On the On-Off Problem: An Objective Bayesian Analysis

Claiming detections, setting credibility intervals, and setting upper limits.
A unified, single and consistent method — valid everywhere.

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Introduction

Typical counting experiments measure discrete sets of events. Such data are often [12, 7] modeled with the Poisson distribution. The Poisson distribution may be approximated by a normal distribution when measuring many events. However, when data is rare, such an approximation is not good enough. In Fig. 1, a typical example of a low count data sample is shown. The question arises: What do you do when it simply impossible to "go out and get more data"?

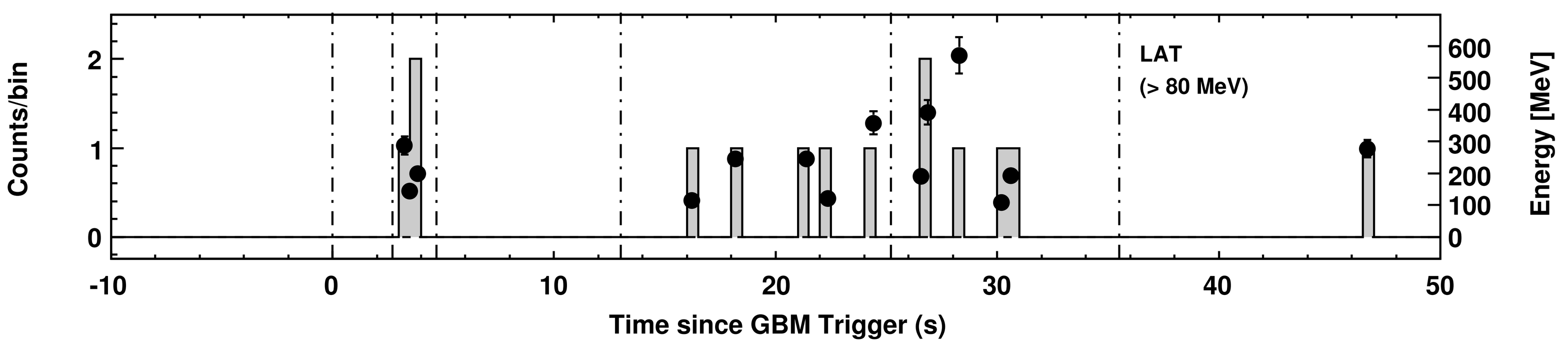

Figure 1: A typical low count high-energy astrophysics data set. It shows gamma-rays measured from GRB080825C as observed by Fermi-LAT. Black dots represent the energy measurement, the grey bars represent the number of photons. Figure reproduced from [1]

The On-Off Problem

In the On-Off problem, also known as Li-Ma problem, one would like to infer a signal rate in the presence of an imprecisely known background rate. It is a classical counting experiment with parameters:

1. N_{on} events in some "on" region with potential signal
2. N_{off} events in some "off" region, known to be signal free
3. α – the ratio of exposures, known with negligible uncertainty

In the case of gamma-ray astronomy, Berge et al. [4] explain the problem and its parameters well. The common frequentist and Bayesian methods all have drawbacks, see [11]. None of the methods in the literature so far cover the full range of the problem:

1. calculate the significance of a detection
2. estimate the signal

Here I present a two-step objective Bayesian solution to the full On-Off problem [11], inspired by [6], that adresses these issues in a unified a way.

Methods Development

The idea behind objective Bayesian analysis simple. One takes, in a sense, "flat" priors representing the lack of knowledge. One interesting objective Bayesian prior: Jeffreys's rule [10, 5]. This prior is a keystone in this analysis. The analysis follows [6] and is done in two steps.

1. The probability that the observed counts are due to background only is calculated. If this is smaller than a previously defined value, the signal is said to be detected.
2. The signal contribution is to be estimated or upper limit is calculated, depending on whether the detection limit has been reached.

Methods: objective Bayesian hypothesis testing via Bayes factors and objective Bayesian estimation.

First step: Hypothesis testing

One problem: objective priors are only defined up to a proportionality constant. Those constants become relevant in this case. A full discussion of the topic can be found in [11, 3]. Fix suggestion: "minimal sample device" [14, 8]. Calculations can be found in [11, 3]. The odds (Bayes factor) of the background model over the signal model, using the above assumptions, turn out to be

$$B_{01} = \frac{c_0 \gamma}{c_1 \delta}, \tag{1}$$

where γ and δ are defined using the Gamma function $\Gamma(x)$ and the hypergeometric function ${}_2\mathrm{F}_1(a, b; c; z)$:

$$\gamma := (1 + 2N_{\text{off}})\,\alpha^{\frac{1}{2}+N_{\text{on}}+N_{\text{off}}}\,\Gamma\left(\frac{1}{2} + N_{\text{on}} + N_{\text{off}}\right)$$
$$\delta := 2(1+\alpha)^{N_{\text{on}}+N_{\text{off}}}\,\Gamma(1 + N_{\text{on}} + N_{\text{off}})\,{}_2\mathrm{F}_1\left(\frac{1}{2} + N_{\text{off}}, 1 + N_{\text{on}} + N_{\text{off}}; \frac{3}{2} + N_{\text{off}}; -\frac{1}{\alpha}\right)$$
$$\frac{c_0}{c_1} = \frac{2\arctan\left(\frac{1}{\sqrt{\alpha}}\right)}{\sqrt{\pi}}. \tag{2}$$

A signal detection based on Eqn. 1 may be claimed when the resulting odds of the background model are low. Proposal: use a "Bayesian z-value", similar to [9]

$$S_{\text{b}} = \sqrt{2}\,\mathrm{erf}^{-1}(1 - B_{01}), \tag{3}$$

where $B_{01} = 5.710^{-7}$ would correspond to $S_{\text{b}} = 5$ or "5 sigma".

