Analytical Description of the Earth Matter Effect on Neutrino Oscillations

G. Medina-Tanco, D. Supanitsky and J.C.D
ICN-UNAM

DISCRETE'08, Valencia,Spain
December 2008

Introduction

In most oscillation experiments neutrinos propagate considerable distances in matter

An accurate description of matter neutrino oscillations is an important ingredient in the analysis of the data

For an arbitrary density profile the evolution equation for the flavor amplitudes admit no closed form solution

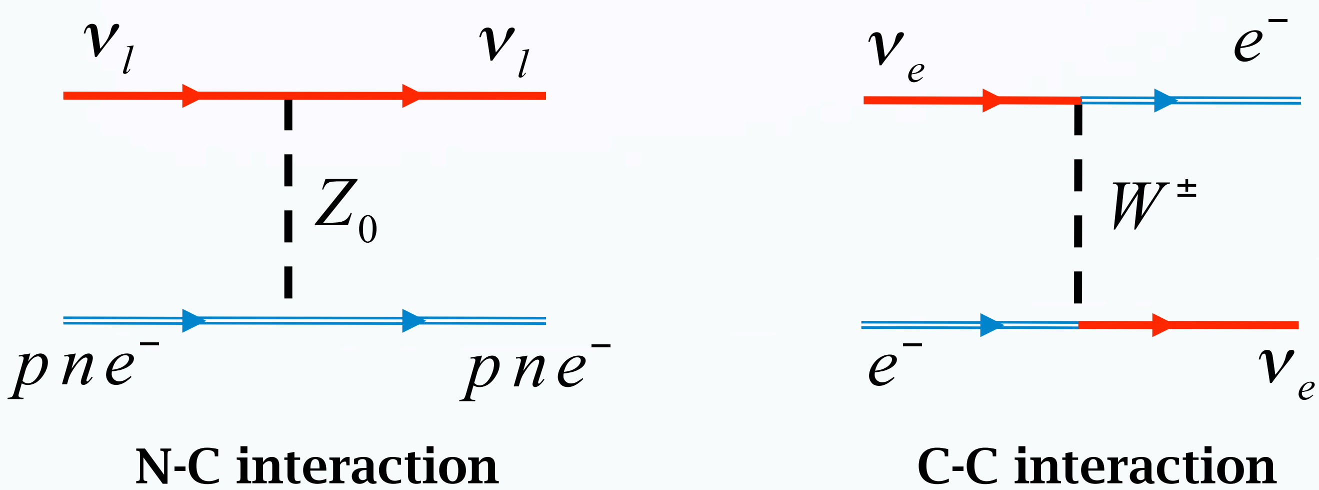
Numerical integrations have been extensively used to examine the phenomenon

Analytic solutions still useful:

- Significant insight into the physics of the problem
- Better understanding of the dependence of the oscillation on the neutrino parameters and the properties of the medium
- Help to save the CPU time

Neutrino Oscillations in Matter

Propagation of neutrinos in normal matter (*p*, *n*, *e*)



$$|\nu(t)\rangle = \zeta_e(t)\,|\nu_e\rangle + \zeta_a(t)\,|\nu_a\rangle \qquad \begin{pmatrix}\zeta_e(t)\\ \zeta_a(t)\end{pmatrix} = U(t,t_0)\begin{pmatrix}\zeta_e(t_0)\\ \zeta_a(t_0)\end{pmatrix}$$

$L \cong t - t_0$ **Distance from the production point**

Evolution Operator

$$i\hbar \frac{d\mathcal{U}}{dt}(t,t_0) = H(t)\,\mathcal{U}(t,t_0)$$

$$\mathcal{U}(t_0,t_0) = I$$

$$H(t) = \frac{\Delta_0}{2}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \frac{V(t)}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

θ mixing angle

$$\Delta_0 \equiv \delta m^2/2E = (m_i^2 - m_j^2)/2E$$

$$V(t) = \sqrt{2}G_F n_e(t)$$

Effective potential

$$P_{\nu_e} = |\langle \nu_e | \nu(t_f)\rangle|^2 = \langle \nu_e | \mathcal{U}(t_f,t_0) | \nu(t_0)\rangle|^2$$

Initial state $|\nu(t_0)\rangle = \alpha|\nu_e\rangle + \beta|\nu_a\rangle$

$$\mathcal{U}(t_f, t_0) = U_m(t_f)\, \mathcal{U}^{\mathcal{A}}(t_f, t_0)\, U_m^\dagger(t_0)$$

