}{\beta} + \frac{1}{a(\epsilon)}\right)\left[\frac{1}{\epsilon} V_{-1}\left(\frac{x + a(\epsilon)}{\epsilon}\right) + \frac{1}{\epsilon} V_1\left(\frac{x - a(\epsilon)}{\epsilon}\right)\right]$$

where $\int_{\mathbb{R}} V_j(x)dx = 1$, $\int_{\mathbb{R}} |x|^{1/2}|V_j(x)|dx < \infty$ for $j \in \{1, 2, 3\}$, $\lim_{\epsilon\to 0} \frac{\epsilon}{a(\epsilon)^{12}} = 0$ and $a(0) = 0$

δ' interaction on a line

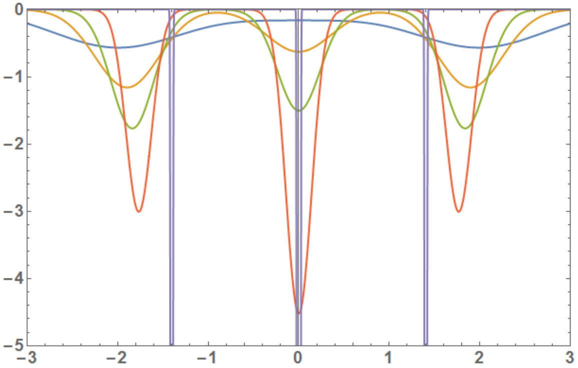

Figure: Approximating potential $W_{\epsilon,0}^{\epsilon^{1/13},1}(0)$ for $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{100}$

δ' interaction-definition, properties

- Operator acts as
$$H = -\Delta$$
with the domain
$$\mathcal{D}(H) = \{\psi \in \mathcal{H}^2(\mathbb{R} \setminus \{0\}) | -\beta\psi'(0+) = -\beta\psi'(0-) = \psi(0+) - \psi(0-)\}$$
- Sesqilinear form associated with this operator
$$q_{\beta,\delta'}(\psi,\phi) = (\nabla\psi, \nabla\phi) - \beta^{-1}\overline{(\psi(0+) - \psi(0-))}(\phi(0+) - \phi(0-))$$
with the domain $\mathcal{D}(q_{\beta,\delta'}) = \mathcal{H}^1(\mathbb{R} \setminus 0)$
- Attractive δ' interactions, i.e. $\beta > 0$
- Ground state eigenvalue $-\frac{4}{\beta^2}$
- Essential spectrum $\sigma_{ess}(H) = [0, \infty)$

Definition of the operator-δ' interaction supported by non-closed curve

- The symmetric sesqulinear form

$$\mathfrak{a}^{\Sigma}_{\delta',\omega}[f,g] := (\nabla f_{\rm i}, \nabla g_{\rm i})_{\rm i} + (\nabla f_{\rm e}, \nabla g_{\rm e})_{\rm e} - (\omega(f_{\rm e}|_\Sigma - f_{\rm i}|_\Sigma), g_{\rm e}|_\Sigma - g_{\rm i}|_\Sigma)_\Sigma,$$
$$\operatorname{dom} \mathfrak{a}^{\Sigma}_{\delta',\omega} := \mathcal{H}^1(\Omega_{\rm e}) \oplus \mathcal{H}^1(\Omega_{\rm i}),$$

  is closed, densely defined and lower-semibounded in $L^2(\mathbb{R}^2)$
- Linear mapping

$$\Gamma \colon \mathcal{H}^1(\Omega_{\rm e}) \oplus \mathcal{H}^1(\Omega_{\rm i}) \to L^2(\Sigma \setminus \Lambda), \qquad \Gamma f := f_{\rm e}|_{\Sigma\setminus\Lambda} - f_{\rm i}|_{\Sigma\setminus\Lambda},$$

- Symmetric, densely defined and lower-semibounded form

$$\mathfrak{a}^{\Lambda}_{\delta',\omega}[f,g] := \mathfrak{a}^{\Sigma}_{\delta',\omega}[f,g], \qquad \operatorname{dom} \mathfrak{a}^{\Lambda}_{\delta',\omega} := \{f \in \operatorname{dom} \mathfrak{a}^{\Sigma}_{\delta',\omega} \colon \Gamma f = 0\}.$$

- The self-adjoint operator $-\Delta^{\Lambda}_{\delta',\omega}$ in $L^2(\mathbb{R}^2)$ induced by the form $\mathfrak{a}^{\Lambda}_{\delta',\omega}$

Curve preliminaries

Hypothesis 1

Let $\Omega_+ \subset \mathbb{R}^2$ be a simply connected Lipschitz domain from the above class, whose complement $\Omega_- := \mathbb{R}^2 \setminus \overline{\Omega}_+$ is a Lipschitz domain from the same class. Set $\Sigma := \Omega_+ = \Omega_-$ and suppose that $\Lambda \subset \Sigma$ is a connected subarc of Σ, which is not necessarily bounded if Σ is unbounded.

