Absence of the Gribov ambiguity in a quadratic gauge

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Outlines

1. Gribov ambiguity and its origin
2. A Quadratic gauge and its suitability for the nonperturbative problems in QCD
3. Absence of the ambiguity in a quadratic gauge
4. Spherically symmetric gauge field configuration and the quadratic Gauge
5. BRST symmetry in a quadratic gauge

What is the Gribov ambiguity?

It is an inability of a covariant gauge to choose a gauge configuration uniquely.

Its origin

- The path integral in gauge theories is given by

$$Z = N \int \mathcal{D}A_\mu e^{-S_{YM}} \tag{1}$$

  Where

$$S_{YM} = \int d^4x \Big(-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \Big) \tag{2}$$

  is the Yang-Mills action.

- $S_{YM}[\tilde{A}_\mu] = S_{YM}[A_\mu] \Rightarrow$ infinite redundant functional integrations in (1) $\because \int \mathcal{D}A_\mu$ includes configurations related by the gauge transformation.
- The issue is addressed by invoking a gauge condition such as the Landau gauge $\partial_\mu A^\mu = f$

- However, equivalent configurations still exist since

$$\partial^\mu (D_\mu \alpha)^a = 0 \tag{3}$$

has non-trivial solutions belonging to non-trivial configurations. [1]

- It is an ambiguity common in covariant gauges[2].
- Sol: Restrict the space of integration to the fundamental modular region C^0. However, the region C^0 still contains Gribov copies.
- Restriction modifies the action, known as Gribov-Zwanziger action [3]. The GZ action is not $BRST$ invariant [4].
- An attempt to eliminate copies → loss of the $BRST$ invariance of the theory whereas the quadratic gauge is $BRST$ invariant.

[1] V. N. Gribov, *Nucl. Phys.* **B139** 1 (1978) [2] I. Singer, *Comm. Math. Phys.* **60** 7 (1978) [3] D. Zwanziger *Nucl. Phys.* **B323** 513-544 (1989) [4] N. Maggiore and M. Schadent, *Phys. Rev. D* **50**, 10 (1994)

A quadratic gauge

- $$H^a[A^\mu(x)] = A^a_\mu(x) A^{\mu a}(x) = f^a(x); \quad \text{for each } a \tag{4}$$

  where $f^a(x)$ is an arbitrary function of x.

- **The effective Lagrangian**

  $$\mathcal{L}_{eff} \quad = \quad -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{\zeta}{2} F^{a2} + F^a \; A^a_\mu A^{\mu a} - \overline{c^a} A^{\mu a} (D_\mu c)^a \tag{5}$$

  where second, third and fourth terms are gauge fixing and ghost Lagrangian resp. The field F^a is auxiliary, $(D_\mu c)^a = \partial_\mu c^a - g f^{abc} A^b_\mu c^c$.

- USP:This gauge is Lorentz invariant. In general, algebraic gauges are not.

Suitability for infra-red phenomena

- The ghost Lagrangian contains a term $gf^{abc}\overline{c^a}c^c A^{\mu a} A^b_\mu$.
- In the ghost condensed phase as shown in ref. [5], the term gives two strong signatures of the color confinement: 1. Abelian dominance and 2. A pole of the off-diagonal gluon propagator is on imaginary p^2 axis [6].
- Here we aim to address another crucial non-perturbative phenomenon, the Gribov ambiguity.
- Thus, the purpose of this work with results mentioned above is to justify the motivation and suitability of a choice of the quadratic gauge for the nonperturbative problems in QCD

[5] Haresh Raval and Urjit A. Yajnik, Phys. Rev. D **91**, 085028 (2015) [6] C. D. Roberts, A. G. Williams, and G. Krein, Int. J. Mod.Phys. **A 07**, 5607 (1992)

Why absent?