$\{|\nu_{1m}(t)\rangle, |\nu_{2m}(t)\rangle\}$ **Adiabatic basis (Instantaneous energy eigenstates)**

$$U_m(t) = \begin{pmatrix} \cos\theta_m(t) & \sin\theta_m(t) \\ -\sin\theta_m(t) & \cos\theta_m(t) \end{pmatrix}$$

$$\sin 2\theta_m(t) = \frac{\Delta_0 \sin 2\theta}{\Delta_m(t)}$$

$$H^{\mathcal{A}}(t) = H_D(t) - i\underbrace{U_m^\dagger(t)\dot{U}_m(t)}_{i\,\dot{\theta}_m(t)\,\sigma_y}$$

$$H_D(t) = -\frac{1}{2}\Delta_m(t)\sigma_z$$

$$\Delta_m(t) = \Delta_0 \sqrt{(\varepsilon(t) - \cos 2\theta)^2 + \sin^2 2\theta}$$

Difference between the energy eigenvalues

$$\varepsilon(t) = 2EV(t)/\delta m^2$$

$$\mathcal{U}^{\mathcal{A}}(t,t_0) = \mathcal{P}(t,t_0)\,\mathcal{U}^{\mathcal{A}}_{\mathcal{P}}(t,t_0)$$

$$\mathcal{P}(t,t_0) \;=\; \exp\left(-i\int_{t_0}^{t} dt'\,H_D(t')\right) = \begin{pmatrix} e^{-\frac{i}{2}\phi_{t_0\to t}} & 0 \\ 0 & e^{\frac{i}{2}\phi_{t_0\to t}} \end{pmatrix}$$

$$i\frac{d\mathcal{U}^{\mathcal{A}}_{\mathcal{P}}}{dt} = H^{\mathcal{A}}_{\mathcal{P}}(t)\,\mathcal{U}^{\mathcal{A}}_{\mathcal{P}}$$

$$\phi_{x\to y} = \int_{x}^{y} dt'\;\Delta_m(t')$$

$$H^{\mathcal{A}}_{\mathcal{P}}(t) = i\,\dot{\theta}_m(t)\begin{pmatrix} 0 & -e^{-i\phi_{t_0\to t}} \\ e^{i\phi_{t_0\to t}} & 0 \end{pmatrix}$$

Magnus Expansion

"True" exponential expansion of the evolution operator

$$\mathcal{U} = \exp(\Omega)$$

$$\Omega = \sum_{n=1}^{\infty} \Omega_n$$

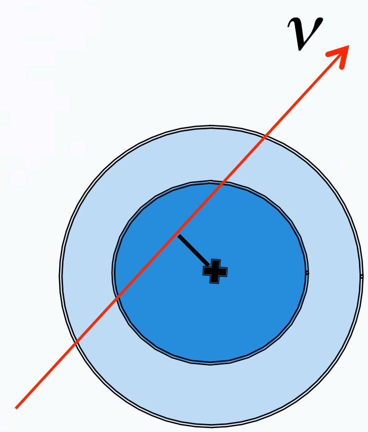
Ω_n **Anti-Hermitean** $\Longrightarrow$ **Unitarity preserved order by order**

$$\begin{aligned}
\Omega_1 &= -\frac{i}{\hbar} \int_{t_0}^{t} dt' H(t'), \\
\Omega_2 &= -\frac{1}{2\hbar^2} \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' [H(t'), H(t'')]
\end{aligned}$$

(S. Blanes, F. Casas, J. A. Oteo, and J. Ross, math-ph/0810.5488v1)

Potential symmetric with respect to $\bar{t} = (t_f + t_0)/2$

$$\theta_m(t_f) = \theta_m(t_0) \equiv \theta_m^0$$



$$\mathcal{U}(t_f, t_0) = U_m(t_0)\, \mathcal{U}^{\mathcal{A}}(t_f, t_0)\, U_m^\dagger(t_0)$$

