Adiabatic Parasites
&
Deformed Quantization

Chryssomalis Chryssomalakos

Instituto de Ciencias Nucleares
UNAM

Joint work with P. Aguilar and H. Hernandez (ICN)

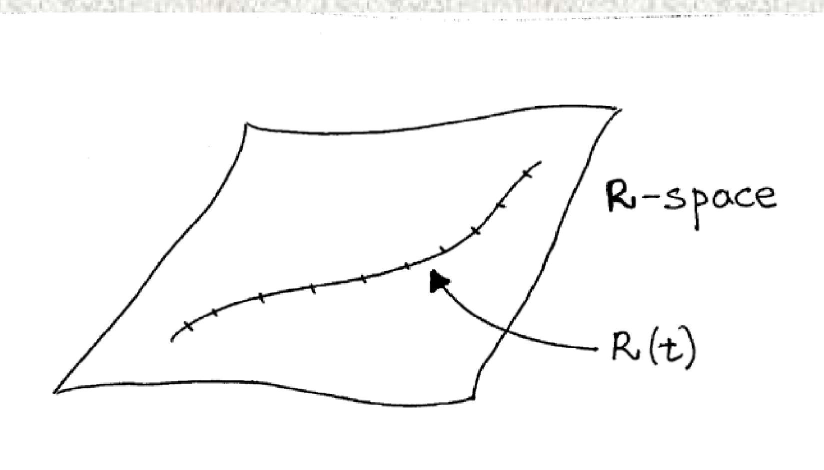
Geometric Phases

H(**R**), **R**: parameters
R(t): slow curve in parameter space

Isolated, non-degenerate eigenstate:

$$H(\mathbf{R})\,|n;\mathbf{R}\rangle = E_n(\mathbf{R})\,|n;\mathbf{R}\rangle$$

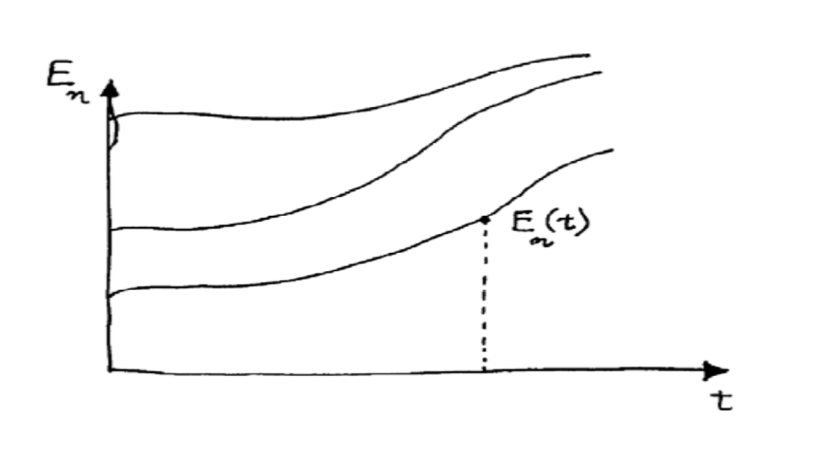
$$\langle n;\mathbf{R}|n;\mathbf{R}\rangle = 1$$



SE: $i\, d\psi(t)/dt = H(\mathbf{R}(t))\,\psi(t)$

$$\psi(t=0) = |n;\mathbf{R}(t=0)\rangle$$

$$\psi(t) \sim |n;\mathbf{R}(t)\rangle \quad \text{(adiabaticity)}$$



Geometric Phases II

Naive guess

$$|\psi(t)\rangle = \exp\left(-\mathrm{i}\int_0^t \mathrm{d}s E_n(\boldsymbol{R}(s))\right)|n, \boldsymbol{R}(t)\rangle$$

where $H(\boldsymbol{R}(t))|n, \boldsymbol{R}(t)\rangle = E_n(\boldsymbol{R}(t))|n, \boldsymbol{R}(t)\rangle$ **does not work..**