Definition 1

A non-closed curve $\Lambda \subset \mathbb{R}^2$ satisfying Hypothesis 1 is called *piecewise-C^1* if it can be parametrized via a piecewise-C^1 mapping

$$\lambda\colon I \to \mathbb{R}^2, \qquad \lambda(s) := (\lambda_1(s), \lambda_2(s)), \quad I := (0, L),\ L \in (0, +\infty],$$

such that $\lambda(I) = \Lambda$ and λ is injective. If, moreover, $|\lambda'(s)| = 1$ for almost all $s \in I$, then such a parametrization is called *natural* and L is then called the length of Λ.

Curve preliminaries

Definition 2

A non-closed piecewise-C^1 curve $\Lambda \subset \mathbb{R}^2$ is monotone if it can be parametrized via the piecewise-C^1 mapping $\phi\colon (0, R) \to \mathbb{R}$, $R \in (0, +\infty]$, as

$$\Lambda = \left\{x_0 + (r\cos\phi(r), r\sin\phi(r)) \in \mathbb{R}^2\colon\ r \in (0, R)\right\};$$

here $x_0 \in \mathbb{R}^2$ is fixed.

Essential spectrum

Theorem

Let the curve $\Lambda \subset \mathbb{R}^2$ as in Hypothesis 1 be such that the domain $\mathbb{R}^2 \setminus \Lambda$ is quasi-conical. Then the spectrum of the self-adjoint operator $-\Delta^{\Lambda}_{\delta',\omega}$ satisfies

$$-\Delta^{\Lambda}_{\delta',\omega} \supseteq [0, +\infty).$$

Theorem

Let the bounded curve $\Lambda \subset \mathbb{R}^2$ be as in Hypothesis 1 and let the self-adjoint operator $-\Delta^{\Lambda}_{\delta',\omega}$ be as above. Then its essential spectrum is characterized as

$$\sigma_{ess}(-\Delta^{\Lambda}_{\delta',\omega}) = [0, +\infty).$$

δ' interaction on a loop

- auxiliary self-adjoint Schrödinger operator $T_{d,\beta}$

$$T_{d,\beta}\psi = -\psi''$$
$$\mathrm{dom}\, T_{d,\beta} = \{\psi \in H^2(0,d) \colon \psi'(0) = \psi'(d), \psi(d-) - \psi(0+) = \omega^{-1}\psi'(d)\}$$

- associated with the following form

$$\mathfrak{t}_{d,\beta}[f,g] := (f',g')_{L^2(0,d)} - \omega\overline{(f(d-) - f(0+))}(g(d-) - g(0+)),$$
$$\mathrm{dom}\,\mathfrak{t}_{d,\beta} := H^1(0,d).$$

Lemma

If $d\omega \leq 1$, then the above self-adjoint operator $T_{d,\beta}$ is non-negative.

Additional properties of the curve

Hypothesis 2

(A) Let a monotone piecewise-C^1 curve Λ be parametrized via the mapping $\phi\colon (0,R) \to \mathbb{R}$, $R \in (0,+\infty]$, as in Definition above with $x_0 = 0$.
(B) Suppose that piecewise-C^1 domains $G_\pm \subset D_R$ satisfy the following conditions:

$$G_+ \cap G_- = \varnothing, \quad \overline{D_R} = \overline{G_+ \cup G_-}, \quad \text{and} \quad \Lambda \subset \overline{G_+} \cap \overline{G_-}.$$

Set $\Sigma := \overline{G_+} \cap \overline{G_-}$. In particular, the inclusion $\Lambda \subset \Sigma$ holds.
(C) Let the function $\omega \in L^\infty(\Lambda;\mathbb{R})$ as a function of the distance r from the origin satisfy

$$\omega(r) \le \frac{1}{2\pi r\sqrt{1+(r\varphi'(r))^2}}, \qquad \text{for all} \quad r \in (0,R).$$

Absence of the discrete spectrum

Theorem

Let a monotone piecewise-C^1 curve $\Lambda \subset \mathbb{R}^2$ be parametrized via $\phi\colon (0,R) \to \mathbb{R}$, $R \in (0,+\infty]$. Then

$$\sigma(-\Delta^{\Lambda}_{\delta',\omega}) \subseteq [0,+\infty) \qquad \text{if} \quad \omega(r) \leq \frac{1}{2\pi r\sqrt{1+(r\phi'(r))^2}}, \quad \text{for all} \quad r \in (0,R).$$

If ω is majorized as above, and additionally, the domain $\mathbb{R}^2 \setminus \Lambda$ is quasi-conical, then $\sigma(-\Delta^{\Lambda}_{\delta',\omega}) = [0,+\infty)$.