Up to 2nd order $\Omega \cong \Omega_1 + \Omega_2$

$$\mathcal{U}^{\mathcal{A}}(t_f, t_0) \cong \begin{pmatrix} \left(\cos\xi - i\,\sin\xi \frac{\xi_{(2)}}{\xi}\right) e^{i\phi_{\bar{t}\to t_f}} & i\,\sin\xi \frac{\xi_{(1)}}{\xi} \\ i\,\sin\xi \frac{\xi_{(1)}}{\xi} & \left(\cos\xi + i\,\sin\xi \frac{\xi_{(2)}}{\xi}\right) e^{-i\phi_{\bar{t}\to t_f}} \end{pmatrix}$$

$$\xi = \sqrt{\xi_{(1)}^2 + \xi_{(2)}^2}$$

$$\begin{aligned} \xi_{(1)} &= 2\int_{\bar{t}}^{t_f} dt' \, \frac{d\theta_m}{dt'} \, \sin\phi_{\bar{t}\to t'}, \\ \xi_{(2)} &= \int_{t_0}^{t_f} dt' \int_{t_0}^{t'} dt' \, \frac{d\theta_m}{dt'} \, \frac{d\theta_m}{dt''} \, \sin\phi_{t'\to t''} \end{aligned}$$

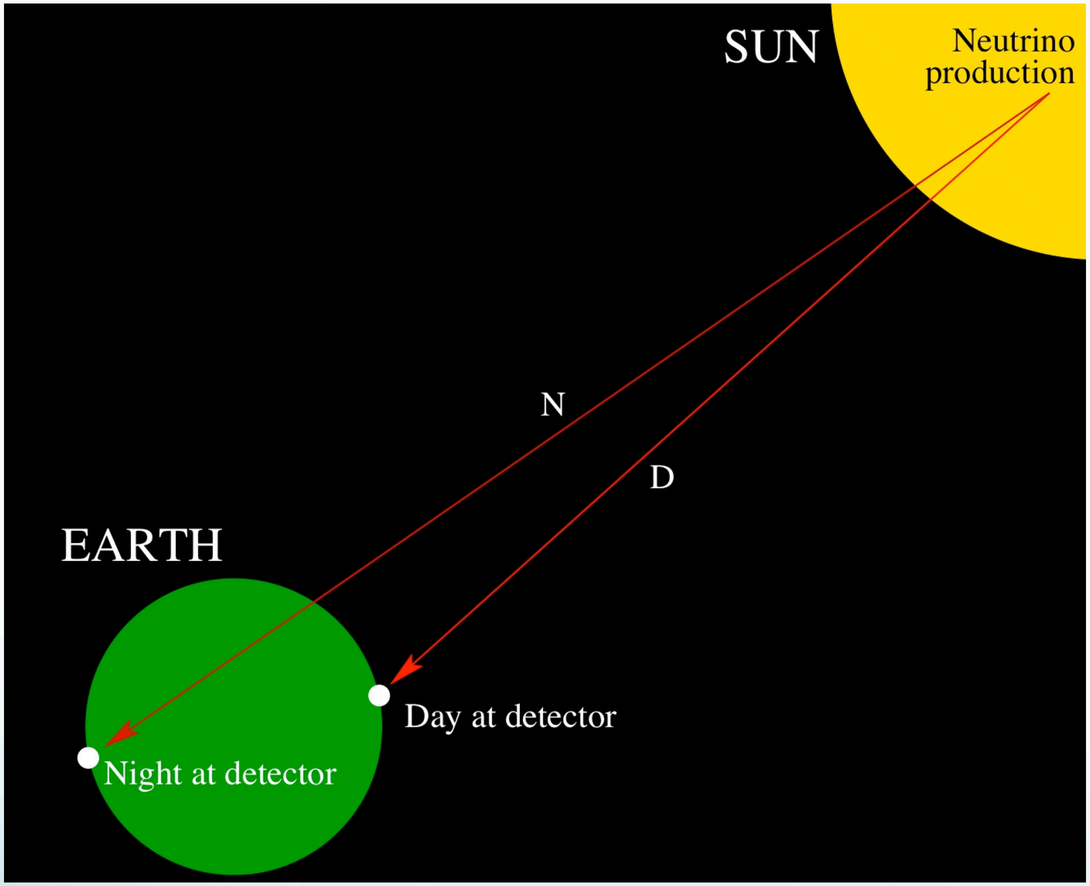
$$P_{\nu_e} = \alpha^2 + (\beta^2 - \alpha^2)(\mathrm{Im}\,\mathcal{U}_{ea})^2 + 2\alpha\beta\,(\mathrm{Im}\,\mathcal{U}_{ee})\,(\mathrm{Im}\,\mathcal{U}_{ea}),$$

$$\begin{aligned}
\mathrm{Im}\,\mathcal{U}_{ee} &= \cos 2\theta_m^0\,\mathrm{Im}\,\mathcal{U}_{11}^{\mathcal{A}} + \sin 2\theta_m^0\,\mathrm{Im}\,\mathcal{U}_{12}^{\mathcal{A}}\,, \\
\mathrm{Im}\,\mathcal{U}_{ea} &= -\sin 2\theta_m^0\,\mathrm{Im}\,\mathcal{U}_{11}^{\mathcal{A}} + \cos 2\theta_m^0\,\mathrm{Im}\,\mathcal{U}_{12}^{\mathcal{A}}\,,
\end{aligned}$$