Berry tried…

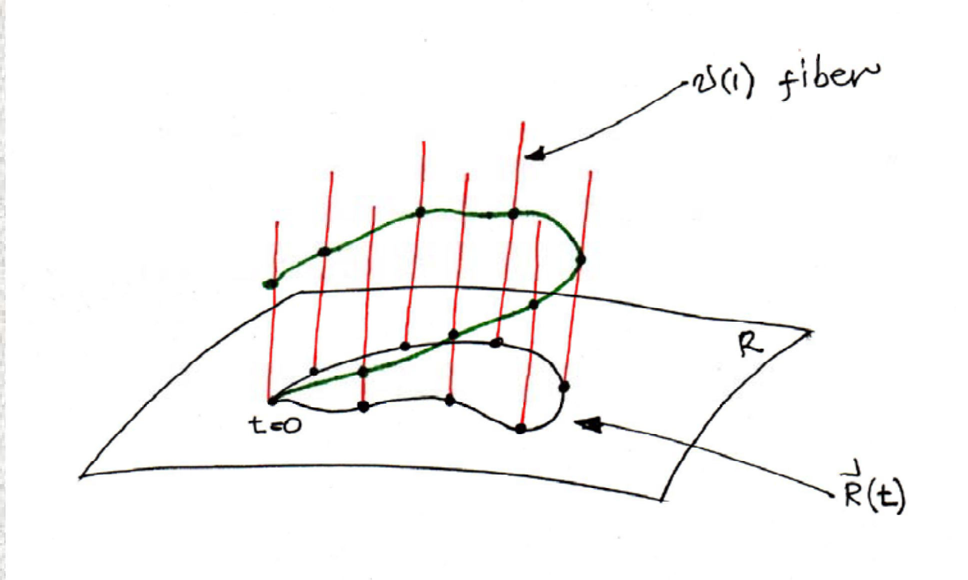
$$|\psi(t)\rangle = \exp\left(\mathrm{i}\gamma_n(t) - \mathrm{i}\int_0^t E_n(\boldsymbol{R}(s))\,\mathrm{d}s\right)|n, \boldsymbol{R}(t)\rangle$$

…and found

$$\gamma_n(t) = \mathrm{i}\int_0^t \left\langle n, \boldsymbol{R}(s)\left|\frac{\mathrm{d}}{\mathrm{d}s}\right|n, \boldsymbol{R}(s)\right\rangle \mathrm{d}s$$

Geometric Phases III

For a closed loop in R-space,
γ may be nonzero - U(1) holonomy
of Berry's (Simon's) connection



$$\mathcal{A}_\mu(\boldsymbol{R}) = \left\langle \boldsymbol{R} \left| \frac{\partial}{\partial R^\mu} \right| \boldsymbol{R} \right\rangle.$$

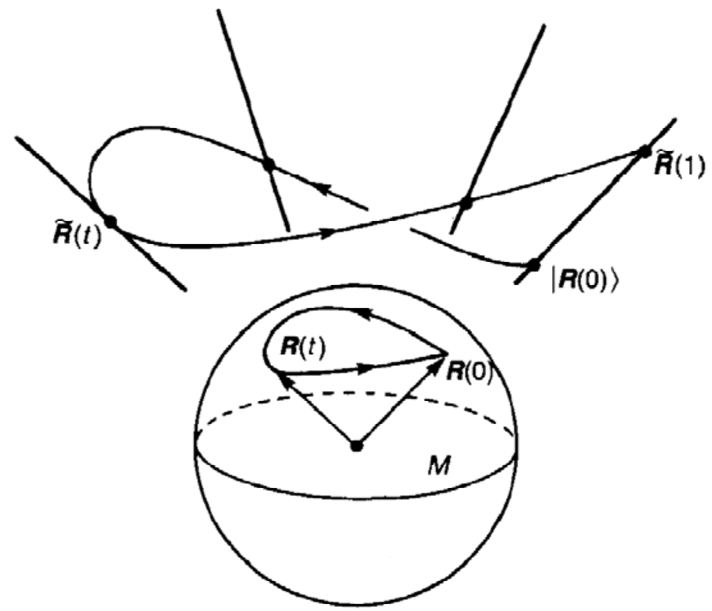
with curvature

$$\mathcal{F} = \mathrm{d}\mathcal{A} = (\mathrm{d}\langle \boldsymbol{R}|) \wedge (\mathrm{d}|\boldsymbol{R}\rangle) = \left(\frac{\partial \langle \boldsymbol{R}|}{\partial R^\mu}\right)\left(\frac{\partial |\boldsymbol{R}\rangle}{\partial R^\nu}\right) \mathrm{d}R^\mu \wedge \mathrm{d}R^\nu.$$