- For the infinitesimal gauge transformation the gauge in (4) gives

$$\tilde{A}^a_\mu \tilde{A}^{\mu a} = (A^a_\mu + (D_\mu\alpha)^a)(A^{\mu a} + (D^\mu\alpha)^a)$$
$$= A^a_\mu A^{\mu a} + 2A^{\mu a}(D_\mu\alpha)^a + (D_\mu\alpha)^a(D^\mu\alpha)^a \qquad (6)$$

Hence for $\tilde{A}^a_\mu \tilde{A}^{\mu a} = A^a_\mu A^{\mu a}$ to happen

$$2A^{\mu a}(D_\mu\alpha)^a + (D_\mu\alpha)^a(D^\mu\alpha)^a = 0 \qquad (7)$$

Therefore, $(D_\mu\alpha)^a = 0 \Rightarrow \tilde{A}^a_\mu = A^a_\mu$ or
$(D_\mu\alpha)^a = -2A^a_\mu \Rightarrow \tilde{A}^a_\mu = -A^a_\mu$

- So in neither case any new configuration is present.

- For the finite transformation also the same result is true. The finite transformation on A_μ can be given as

$$\tilde{A}^a_\mu = A^a_\mu + (D_\mu \alpha)^a + G^a_\mu \tag{8}$$

  where G^a_μ accounts for the finiteness.
- Since the gauge is quadratic, the equivalence of the gauge $\tilde{A}^a_\mu \tilde{A}^{\mu a} = A^a_\mu A^{\mu a}$ yields the same two configuration.
- Thus, the gauge is Gribov ambiguity free.

On $\mathbb{S}^3$, no $O(3)$ symmetric field exists

- Adopting a parameterization for a vector potential shown in ref [1].

$$A_i = f_1(r)\frac{\partial \hat{n}}{\partial x_i} + f_2(r)\hat{n}\frac{\partial \hat{n}}{\partial x_i} + f_3(r)\hat{n}n_i, \qquad i = 1, 2, 3 \tag{9}$$

  Where $n_i = \frac{x_i}{r}$, $r = \sqrt{\Sigma x_i^2}$, $\hat{n} = i n_j \sigma_j$ σ_j are Pauli matrices , $\hat{n}^2 = -1$. For simplicity, $A_0 = 0$.

- The spherically symmetric operator is given by

$$U = exp\left(\frac{\alpha(r)}{2}\hat{n}\right) = \cos\left(\frac{\alpha(r)}{2}\right) + \hat{n}\sin\left(\frac{\alpha(r)}{2}\right) \tag{10}$$

- Compactification of a $\mathbb{R}^N$ to a compact manifold $\mathbb{S}^N$ is achieved by the condition $U(\infty) = I$, which implies $\alpha(\infty) = 4\pi n$;
  n is an integer.

[1] V. N. Gribov, *Nucl. Phys.* **B139** 1

- The gauge transformation results in transformations of f_1, f_2 and f_3 as follows

$$\begin{aligned} \tilde{f}_1 &= f_1 \cos\alpha + (f_2 + \frac{1}{2}) \sin\alpha \\ \tilde{f}_2 + \frac{1}{2} &= -f_1 \sin\alpha + (f_2 + \frac{1}{2}) \cos\alpha \\ \tilde{f}_3 &= f_3 + \frac{1}{2}\dot{\alpha} \end{aligned} \tag{11}$$

  where overdot indicates differentiation with respect to r.

- Now, *a th* component of A_i can be derived using following formula

$$A_i^a = \frac{1}{2} Tr(A_i \sigma_a) \tag{12}$$

(13)

  After some algebra

$$A_1^1 = i[f_1(\frac{1}{r} - \frac{x_1^2}{r^3}) + f_3\frac{x_1^2}{r^2}] \tag{14a}$$

$$A_2^1 = i[-f_1\frac{x_1x_2}{r^3} + f_2\frac{x_3}{r^2} + f_3\frac{x_1x_2}{r^2}] \tag{14b}$$

$$A_3^1 = i[-f_1\frac{x_1x_3}{r^3} - f_2\frac{x_2}{r^2} + f_3\frac{x_1x_3}{r^2}] \tag{14c}$$

$$A_1^2 = i[-f_1\frac{x_1x_2}{r^3} - f_2\frac{x_3}{r^2} + f_3\frac{x_1x_2}{r^2}] \tag{15a}$$

$$A_2^2 = i[f_1(\frac{1}{r} - \frac{x_2^2}{r^3}) + f_3\frac{x_2^2}{r^2}] \tag{15b}$$