(α and β real numbers)

$$\begin{aligned}
\mathrm{Im}\,\mathcal{U}_{11}^{\mathcal{A}} &= \cos\xi\,\sin\phi_{\bar{t}\to t_f} - \sin\xi\,\frac{\xi_{(2)}}{\xi}\cos\phi_{\bar{t}\to t_f} \\
\mathrm{Im}\,\mathcal{U}_{12}^{\mathcal{A}} &= \sin\xi\,\frac{\xi_{(1)}}{\xi}\,.
\end{aligned}$$

Day-Night Asymmetry



At night, neutrinos coming from the Sun reach the detector after they propagate through the Earth.

$$A_{DN} = 2\frac{D-N}{D+N}$$

D(N): neutrino rate at day (night)

Key (still no established) signature of the LMA solution

Solar neutrinos parameters:

$$\delta m^2_{21} = 8 \times 10^{-5}\ \mathrm{eV}^2$$

$$(\delta m^2 = \delta m^2_{12},\ \theta = \theta_{12})$$

$$\tan^2\theta_{12} = 0.4$$

- Adiabatic transformation inside the Sun
- Lose of coherence in the propagation from the Sun to the Earth

$$\overline{P}(\nu_e \to \nu_e) = \sin^2\theta + \cos 2\theta \cos^2\theta^0_\odot - \cos 2\theta^0_\odot\, f_{reg}$$

$\theta^0_\odot$ Matter mixing angle at the production point in the solar core

$$f_{reg} = P_{2e} - \sin^2\theta$$

$$|\langle \nu_e | \mathcal{U}(t_f, t_0) | \nu_2 \rangle|^2 \qquad |\langle \nu_e | \nu_2 \rangle|^2$$

$$|\nu(t_0)\rangle = |\nu_2\rangle = \sin\theta|\nu_e\rangle + \cos\theta|\nu_\mu\rangle$$

$$f_{\mathrm{reg}} = (\mathrm{Im}\,U_{12})(\mathrm{Im}\,U_{e\mu})$$

$$\begin{aligned} f_{reg} \;=\;& \cos 2\tilde{\theta}_m^0 \cos 2\theta_m^0 (\mathrm{Im}\,\mathcal{U}_{12}^{\mathcal{A}})^2 + \sin 2\tilde{\theta}_m^0 \sin 2\theta_m^0 (\mathrm{Im}\,\mathcal{U}_{11}^{\mathcal{A}})^2 \\ & - \sin(2\tilde{\theta}_m^0 + 2\theta_m^0)(\mathrm{Im}\,\mathcal{U}_{12}^{\mathcal{A}})(\mathrm{Im}\,\mathcal{U}_{12}^{\mathcal{A}})\,. \end{aligned}$$

$\tilde{\theta}_m^0 = \theta_m^0 - \theta$ **Mixing between** $\{|\nu_1\rangle, |\nu_2\rangle\}$ **and** $\{|\nu_{1m}(t_0)\rangle, |\nu_{1m}(t_0)\rangle\}$

For a constant potential we recover the exact result

$$f_{reg} = \varepsilon_0 \sin^2 2\theta_m^0 \sin^2\left[\frac{\Delta_m}{2}(t_f - t_0)\right]$$