Geometric Phases IV

A simple example: spin ½ in **R**

$$H(\boldsymbol{R}) = \boldsymbol{R}\cdot\boldsymbol{\sigma} = \begin{pmatrix} R_3 & R_1 - \mathrm{i}R_2 \\ R_1 + \mathrm{i}R_2 & -R_3 \end{pmatrix}.$$



Eigenstate 'up'

$$|\boldsymbol{R}\rangle_{\mathrm{N}} = [2R(R + R_3)]^{-1/2}\begin{pmatrix} R + R_3 \\ R_1 + \mathrm{i}R_2 \end{pmatrix}.$$

Berry's connection

$$\mathcal{A}_{\mathrm{N}} = {}_{\mathrm{N}}\langle \boldsymbol{R}|\mathrm{d}|\boldsymbol{R}\rangle_{\mathrm{N}} = -\mathrm{i}\,\frac{R_2\,\mathrm{d}R_1 - R_1\,\mathrm{d}R_2}{2R(R + R_3)}.$$

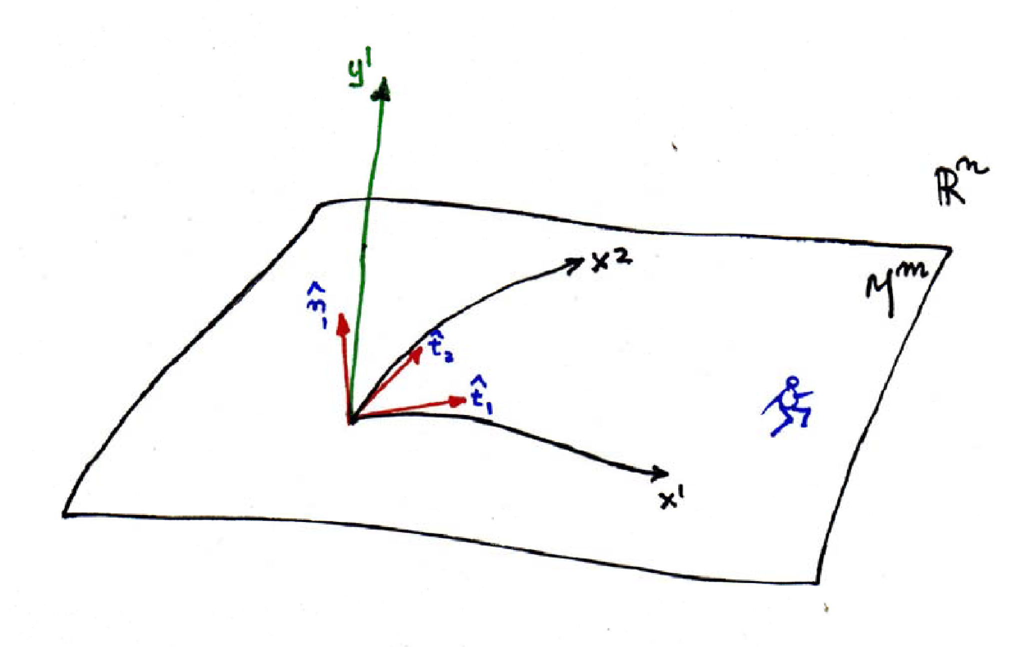
i.e., phase ~ solid angle

Berry's curvature

$$\mathcal{F} = \mathrm{d}\mathcal{A} = \frac{\mathrm{i}}{2}\,\frac{R_1\,\mathrm{d}R_2 \wedge \mathrm{d}R_3 + R_2\,\mathrm{d}R_3 \wedge \mathrm{d}R_1 + R_3\,\mathrm{d}R_1 \wedge \mathrm{d}R_2}{R^3}.$$

QM on Hypersurfaces

- Intrinsic quantization: coordinates on M, ignore ambient space (unphysical)
- Confining potential: $H_E = -\frac{1}{2}\partial_A^E \partial_A^E + V$ with $V \equiv V(y)$



Frenet-Serret:

$$\partial_\mu \hat{\mathbf{n}}^i = -\alpha_\mu^{i\nu} \mathbf{t}_\nu - A_\mu^{ij} \hat{\mathbf{n}}^j$$
$$\partial_\mu \mathbf{t}_\nu = \Gamma^\rho_{\mu\nu} \mathbf{t}_\rho + \alpha^i_{\mu\nu} \hat{\mathbf{n}}^i$$

$$g_{\mu\nu} = \mathbf{t}_\mu \cdot \mathbf{t}_\nu$$
$$\alpha^i_{\mu\nu} = \mathbf{t}_\mu \cdot \partial_\nu \hat{\mathbf{n}}^i$$
$$A^{ij}_\mu = \hat{\mathbf{n}}^i \cdot \partial_\mu \hat{\mathbf{n}}^j$$

