

# Small-scale anisotropies of cosmic rays from relative diffusion

**Philipp Mertsch and Markus Ahlers**

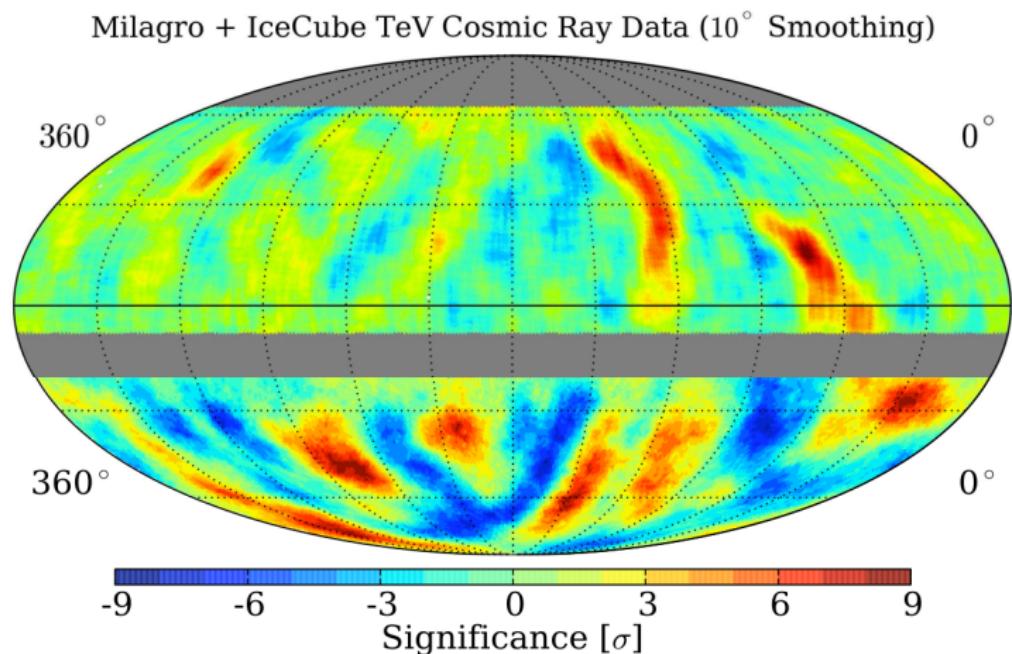
Kavli Institute for Particle Astrophysics and Cosmology, Stanford University

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# Small-scale anisotropies

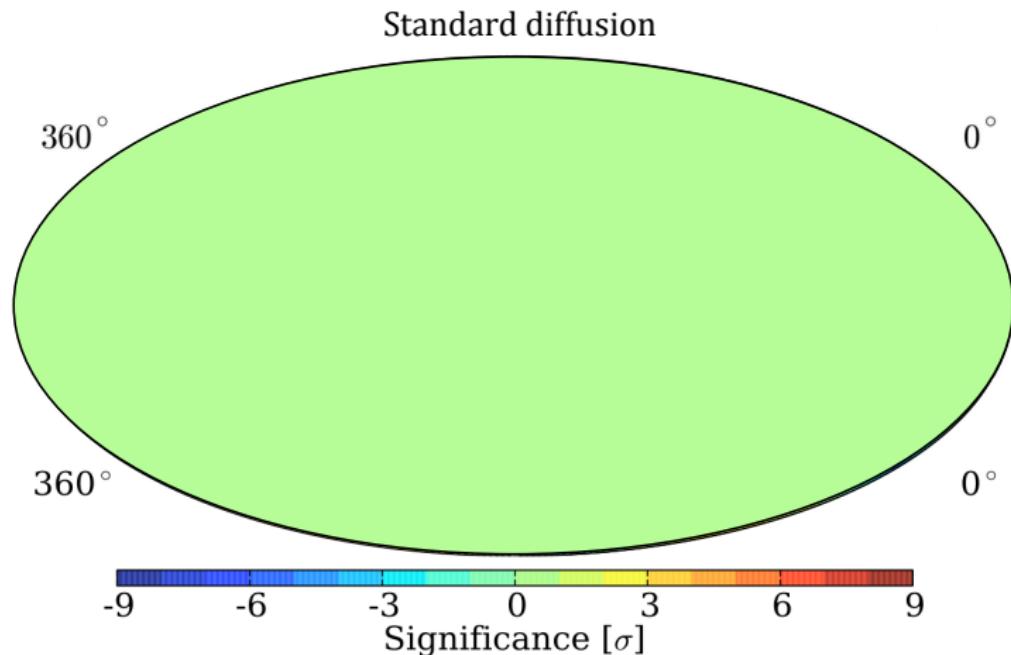
after subtraction of dipole and quadrupole:



Abdo et al., *PRL*, **101**, 221101, 2008; Santander et al., *Proc. ICRC 2013*; slide concept: P. Desai

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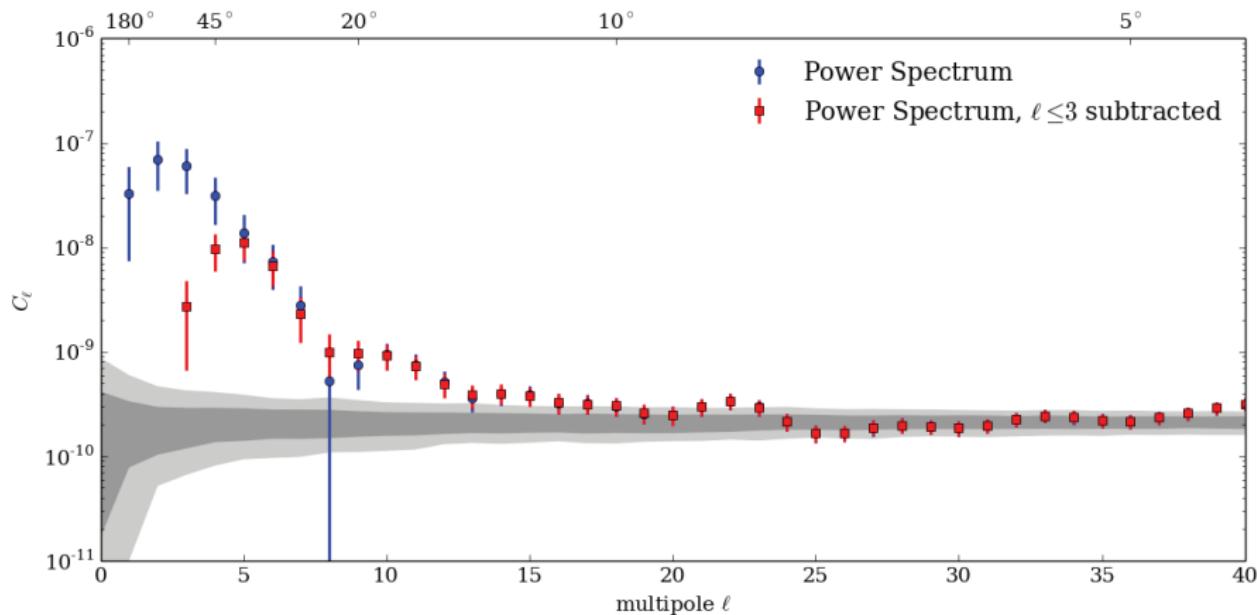
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# Angular power spectrum