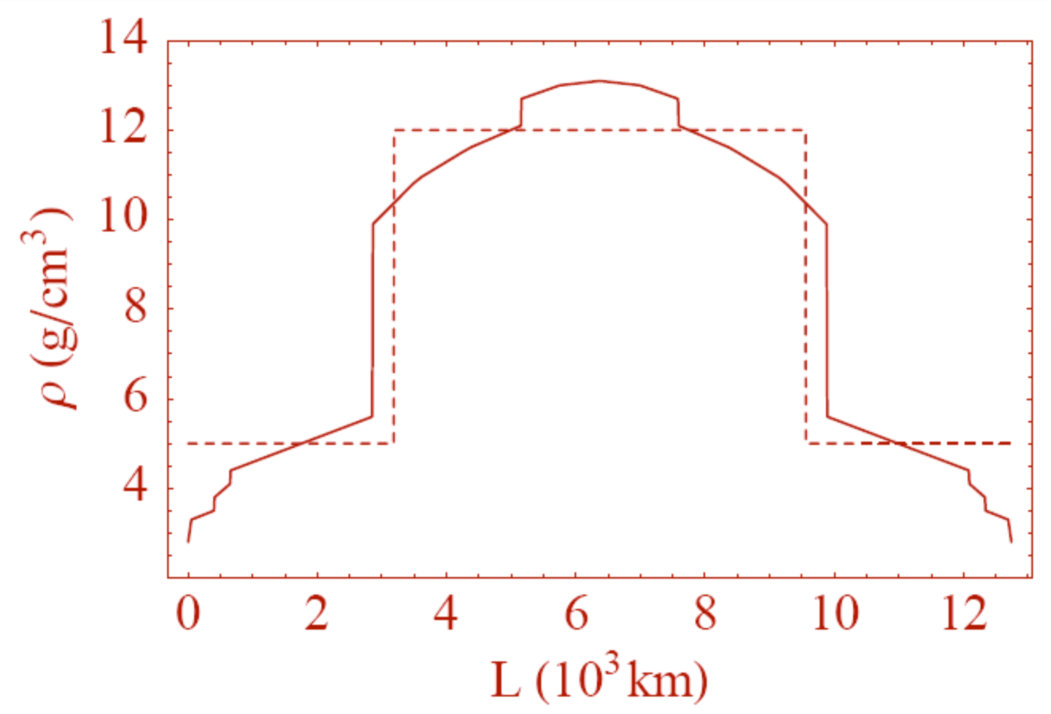
$\varepsilon << 1$

$$f_{reg} \cong \frac{1}{2}\sin 2I \sin 2\theta \sin\phi_{\bar{t}\to t_f} + \sin^2 I \cos 2\theta$$

$$I = \sin 2\theta \int_{\bar{t}}^{t_f} dt' V(t') \cos\phi_{\bar{t}\to t'}$$

(D.S., G.M.T.&J.C.D., astro-ph/0708.0629)

Earth Potential



Mantle-Core-Mantle Model

$$n_e(r) = N_A \begin{cases} 5.95 \text{ cm}^{-3}, & r \le R_\oplus/2 \\ 2.48 \text{ cm}^{-3}, & R_\oplus/2 < r \le R_\oplus \end{cases}$$

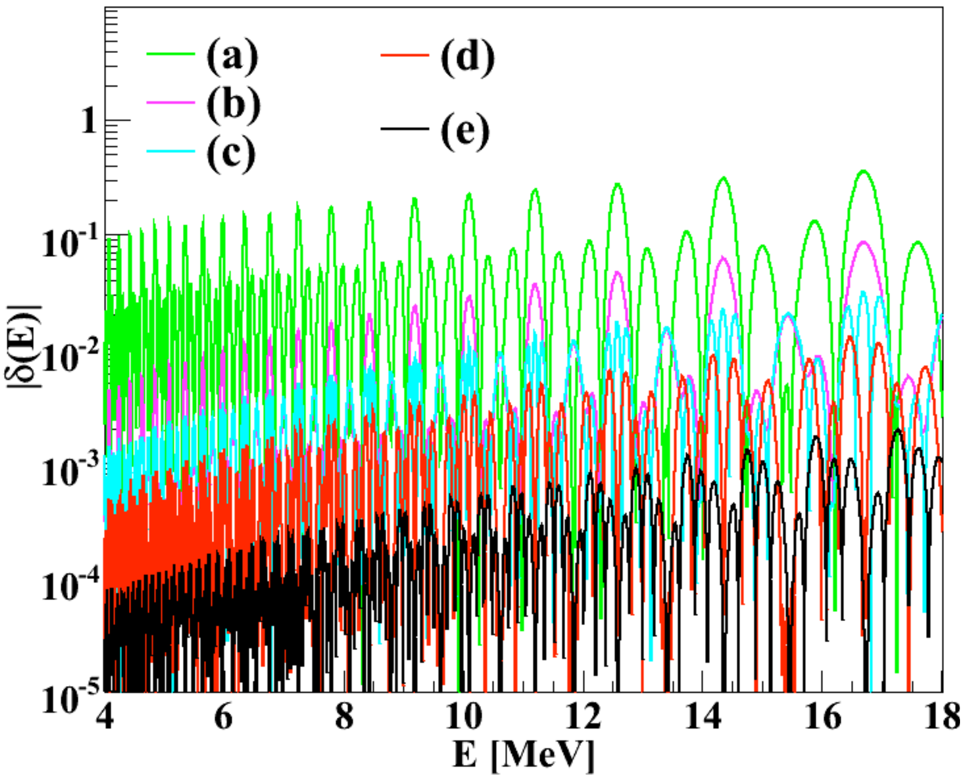
A. N. Ioannisian et al., Phys. Rev. D71, 0033006 (2005).