Metric:

$$G_{AB} = \begin{pmatrix} \gamma_{\mu\nu} + y^k y^l A_\mu^{kh} A_\nu^{lh} & y^k A_\mu^{jk} \\ y^k A_\nu^{ik} & \delta^{ij} \end{pmatrix}, \quad \gamma_{\mu\nu} = g_{\mu\nu} - 2y^k \alpha^k_{\mu\nu} + y^k y^l \alpha^k_{\mu\rho} g^{\rho\sigma} \alpha^l_{\sigma\nu}.$$

QM on Hypersurfaces II

Total (3D) hamiltonian

$$H = -\frac{1}{2|\gamma|^{1/4}}\partial_i|\gamma|^{1/2}\partial_i\frac{1}{|\gamma|^{1/4}} - \frac{1}{2|g|^{1/4}|\gamma|^{1/4}}\Big(\partial_\mu\lambda^{\mu\nu}|\gamma|^{1/2}\partial_\nu + y^k y^l A_\mu^{ik}A_\nu^{jl}\partial_i\lambda^{\mu\nu}|\gamma|^{1/2}\partial_j + \partial_\mu\lambda^{\mu\rho}y^k A_\rho^{kj}|\gamma|^{1/2}\partial_j + \partial_i\lambda^{\nu\rho}y^k A_\rho^{ki}|\gamma|^{1/2}\partial_\nu\Big)\frac{|g|^{1/4}}{|\gamma|^{1/4}} + V(y).$$