Conformal mapping-Definition

- Function M is smooth analytic complex function with non-zero derivative everywhere in $S \subset \mathbb{C}$
- conformal map acts as follows

$$\tilde{x} = \Re(M(x+iy))$$
$$\tilde{y} = \Im(M(x+iy))$$

- Cauchy-Riemann equations

$$\partial_x \tilde{x} = \partial_y \tilde{y} \quad \partial_x \tilde{y} = -\partial_y \tilde{x}$$

Linear fractional transformation-LFT

For $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$ the mapping $M\colon \widehat{\mathbb{C}} \to \mathbb{C}$ is an LFT if one of the two conditions holds:
1) $c = 0$, $d \neq 0$, $M(\infty) := \infty$, and $M(z) := (a/d)z + (b/d)$ for $z \in \mathbb{C}$.
2) $c \neq 0$, $M(\infty) := a/c$, $M(-d/c) := \infty$, and $M(z) := \frac{az+b}{cz+d}$ for $z \in \mathbb{C}$, $z \neq -d/c$.

Generalization of previous result

Theorem

Let $\Lambda \subset \mathbb{R}^2$ be a bounded piecewise-C^1 curve and let the self-adjoint operator $-\Delta^{\Lambda}_{\delta',\omega}$ in $L^2(\mathbb{R}^2)$. Suppose that there exists an LFT $M\colon \mathbb{C} \to \mathbb{C}$ such that:
(a) $M(\infty), M^{-1}(\infty) \notin \Lambda$;
(b) $\Gamma := M^{-1}(\Lambda)$ is monotone.
Then it holds that

$$\sigma(-\Delta^{\Lambda}_{\delta',\omega}) = [0, +\infty)$$

where $\omega \leq \frac{1}{\sqrt{J_m(z) 2\pi r \sqrt{1+(r\phi'(r))^2}}}$.

Circle arc

Theorem

Let $-\Delta^{\Sigma}_{\delta',\omega}$ be as before with constant ω and Σ as below. Then for $\omega \in (-\infty, \frac{\tan(\epsilon/2)}{8\pi R}]$ the operator $-\Delta^{\Lambda}_{\delta',\omega}$ has no negative eigenvalues.

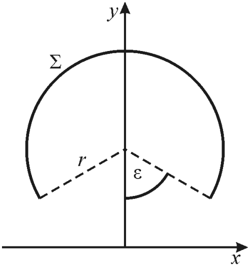

Figure: Sketch of a non-closed segment of the circle in the coordinate system

Line segment

Theorem

Assume that $\Lambda := \{(x,0) : 0 < x < L\} \subset \mathbb{R}^2$ is an interval of length $L > 0$. Let the self-adjoint operator $-\Delta^{\Lambda}_{\delta',\omega}$ be as before. Under assumption $\frac{1}{2\pi L} > \omega$ the operator $-\Delta^{\Lambda}_{\delta',\beta}$ has no negative eigenvalues.

Theorem

Assume that $\Lambda := \{(x,0) : 0 < x < L\} \subset \mathbb{R}^2$ is an interval of length $L > 0$. Let the self-adjoint operator $-\Delta^{\Lambda}_{\delta',\omega}$ be as before. Under assumption $\frac{\pi}{2L} < \omega$ the operator $-\Delta^{\Lambda}_{\delta',\beta}$ has at least one negative eigenvalue.

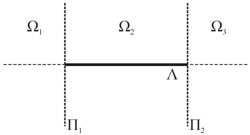

Figure: Separation of $\mathbb{R}^2$ into three regions Ω_k

Summary

- Definition of δ' interaction supported by a compact non-closed curve
- Behavior of the point spectrum governed by a behavior of a δ' interaction on a circle
- Absence of the negative eigenvalues state for sufficiently small coupling
- Existence of the ground state for a large coupling

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Thank you for your attention