$$\delta(E) = \frac{1}{\bar{f}_{reg}(E)} \left[f_{reg}^{(appr)}(E) - f_{reg}^{(exact)}(E) \right]$$

Relative Error

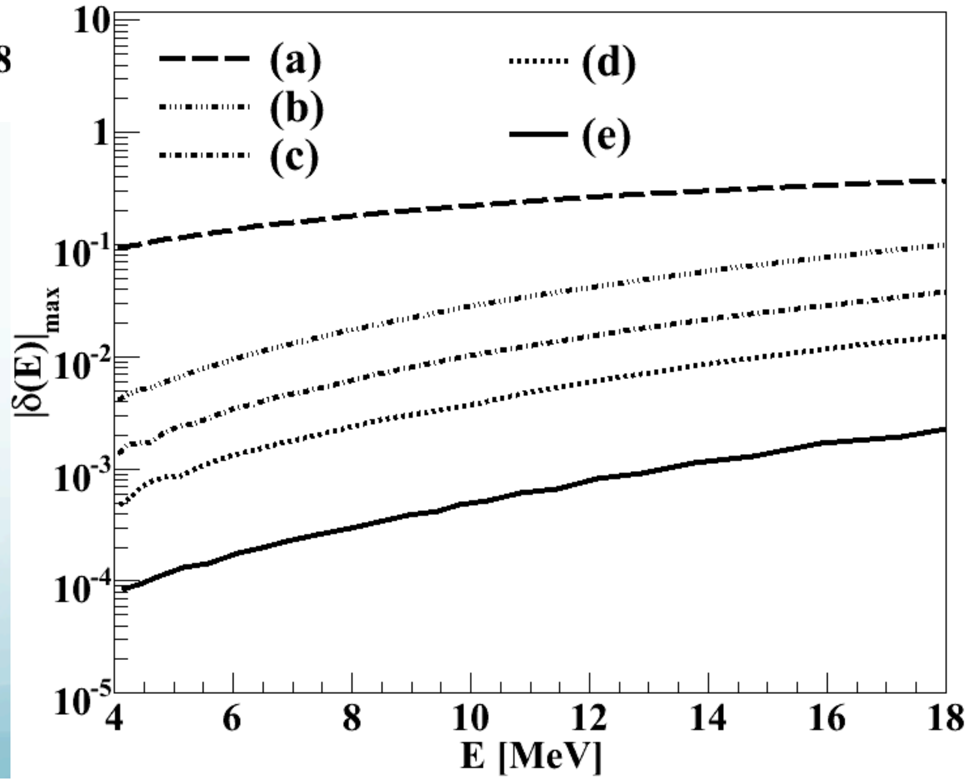
$$\bar{f}_{reg}(E) = 1/2\, \varepsilon_0 \sin^2\theta$$