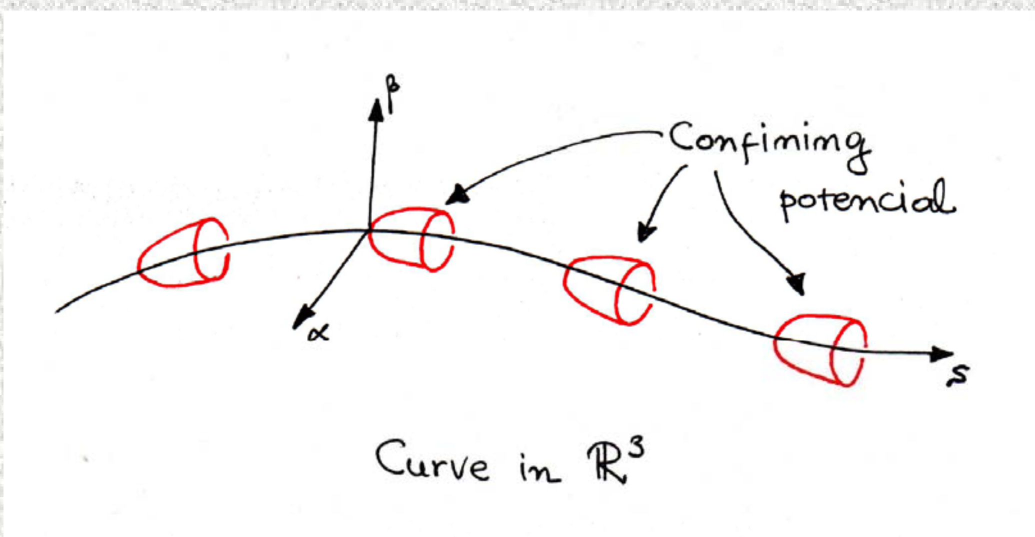
giving rise to normal & tangent SE's

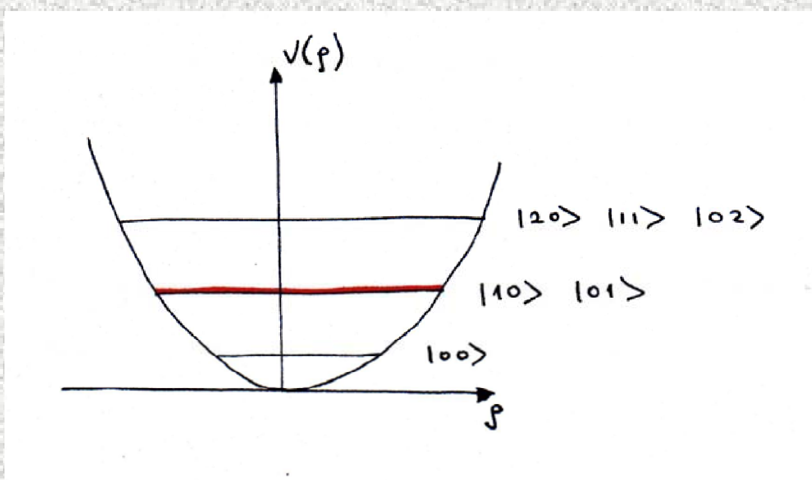
$$\hat{H}_0 = \tfrac{1}{2}\left(-\partial_i\partial_i + \omega^{i2}y^{i2}\right)$$ (harmonic oscillator)

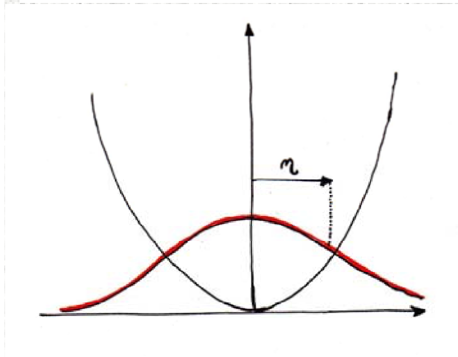
$$\hat{H} = -\frac{1}{2g^{1/2}}\Big(\partial_\mu + \frac{i}{2}A_\mu^{ij}L_{ij}\Big)g^{\mu\nu}g^{1/2}\Big(\partial_\nu + \frac{i}{2}A_\nu^{kl}L_{kl}\Big) + \tfrac{1}{8}g^{\mu\nu}g^{\rho\sigma}\left(\alpha^i_{\mu\nu}\alpha^i_{\rho\sigma} - 2\alpha^i_{\mu\rho}\alpha^i_{\nu\sigma}\right).$$

Effective hamiltonian for motion on M (Maraner, Destri 93)

Cables with quantum memory







3D — normal — on M

$$\eta^2 H = \hat{H}_0 + \eta^2 \hat{H} + \mathcal{O}(\eta^3)$$

Choose: n, b frame, |+>, |-> states

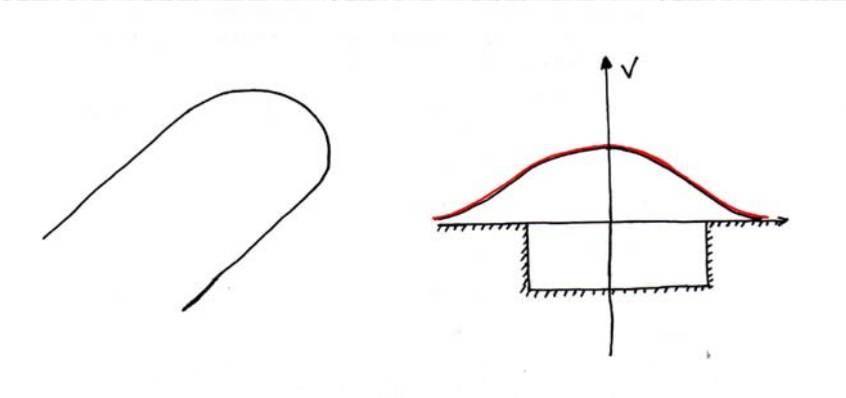
$$\mathbf{H} = \begin{pmatrix} \mathcal{H}_+ & 0 \\ 0 & \mathcal{H}_- \end{pmatrix}$$