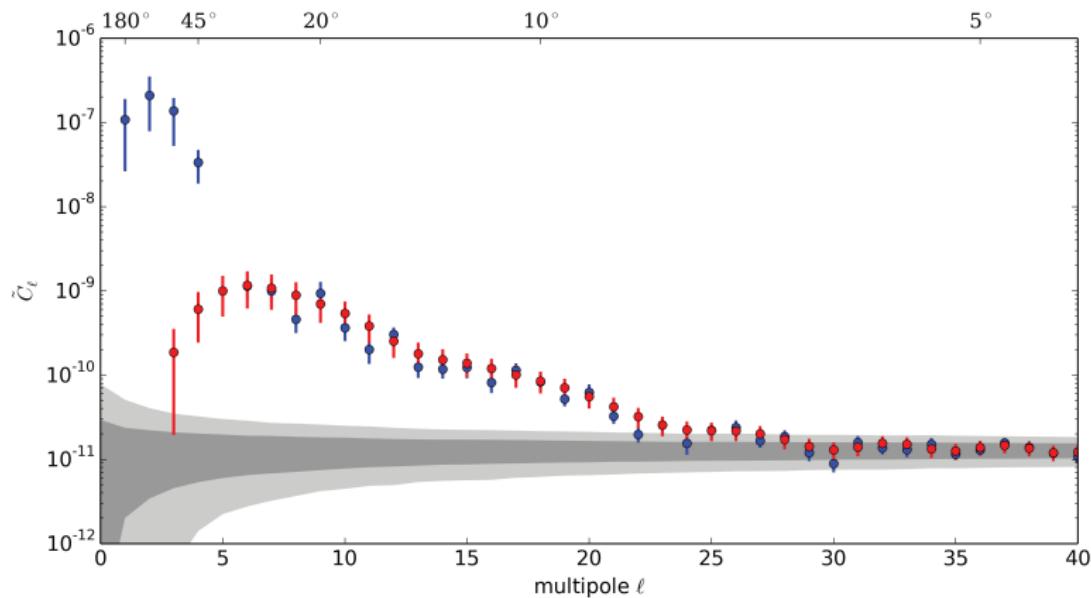
$$C_\ell = \frac{1}{4\pi} \int d\hat{\mathbf{p}}_1 \int d\hat{\mathbf{p}}_2 P_\ell(\hat{\mathbf{p}}_1 \hat{\mathbf{p}}_2) f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2)$$



Fiorino et al., PoS 241

# Angular power spectrum

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Westerhoff et al., PoS 274

# Standard diffusion

e.g. Jokipii, *Rev. Geophys.* 9 (1971) 27

- Liouville's theorem:

$$\frac{df}{dt} = 0 = \frac{\partial f}{\partial t} + \dot{\mathbf{x}} \nabla_{\mathbf{x}} f + \dot{\mathbf{p}} \nabla_{\mathbf{p}} f$$

- regular and turbulent magnetic field:  $\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + \delta\mathbf{B}(\mathbf{r}) = p_0/e (\boldsymbol{\Omega} + \boldsymbol{\omega})$
- ensemble averaged phase space density and fluctuations:  $f = \langle f \rangle + \delta f$
- BGK-ansatz; drive  $\langle f \rangle$  to isotropic distribution  $n$ :

$$\frac{\partial \langle f \rangle}{\partial t} + \hat{p} \nabla_{\mathbf{x}} \langle f \rangle - i\boldsymbol{\Omega} \mathbf{L} \langle f \rangle = \langle i\boldsymbol{\omega} \mathbf{L} \delta f \rangle \rightarrow -\nu \left( \langle f \rangle - \frac{n}{4\pi} \right)$$

Bhatnagar, Gross, Krook *Phys. Rev.* 94 (1954) 511

- diffusion equation for  $n$ :

$$\frac{\partial n}{\partial t} - \nabla_{\mathbf{x}} (K \nabla_{\mathbf{x}} n) = 0$$

- dipole from CR gradient:  $\Phi = -K \nabla_{\mathbf{x}} n$ ,  $C_1 = 4\pi |K \nabla n/n|^2$

# Ensemble averaging

- in standard diffusion, compute  $C_\ell$  from  $\langle f \rangle$ :

$$C_\ell^{\text{std}} = \frac{1}{4\pi} \int d\hat{\mathbf{p}}_1 \int d\hat{\mathbf{p}}_2 P_\ell(\hat{\mathbf{p}}_1 \hat{\mathbf{p}}_2) \langle f(\hat{\mathbf{p}}_1) \rangle \langle f(\hat{\mathbf{p}}_2) \rangle$$