Second Step: Signal Estimation

After determining the Bayes factor of the background model over the signal model, one proceeds to estimating the signal contribution. Two cases:

1. Detection: then assume signal model and calculate mode and credible interval
2. No detection: calculate upper limit, assuming the signal is there, but too weak to be measured

In both cases one needs the conditional probability $P(\lambda_{\text{s}}|N_{\text{on}}, N_{\text{off}}, H_1)$ of the signal λ_{s}, given the number counts and the signal model H_1. Here, the improper Jeffreys's prior cancel and the posterior is proper. After marginalization over the background parameter λ_{bg}, the result is (calculation in [11])

$$P(\lambda_{\text{s}}|N_{\text{on}}, N_{\text{off}}, H_1) = P_{\text{P}}(N_{\text{on}} + N_{\text{off}}|\lambda_{\text{s}})\,\frac{\mathrm{U}\left[\frac{1}{2} + N_{\text{off}}, 1 + N_{\text{off}} + N_{\text{on}}, \left(1 + \frac{1}{\alpha}\right)\lambda_{\text{s}}\right]}{{}_2\tilde{\mathrm{F}}_1\left(\frac{1}{2} + N_{\text{off}}, 1 + N_{\text{off}} + N_{\text{on}}; \frac{3}{2} + N_{\text{off}}; -\frac{1}{\alpha}\right)}, \tag{4}$$

as expressed in terms of three functions, namely the Poisson distribution $P_{\text{P}}(N|\lambda)$, the regularized hypergeometric function ${}_2\tilde{\mathrm{F}}_1(a, b; c; z) = \frac{{}_2\mathrm{F}_1(a,b;c;z)}{\Gamma(c)}$, and the Tricomi confluent hypergeometric function $\mathrm{U}(a, b, z)$.

To state a flux, one should take the mode λ_{s}^*, of the posterior distribution $P(\lambda_{\text{s}}|N_{\text{on}}, N_{\text{off}}, H_1)$, as signal estimator. On interesting choice as credible interval is the highest posterior density interval (HPD) $[\lambda_{\min}, \lambda_{\max}]$ [11] containing 68% probability. The upper limit case is equivalent, but one can directly take the numerical cumulative distribution function. Benefits:

1. marginalization natural in a Bayesian approach of the problem
2. no issues at the border of the parameter space
3. all possible number counts are OK (no approximations of the Likelihood)
4. signal posterior always physically meaningful (i.e. positive $\lambda_{\text{s}}^*, \lambda_{min}, \lambda_{max}, \lambda_{99}, ...$).

Validation

An extensive validation was made [11]. It shows that the two-step method behaves well in all test-case examples, in particular at $N_{\text{on}} \sim \alpha N_{\text{off}}$. The objective Bayesian hypothesis testing converges to the results from other methods for high count numbers. The objective Bayesian signal estimation can reconstruct the true signal parameter λ_{s} with a good error estimate.

Application: Gamma-Ray Bursts

GRBs are extraterrestrial flashes of gamma-rays mostly lasting only a few seconds. Interesting question: How exactly do GRBs produce high energy gamma-rays [1]? These measurements are usually very low count (see Fig. 1).

The first example is the GRB080825C as seen by Fermi-LAT (see Fig. 1, Fig. 2, and [1]):

1. $N_{\text{on}} = 15$
2. $N_{\text{off}} = 19$
3. $\alpha = 33/525$

Analysis resuts compare well to published results:

1. $S_{\text{b}} = 6.11$, published: $S_{\text{Ref.}} = 6.4$
2. $\lambda_{\text{s}} = 13.28^{+4.16}_{-3.49}$, published: $\lambda_{\text{Ref.}} = 13.7$.