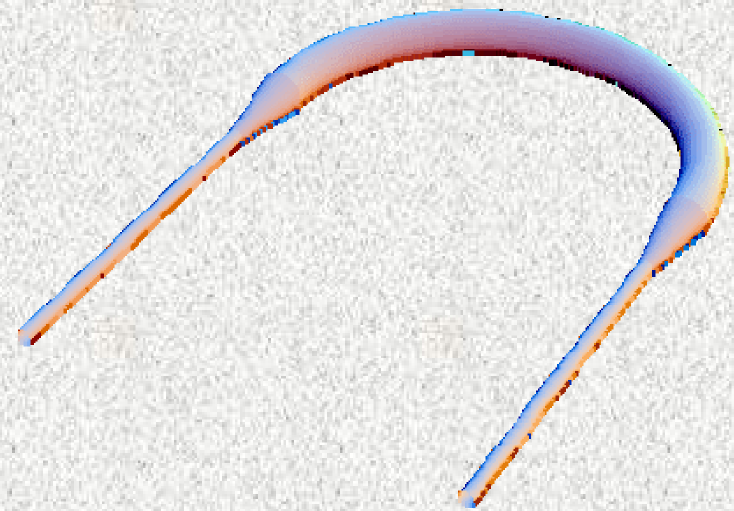
$$\mathcal{H}_\pm = -\frac{1}{2}\partial_s{}^2 \pm i\,\tau\partial_s \pm \frac{i}{2}\dot{\tau} + \frac{1}{2}\tau^2 - \frac{1}{8}\kappa^2$$

Cables with quantum memory II

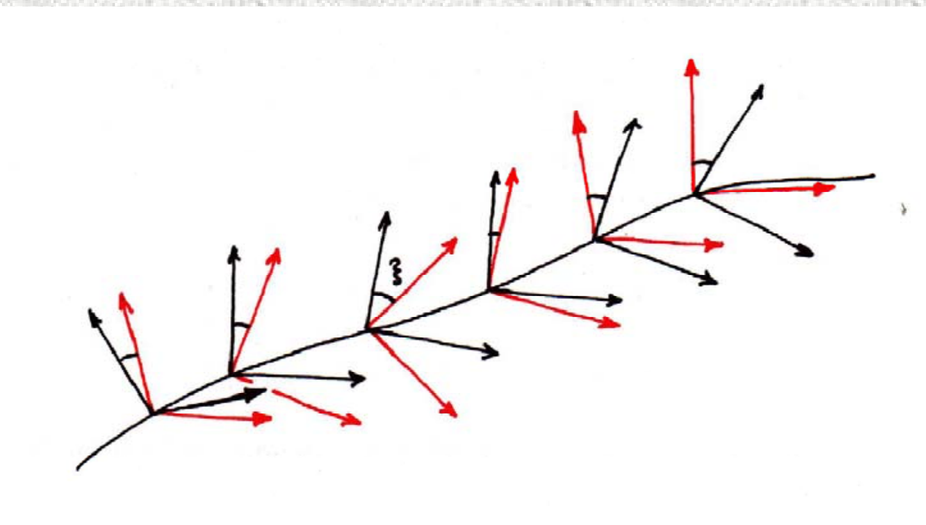
Features of the effective 1D hamiltonian

• Curvafilia $V_{\text{eff}}(s) = -\frac{1}{8}\kappa(s)^2$





• Induced gauge field

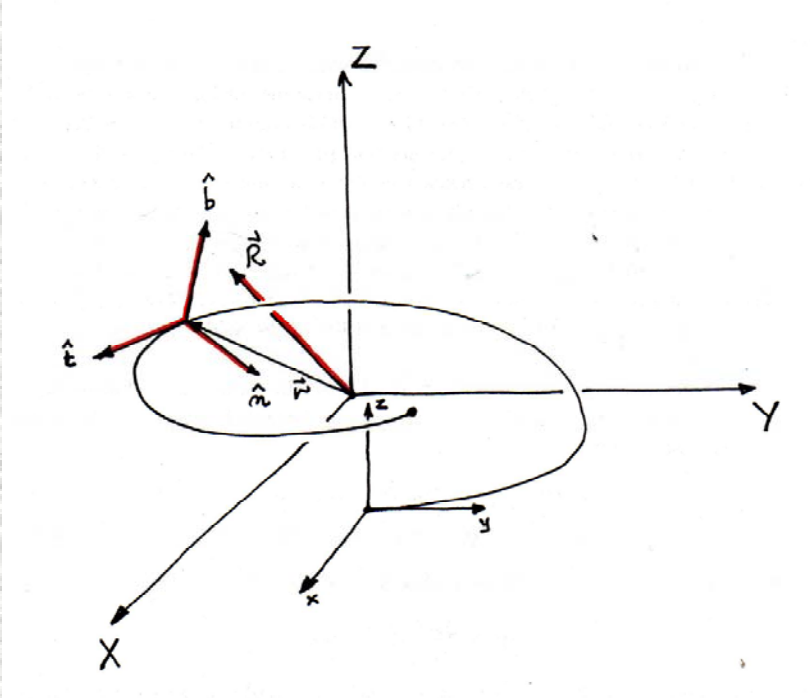


$$\begin{aligned} \psi(s) &\rightarrow e^{i\xi(s)}\psi(s) \\ \tau(s) &\rightarrow \tau(s) + \dot{\xi}(s) \\ (\partial_s - i\tau(s))\psi(s) &\rightarrow e^{i\xi(s)}(\partial_s - i\tau(s))\psi(s) \end{aligned}$$