$$A_3^2 = i[-f_1\frac{x_2x_3}{r^3} + f_2\frac{x_1}{r^2} + f_3\frac{x_2x_3}{r^2}] \tag{15c}$$

$$A_1^3 = i[-f_1\frac{x_1x_3}{r^3} + f_2\frac{x_2}{r^2} + f_3\frac{x_1x_3}{r^2}] \tag{16a}$$

$$A_2^3 = i[-f_1\frac{x_2x_3}{r^3} - f_2\frac{x_1}{r^2} + f_3\frac{x_2x_3}{r^2}] \tag{16b}$$

$$A_3^3 = i[f_1(\frac{1}{r} - \frac{x_3^2}{r^3}) + f_3\frac{x_3^2}{r^2}] \tag{16c}$$

- Imposing a boundary condition on A^j_k s.

$$A^j_k \to 0 \quad \text{as} \quad \frac{1}{r}, \quad \text{as} \quad r \to \infty \tag{17}$$

From Eq.s (14), (15), (16), this implies the boundary condition on f_1, f_2 and f_3 to be

$$f_1, f_2 \to const. \quad \text{as} \quad r \to \infty \text{ and } f_3 \to 0 \quad \text{as fast as} \quad \frac{1}{r} \quad \text{as} \quad r \to \infty \tag{18}$$

- Note: Addressing the ambiguity on $\mathbb{S}^3$ requires boundary condition on f_3 to be little stronger (faster than $\frac{1}{r}$ as $r \to \infty$) because of Eq. (28) that we shall come across later. Hence, we consider a stronger condition on f_3 only.
- Evaluating the quadratic gauge: E.g. $a = 1$, the gauge takes the form

$$\begin{aligned} A^1_i A^{i1} &= A^1_1 A^{11} + A^1_2 A^{21} + A^1_3 A^{31} \\ &= (A^1_1)^2 + (A^1_2)^2 + (A^1_3)^2 \end{aligned} \tag{19}$$

In spherical polar coordinates, the condition can be written as

$$= \quad -\frac{1}{r^2}(f_1^2+f_2^2)+\sin^2\theta\cos^2\phi\Big(\frac{1}{r^2}(f_1^2+f_2^2)-f_3^2\Big) \tag{20}$$

Hence

$$\tilde{A}_i^1\tilde{A}^{i1} \quad = \quad -\frac{1}{r^2}(\tilde{f}_1^2+\tilde{f}_2^2)+\sin^2\theta\cos^2\phi\Big(\frac{1}{r^2}(\tilde{f}_1^2+\tilde{f}_2^2)-\tilde{f}_3^2\Big) \tag{21}$$

- For $\tilde{A}_i^1\tilde{A}^{i1}=A_i^1A^{i1}$ to happen

$$\frac{1}{r^2}[(\tilde{f}_1^2+\tilde{f}_2^2)-(f_1^2+f_2^2)]+\sin^2\theta\cos^2\phi[\tilde{f}_3^2-f_3^2-\frac{1}{r^2}\big((\tilde{f}_1^2+\tilde{f}_2^2)-(f_1^2+f_2^2)\big)]=0 \tag{22}$$

Since $\alpha(r)$ only, the first term and the coefficient of $\sin^2\theta\cos^2\phi$ of Eq. (22) must individually vanish, giving us two different copy equations

$$f_2 + \frac{1}{2} = -f_1 \cot \frac{\alpha}{2} \tag{23}$$

$$f_3 \dot{\alpha} + \frac{1}{4} \dot{\alpha}^2 = 0 \;\Rightarrow\; \quad \dot{\alpha} = 0 \;\text{ or }\; \dot{\alpha} = -4f_3 \tag{24}$$

We encounter two copies corresponding to eq.s $\dot{\alpha} = 0$ and $\dot{\alpha} = -4f_3$.