- however, in an individual realisation of  $\delta B$ ,  $\delta f = f - \langle f \rangle \neq 0$

$$\langle C_\ell \rangle = \frac{1}{4\pi} \int d\hat{\mathbf{p}}_1 \int d\hat{\mathbf{p}}_2 P_\ell(\hat{\mathbf{p}}_1 \hat{\mathbf{p}}_2) \langle f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2) \rangle$$

- if  $f(\hat{\mathbf{p}}_1)$  and  $f(\hat{\mathbf{p}}_2)$  are correlated,

$$\langle f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2) \rangle \geq \langle f(\hat{\mathbf{p}}_1) \rangle \langle f(\hat{\mathbf{p}}_2) \rangle \quad \Rightarrow \quad \langle C_\ell \rangle \geq C_\ell^{\text{std}}$$

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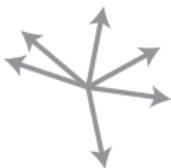
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Source of the small scale anisotropies?

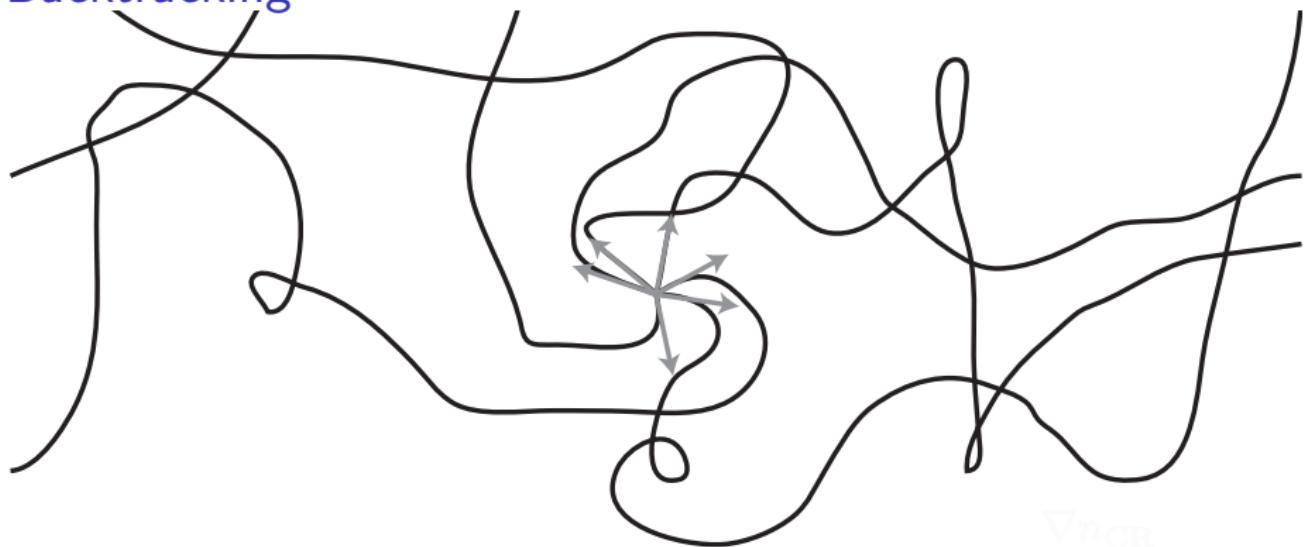
Giacinti & Sigl, *PRL* 109 (2012) 071101; Ahlers, *PRL* 112 (2014) 021101