Cables with quantum memory III

Pre-curve: $\vec{r}(s) = (\cos(s) + \epsilon x s, \sin(s) + \epsilon y s, \epsilon z s)$

Length 2π
Arclength parametrization



$$\vec{R} = \vec{r} + \eta\alpha\hat{n} + \eta\beta\hat{b}$$

$$\begin{aligned}
X(s,\alpha,\beta) &= (1-\eta\alpha)\cos(s) + \frac{\epsilon}{2}\Big(2xs + y(1-\cos(2s)) - 2x\sin(s) + x\sin(2s) \\
&\quad + \eta\big(-2y\alpha(1-\cos(2s)) + 2x\alpha(\sin(s)-\sin(2s)) + 2z\beta\sin(s)\big)\Big), \\
Y(s,\alpha,\beta) &= (1-\eta\alpha)\sin(s) + \epsilon\Big(ys + x\cos(s)(1-\cos(s)) - y\cos(s)\sin(s) \\
&\quad + \eta\big(-x\alpha\cos(s)(1-2\cos(s)) + y\alpha\sin(2s) - z\beta\cos(s)\big)\Big), \\
Z(s,\alpha,\beta) &= \eta\beta + \epsilon z s
\end{aligned}$$

Cables with quantum memory IV

Compute hamiltonian $$\hat{H} = \hat{H}_0 + \epsilon \hat{H}_1 + O(\epsilon^2)$$

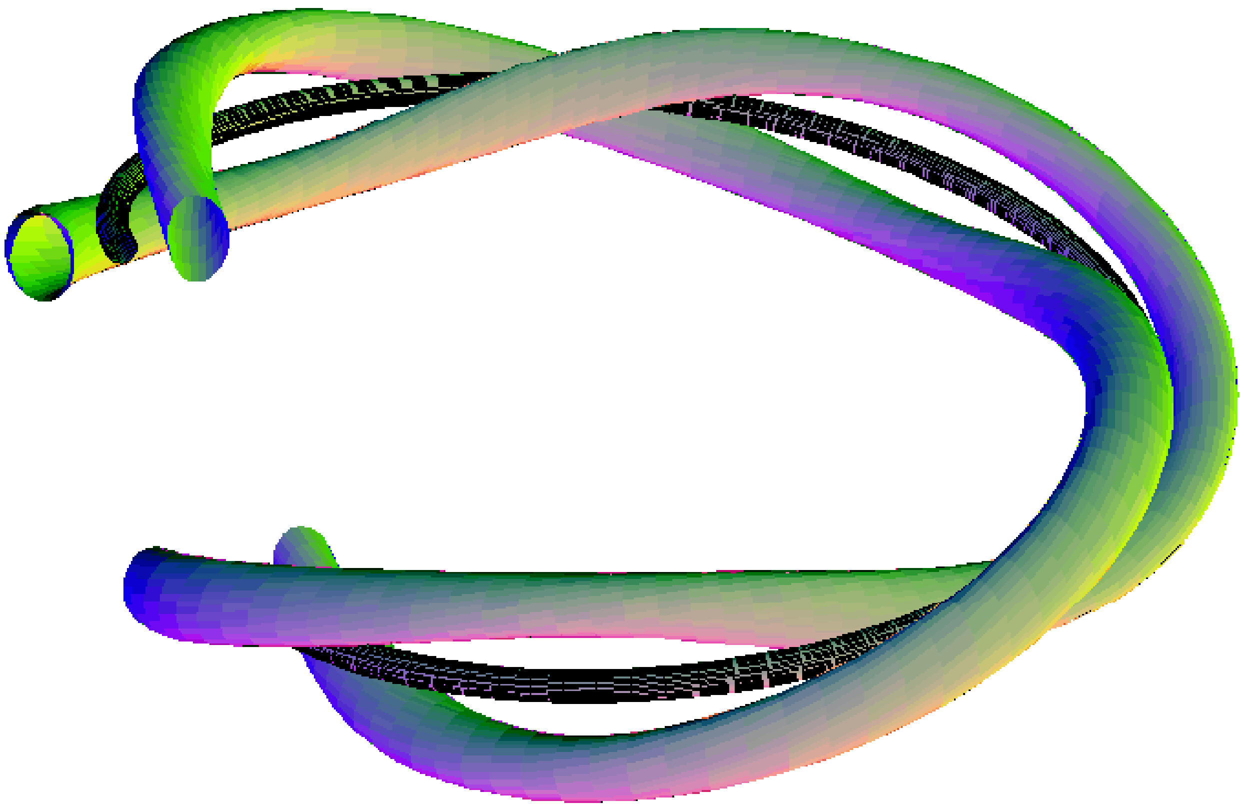
Curvature, torsion
$$\begin{aligned} \kappa(s) &= 1 + \epsilon\,(2x\sin(s) - 2y\cos(s)) + O(\epsilon^2)\,, \\ \tau(s) &= \epsilon z + O(\epsilon^2) \end{aligned}$$