Average regeneration factor at thesurface layer

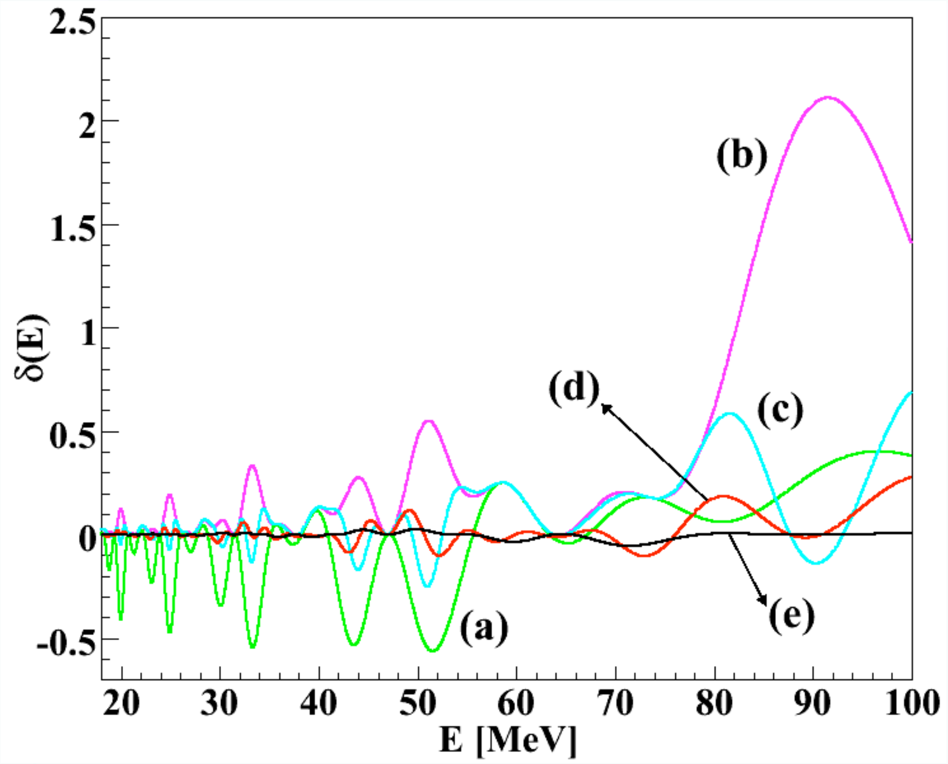


Neutrinos that cross the Earth through its center ($\Theta = 0$)

(a) Perturbation O(1)
(b) Perturbation O(2)
(c) Intermediate formula
(d) Magnus O(1)
(e) Magnus O(2)



Energy range relevant for solar neutrinos



Neutrino that goes through the Earth passing by its center, $\Theta = 0$.

(a) Perturbation O(1)
(b) Perturbation O(2)
(c) Intermediate formula
(d) Magnus O(1)
(e) Magnus O(2)

Energy range relevant for supernova neutrinos

High Energy Neutrinos

$\mathrm{E} \gtrsim 1 \ \mathrm{GeV}$

$\{|\nu_e\rangle, |\nu_a\rangle\}$

$\theta = \theta_{31}$

$$|\nu_a\rangle = \sin\theta_{23}\,|\nu_\mu\rangle + \cos\theta_{23}\,|\nu_\tau\rangle$$

$\delta m^2 = \delta m^2_{31}$

$$P(\nu_\mu \to \nu_e) = \sin^2\theta_{23}\, P(\nu_a \to \nu_e)$$

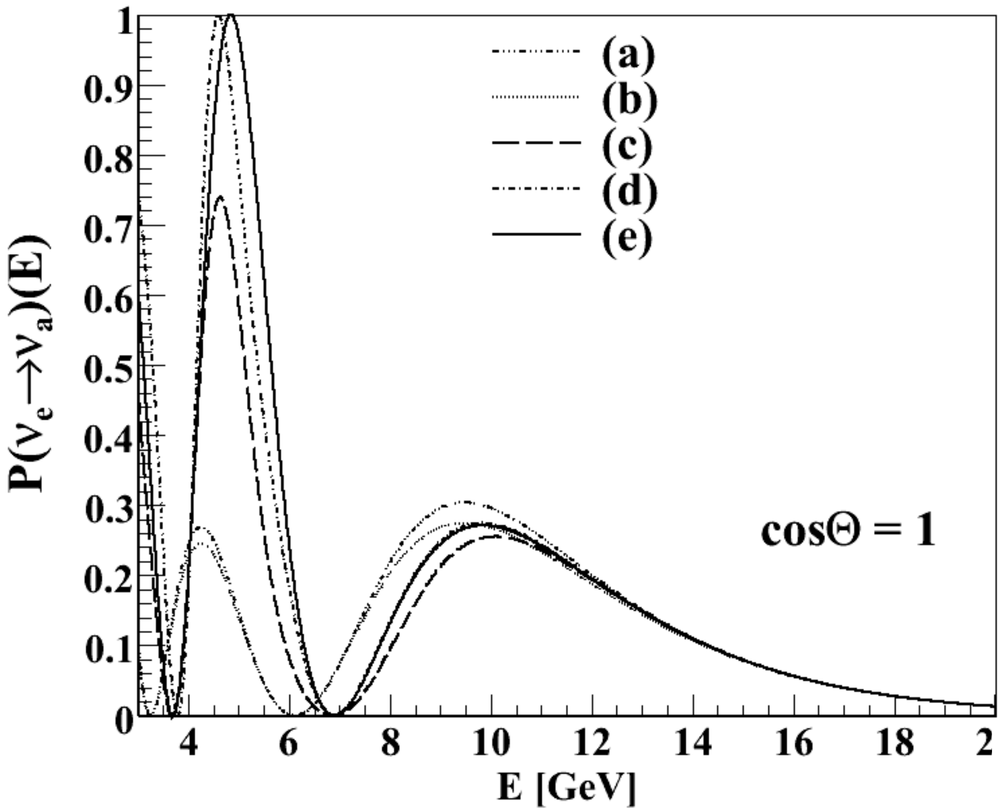
$$P(\nu_\tau \to \nu_e) = \cos^2\theta_{23}\, P(\nu_a \to \nu_e)$$