# Backtracking



$$\nabla n_{\text{CB}}$$

# Backtracking



Navigation

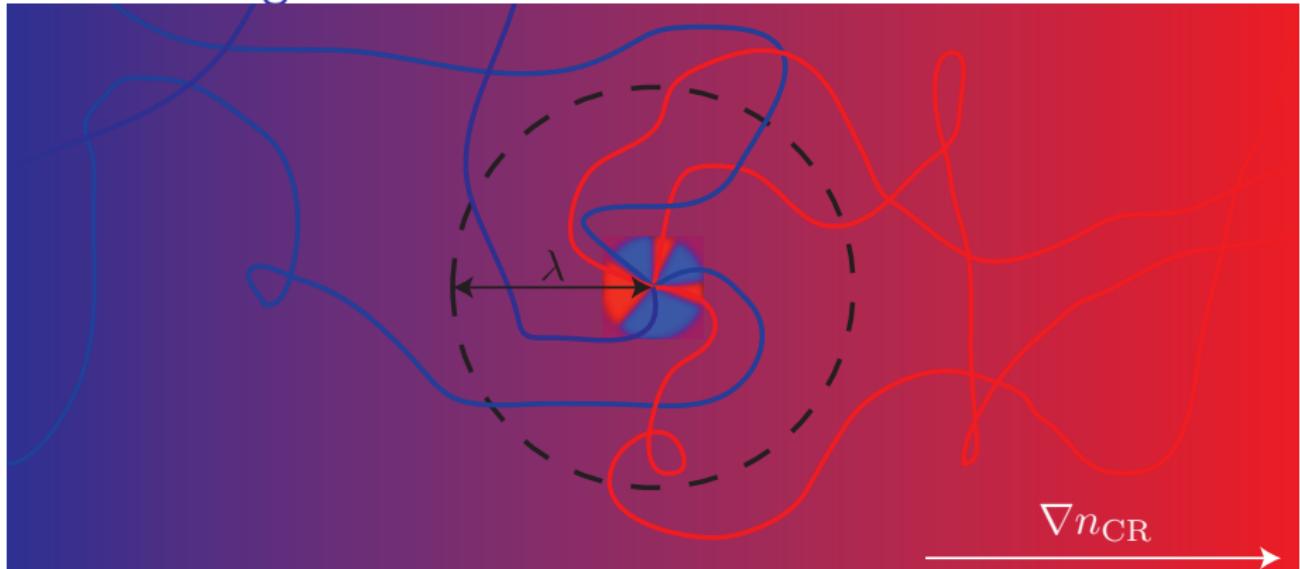
# Backtracking



- quasi-stationary distribution:

$$4\pi \langle f \rangle \simeq n + \mathbf{r} \nabla n - 3\hat{\mathbf{p}} \mathbf{K} \nabla n$$

# Backtracking



- quasi-stationary distribution:

$$4\pi\langle f \rangle \simeq n + \mathbf{r}\nabla n - 3\hat{\mathbf{p}}\mathbf{K}\nabla n$$

- Liouville's theorem

$$df = 0 \quad \Rightarrow \quad f(\mathbf{x} = 0, \hat{\mathbf{p}}_i(t = 0)) = f(\mathbf{x}_i(t = -T), \hat{\mathbf{p}}_i(t = -T))$$

# Backtracking

- quasi-stationary distribution:

$$4\pi \langle f \rangle \simeq n + \mathbf{r} \nabla n - 3\hat{\mathbf{p}} \mathbf{K} \nabla n$$

- the phase space density  $f$  at time  $t = 0$  depends on positions  $\mathbf{r}$  and velocities  $\hat{\mathbf{p}}$  at earlier time  $t = -T$

$$4\pi f \simeq 4\pi \delta f(-T) + n + (\mathbf{r}(-T) - 3\hat{\mathbf{p}}(-T)\mathbf{K}) \nabla n$$

- as before

$$\begin{aligned}\frac{1}{4\pi} \langle C_\ell \rangle &= \int d\hat{\mathbf{p}}_1 \int d\hat{\mathbf{p}}_2 P_\ell(\hat{\mathbf{p}}_1 \hat{\mathbf{p}}_2) \langle f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2) \rangle \\ &\simeq \int \frac{d\hat{\mathbf{p}}_1}{4\pi} \int \frac{d\hat{\mathbf{p}}_2}{4\pi} P_\ell(\hat{\mathbf{p}}_1 \hat{\mathbf{p}}_2) \lim_{T \rightarrow \infty} \langle r_{1i}(-T) r_{2j}(-T) \rangle \frac{\partial_i n \partial_j n}{n^2}\end{aligned}$$