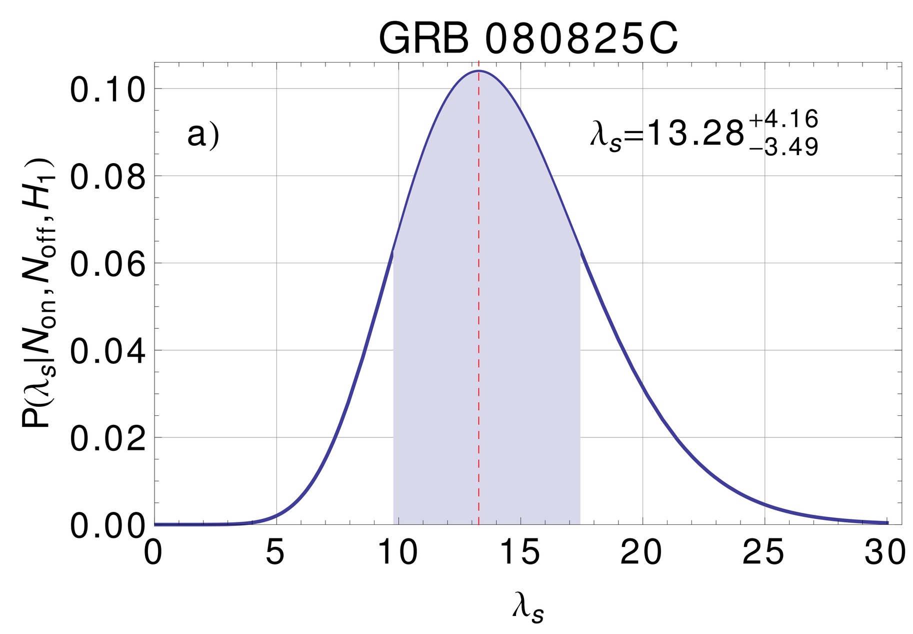

Figure 2: The conditional probability $P(\lambda_{\text{s}}|N_{\text{on}}, N_{\text{off}}, H_1)$ of the signal λ_{s}, given the Fermi-LAT number counts of GRB080825C. The blue band indicates the HPD interval for the signal parameter posterior probability. Figure reproduced from [11].

The second example is the GRB080330 as observed by the VERITAS Cherenkov telescope, (see Fig. 3 and [2]):

1. $N_{\text{on}} = 0$
2. $N_{\text{off}} = 15$
3. $\alpha = 0.123$

The corresponding odds of the background model are $B_{01} = 2.29$, unsurprisingly favouring the null hypothesis as not a single on event was detected. Now, assuming that the source is there one can put an upper limit to λ_{s}. The result is plotted in Fig. 3. The published value uses a frequentist upper limit setting method, popularized by Rolke et al. [13]. Their result is $\lambda_{99}^{\text{Rolke}} = 2.4$ [11], which somewhat lower than the number, calculated from the suggested signal posterior (Eqn. 4), $\lambda_{99} = 4.10$. A detailed analysis indicates [11] that, especially at the border of the parameter space for $N_{\text{on}} \leq \alpha N_{\text{off}}$, Rolke's method is an overestimation and therefore limited. These limits are overcome by the Bayesian method.

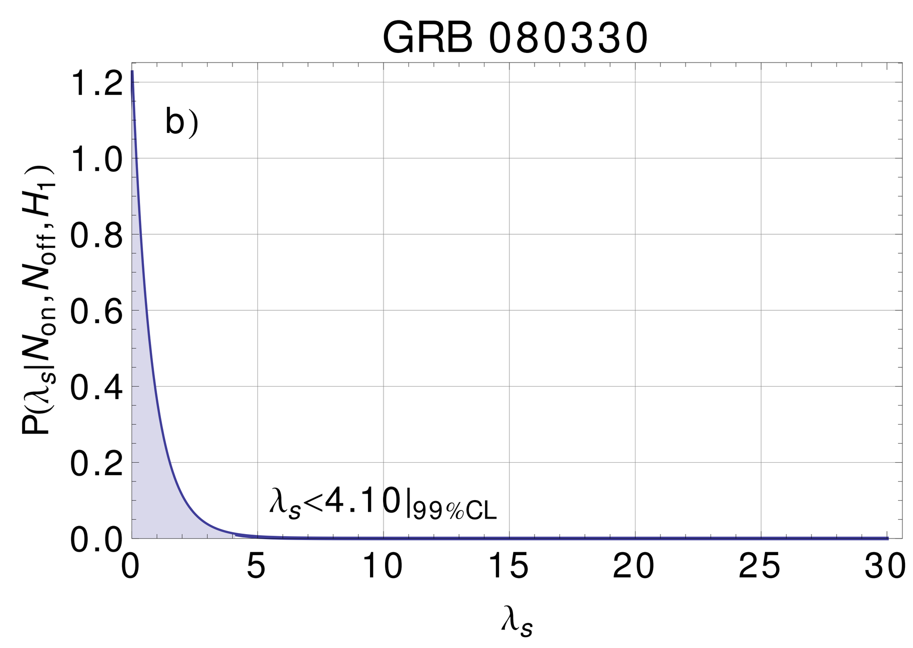

Figure 3: The conditional probability $P(\lambda_{\text{s}}|N_{\text{on}}, N_{\text{off}}, H_1)$ of the signal λ_{s}, given the VERITAS number counts of GRB080330. The blue band indicates the 99% probability upper limit to the signal parameter. Figure reproduced from [11].

Conclusions

Claiming detections, setting credibility intervals, or setting upper limits can be unified over the whole On-Off problem parameter range in one consistent two-step objective Bayesian method.

An example implementation in Python can be downloaded from the public git-repository:

https://bitbucket.org/mknoetig/obayes_onoff_problem

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