$$\begin{aligned} P(\nu_a \to \nu_e) &= (\mathrm{Im}\,\mathcal{U}_{ae})^2 \\ &= (\cos 2\theta_m^0\, \mathrm{Im}\,\mathcal{U}_{12}^{\mathcal{A}} - \sin 2\theta_m^0\, \mathrm{Im}\,\mathcal{U}_{11}^{\mathcal{A}})^2 \end{aligned}$$

$V \gg \Delta_0$ $\quad \varepsilon \gg 1$

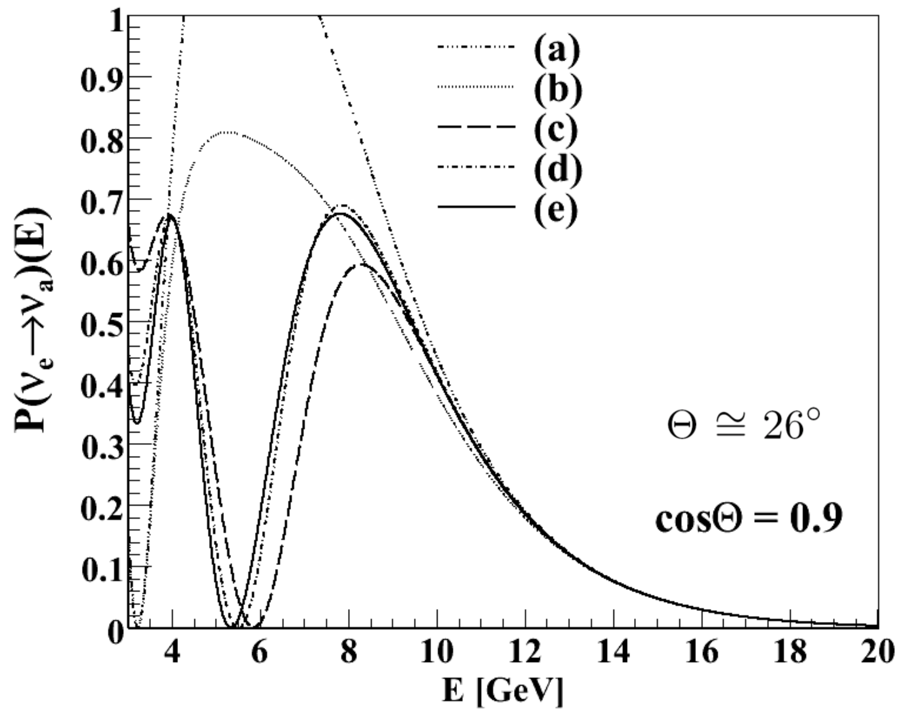
$$P(\nu_a \to \nu_e) = \left[\sin\left(\Delta_0 \sin 2\theta \int_{\bar{t}}^{t_f} dt' \, \cos\phi_{\bar{t}\to t'}\right)\right]^2$$

$\sin x \cong x$ **The perturbative result is recovered**



$\delta m^2_{31} = 2.5 \times 10^{-3}\ \mathrm{eV}^2$

$\theta_{13} = 10^\circ$



(a) Perturbation approach
(b) Intermediate formula
(c) Magnus O(1)
(d) Magnus O(2)
(e) Numerical calculation

Conclusions

•The Magnus expansion for the evolution operator (implemented in the adiabatic basis) provides an efficient formalism to describe neutrino oscillations in a medium with an arbitrary density profile.

•This method takes properly into account the Earth matter effects on the transition probabilities for neutrinos with a wide interval of energies, making possible a simple (and accurate) description of such effects in the case of solar and atmospheric neutrinos.

•The same formalism can be applied to the study of other situations of physical interest (for example, long baseline experiments).

D.S, G.M.T & J.C.D., Phys. Rev. D78, 045024 (2008)
A. N. Ioannisian and A. Yu. Smirnov, hep-ph/0803.1967