# Relative diffusion

- as before:

$$\frac{1}{4\pi} \langle C_\ell \rangle \simeq \int \frac{d\hat{\mathbf{p}}_1}{4\pi} \int \frac{d\hat{\mathbf{p}}_2}{4\pi} P_\ell(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \lim_{T \rightarrow \infty} \langle r_{1i}(-T) r_{2j}(-T) \rangle \frac{\partial_i n \partial_j n}{n^2}$$

- variance from standard diffusion coefficient:

$$\frac{1}{4\pi} \sum_{\ell \geq 0} (2\ell + 1) \langle C_\ell \rangle \simeq \langle r_i(-T) r_j(-T) \rangle \frac{\partial_i n \partial_j n}{n^2} \simeq 2T K_{ij}^s \frac{\partial_i n \partial_j n}{n^2}$$

- monopole from difference of standard and *relative* diffusion coefficients:

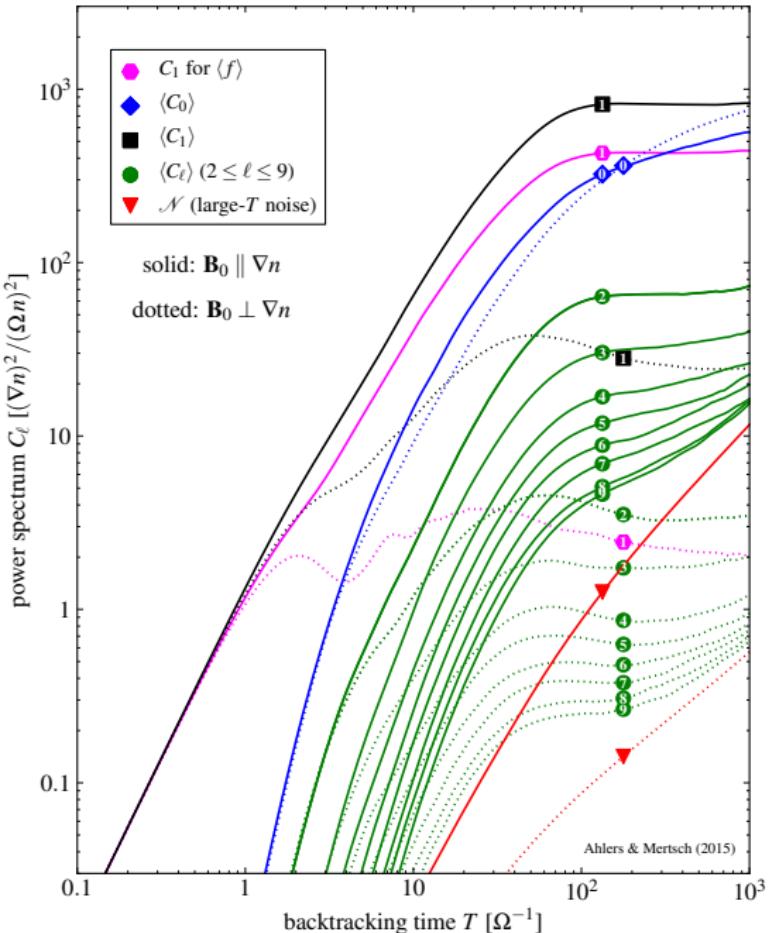
$$\frac{1}{4\pi} \langle C_0 \rangle \simeq 2T \left( K_{ij}^s - \tilde{K}_{ij}^s \right) \frac{\partial_i n \partial_j n}{n^2}$$

where  $\tilde{K}_{ij}^s = \int \frac{d\hat{\mathbf{p}}_1}{4\pi} \int \frac{d\hat{\mathbf{p}}_2}{4\pi} \lim_{T \rightarrow \infty} \frac{1}{4T} \langle \{r_{1i} - r_{2i}\} \{r_{1j} - r_{2j}\} \rangle$