- For $\dot{\alpha} = 0$, we obtain

$$\tilde{f}_1 = -f_1, \quad \tilde{f}_2 = f_2, \quad \tilde{f}_3 = f_3 \tag{25}$$

which yields a copy

$$\tilde{A}^j_j = A^j_j - 2if_1\left(\frac{1}{r} - \frac{x_j^2}{r^3}\right) \tag{26}$$

$$\tilde{A}^j_k = A^j_k + 2if_1\frac{x_j x_k}{r^3} \tag{27}$$

However, on $\mathbb{S}^3$ this copy no longer exists. Because $\dot{\alpha} = 0 \Rightarrow \alpha = const.$ everywhere including infinity. Setting $\alpha(r) = \alpha(\infty) = 4\pi n \Rightarrow$ the copy Eq.(23) implies $f_2 = \infty$ everywhere.

We want finite copies of A_k^j which is well behaved and finite at finite distances, which is not possible for $\dot{\alpha} = 0$ on $\mathbb{S}^3$.
Therefore, the Eq. (23) is not valid on $\mathbb{S}^3$, thus the copy vanishes on it.

- For

$$\dot{\alpha} = -4f_3 \qquad \text{we get} \tag{28}$$

$$\tilde{f}_1 = -f_1, \quad \tilde{f}_2 = f_2, \quad \tilde{f}_3 = -f_3 \tag{29}$$

which yields a copy

$$\tilde{A}_j^j = -A_j^j \tag{30}$$

$$\tilde{A}_k^j = -A_j^k \tag{31}$$

It can also be removed on $\mathbb{S}^3$. Since $f_3 \to 0$ faster than $\frac{1}{r}$ as $r \to \infty$, Eq. (28) $\Rightarrow \alpha(\infty) = const.$. Setting $\alpha(\infty) = 4\pi n$ for which Eq. (23) implies $f_2 \to \infty$ as $r \to \infty$.

- So on $\mathbb{S}^3$, Eq. (23) is an obstruction for the boundary condition on f_2 (Eq. (18)) to be satisfied therefore not valid. Therefore this copy does not exist on $\mathbb{S}^3$.
- The result is true for stronger conditions such as $\frac{1}{r^2}$ and e^{-r}.
- The condition for other two components, $\tilde{A}_i^2 \tilde{A}^{i2} = A_i^2 A^{i2}$ and $\tilde{A}_i^3 \tilde{A}^{i3} = A_i^3 A^{i3}$, produce same two equations for copy.
- Whereas for coulomb gauge [1]

$$\frac{\partial A_i}{\partial x_i} = \hat{n}(\dot{f}_3 + \frac{2}{r} f_3 - \frac{2}{r^2} f_1) \tag{32}$$

we have, for all three components yields the equation

$$\ddot{\alpha} + \frac{2}{r}\dot{\alpha} - \frac{4}{r^2}\Big((f_2 + \frac{1}{2})\sin\alpha + f_1\cos\alpha\Big) = 0 \tag{33}$$

This equation is known to be solvable and therefore the ambiguity exists.

The BRST invariance of the theory

The $BRST$ transformations in the quadratic gauge:

$$\delta c^d = \frac{\omega}{2} f^{dbc} c^b c^c \tag{34a}$$

$$\delta \overline{c^d} = \frac{2\omega}{g} F^a \tag{34b}$$

$$\delta A_\mu^d = \frac{\omega}{g} (D_\mu c)^d \tag{34c}$$

$$\delta F^a = 0 \tag{34d}$$

The transformations (34) are nil-potent. Under these transformations,

$$\begin{aligned}
\delta \mathcal{L}_{eff} &= \delta \left(\frac{\zeta}{2} F^{a2} + F^a \ A_\mu^a A^{\mu a} - \overline{c^a} A^{\mu a} (D_\mu c)^a \right) \qquad \left(\delta \mathcal{L}_{YM} = 0 \right) \\
&= \frac{2\omega}{g} F^a A^{\mu a} (D_\mu c)^a - \frac{2\omega}{g} F^a A^{\mu a} (D_\mu c)^a - \frac{\omega}{g} \overline{c^a} (D_\mu c)^a (D^\mu c)^a \\
&= -\frac{\omega}{g} \overline{c^a} (D_\mu c)^a (D^\mu c)^a \\
&= 0 \qquad \left((D_\mu c)^a \text{ is a grassmann variable} \right)
\end{aligned}$$

Conclusion

- It is a Lorentz invariant gauge free of the ambiguity.
- Provides a $BRST$ invariant solution.
- More suitable for the study of infra-red phenomena in QCD.

Thank you