- hence, all angular power for  $\ell \geq 1$  must be due to relative diffusion:

$$\frac{1}{4\pi} \sum_{\ell \geq 1} (2\ell + 1) \langle C_\ell \rangle(T) \simeq 2T \tilde{K}_{ij}^s \frac{\partial_i n \partial_j n}{n^2}$$

$$B_0^2 = \langle \delta B^2 \rangle, r_L/L_c = 0.1, \lambda_{\min}/L_c = 0.01, \lambda_{\max}/L_c = 100$$



## angular power spectrum of mean-subtracted map

- at early times, all moments increase; dipole  $\propto T^2$
  - later: asymptotic values
  - finite number of trajectories  
→ **shot noise**
  - variance =  $\sum_\ell (2\ell + 1) C_\ell \propto T$
  - relative difference between  $\mathbf{B}_0 \parallel \nabla n$  and  $\mathbf{B}_0 \perp \nabla n$
  - standard dipole  $C_1 < \langle C_1 \rangle$
  - non-vanishing monopole  $\langle C_0 \rangle$

# Generalised BGK–ansatz

- want to write down *local* ODE for  $C_\ell$ , so need

$$\partial_t \langle f_1 f_2 \rangle = \langle f_1 (-\hat{\mathbf{p}}_1 \nabla_{\mathbf{r}} + i\omega_1 \mathbf{L} + i\Omega_0 \mathbf{L}) f_2 \rangle + (1 \leftrightarrow 2) \quad (1)$$

- BGK–ansatz; drive  $\langle f \rangle$  to isotropic distribution  $n$ :

$$\langle i\omega \mathbf{L} \delta f \rangle \rightarrow -\nu \left( \langle f \rangle - \frac{n}{4\pi} \right)$$

- diffusion on the sphere where Laplacian is  $\nabla^2 = -(\nu/2)\mathbf{L}^2$

$$\langle i\omega \mathbf{L} \delta f \rangle \rightarrow -(\nu/2)\mathbf{L}^2$$

- we therefore make the ansatz

$$\langle (i\omega_1 \mathbf{L}_1 + i\omega_2 \mathbf{L}_2) f_1 f_2 \rangle \simeq - \left[ \nu_r(x) \frac{\mathbf{L}_1^2 + \mathbf{L}_2^2}{2} + \nu_c(x) \mathbf{J}^2 \right] \langle f_1 f_2 \rangle$$

with  $x = \hat{\mathbf{p}}_1 \hat{\mathbf{p}}_2$  and  $\mathbf{J} = \mathbf{L}_1 + \mathbf{L}_2$

# Generalised BGK–ansatz

- gradient term:

$$\langle f_1 \hat{\mathbf{p}}_2 \nabla f_2 \rangle \simeq -3/(4\pi)^2 \hat{\mathbf{p}}_1 \nabla n \hat{\mathbf{p}}_2 \mathbf{K} \nabla n$$

- steady-state solution:

$$K_{ij} \frac{\partial_i n \partial_j n}{6\pi} \delta_{\ell 1} = \sum_k \langle C_\ell \rangle k(k+1) \frac{2k+1}{2} \int dx \nu_r(x) P_\ell(x) P_k(x)$$

- depends on relative scattering rate  $\nu_r$  only:

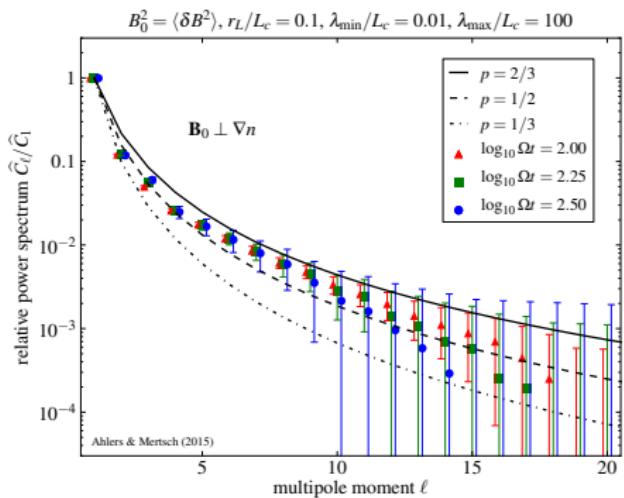
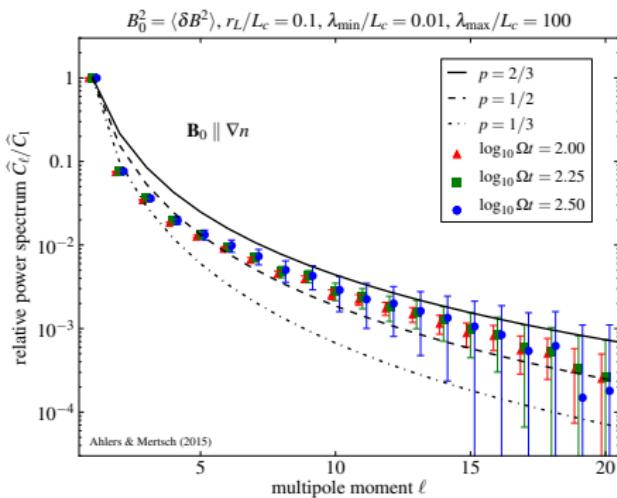
$$\langle C_\ell \rangle = \frac{3}{2} \frac{K_{ij} \partial_i n \partial_j n}{\ell(\ell+1)} \int_{-1}^1 dx \frac{x P_\ell(x)}{\nu_r(x)}$$

# Generalised BGK–ansatz

$$\langle C_\ell \rangle = \frac{3}{2} \frac{Q_1}{\ell(\ell+1)} \int_{-1}^1 dx \frac{x P_\ell(x)}{\nu_r(x)}$$

- ansatz for  $x$ –dependence:

$$\nu_r(x) \propto (1-x)^p$$



# Conclusion

- power in multipole  $C_\ell$  with  $\ell \geq 1$  due to *relative diffusion*
- for large backtracking times  $T$ , angular power spectrum reaches asymptotic value: effect of *local field*
- reproduce concave spectrum with generalised BGK–ansatz

The small-scale anisotropies encode information about the *actual* representation of the turbulent magnetic field in our Galactic neighbourhood

# Simulation: setup

Giacalone & Jokipii, *ApJ* 520 (1999) 204

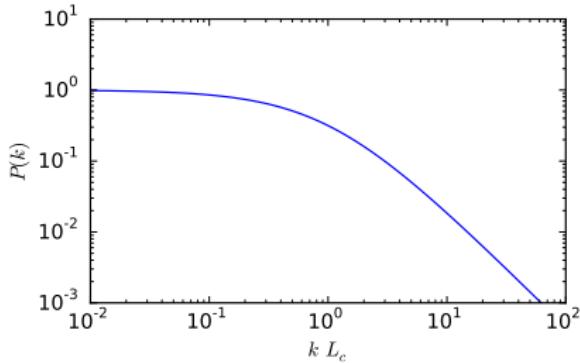
- compute turbulent field as sum of harmonics

$$\delta \mathbf{B}(\mathbf{r}) = \sum_{n=1}^N A(k_n) (\cos \alpha_n \mathbf{x}' + i \sin \alpha_n \mathbf{y}') \exp [ik_n z'_n + i\beta_n]$$

with random phases  $\alpha_n$ ,  $\beta_n$  and random rotations  $\mathbf{r}'_n = \mathcal{R}_n \mathbf{r}$

- $k_n$  are logarithmically spaced

- $A(k_n)$  chosen such that the power spectrum is of Kolmogorov-type with inertial range:



- backtrack for many Larmor times  $\Omega^{-1}$  and compute the  $C_\ell$  with HEALPix

# Simulation: results

- subtract shot noise from simulated  $C_\ell$
- error bars reflects shot noise (no cosmic variance)
- normalised to  $C_1$ , angular power spectra for  $\mathbf{B}_0 \parallel \nabla n$  and  $\mathbf{B}_0 \perp \nabla n$  look very similar