Zeroth and first order hamiltonian
$$\begin{aligned} \hat{H}_0 &= -\frac{1}{2}\frac{\partial}{\partial s^2} - \frac{1}{8} \\ \hat{H}_1 &= i\sigma z\frac{\partial}{\partial s} + \frac{1}{2}(-x\sin(s) + y\cos(s)) \end{aligned}$$

1D wavefunction
$$\psi_m^+(s) = \frac{1}{\sqrt{2\pi}}e^{ims} + \frac{\epsilon}{\sqrt{2\pi}}\left(\frac{-ix+y}{2(2m-1)}e^{i(m-1)s} - \frac{ix+y}{2(2m+1)}e^{i(m+1)s}\right)$$

2D normal wavefunction
$$\varphi(\alpha,\beta) = \frac{1}{\sqrt{\pi}\eta}\sqrt{\alpha^2+\beta^2}e^{-(\alpha^2+\beta^2)/2}e^{i\arctan(\beta/\alpha)}$$

3D total wavefunction
$$\Psi(s,\alpha,\beta) = \psi_m^+(s)\varphi(\alpha,\beta)$$

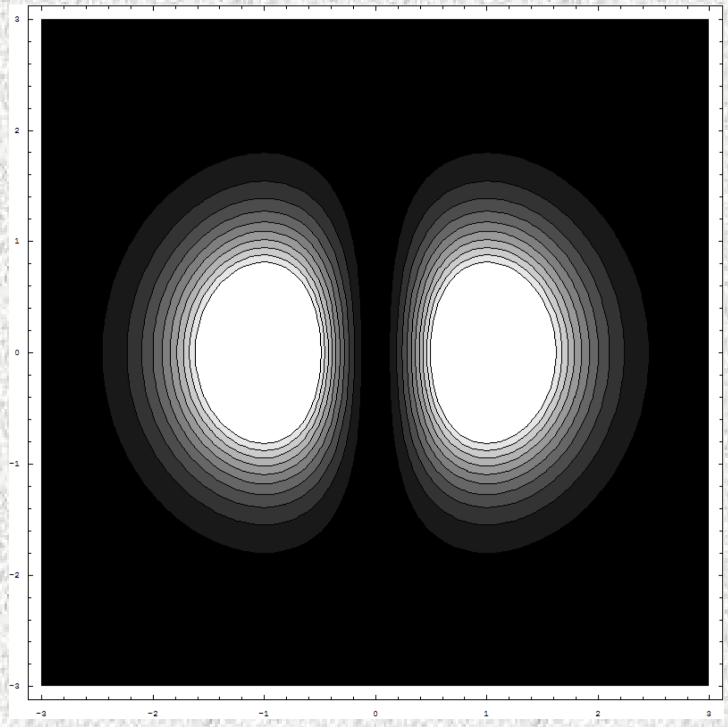


Cables with quantum memory V

$$K = \epsilon^2 (\pi,\ 2,\ \frac{m(21 + 8\pi^2 - 16m^2(3 + 4\pi^2) + 16m^4(3 + 8\pi^2))}{3(4m^2 - 1)^2})$$

(**Notice**: s, α, β, taken as functions of (X,Y,Z; x,y,z))

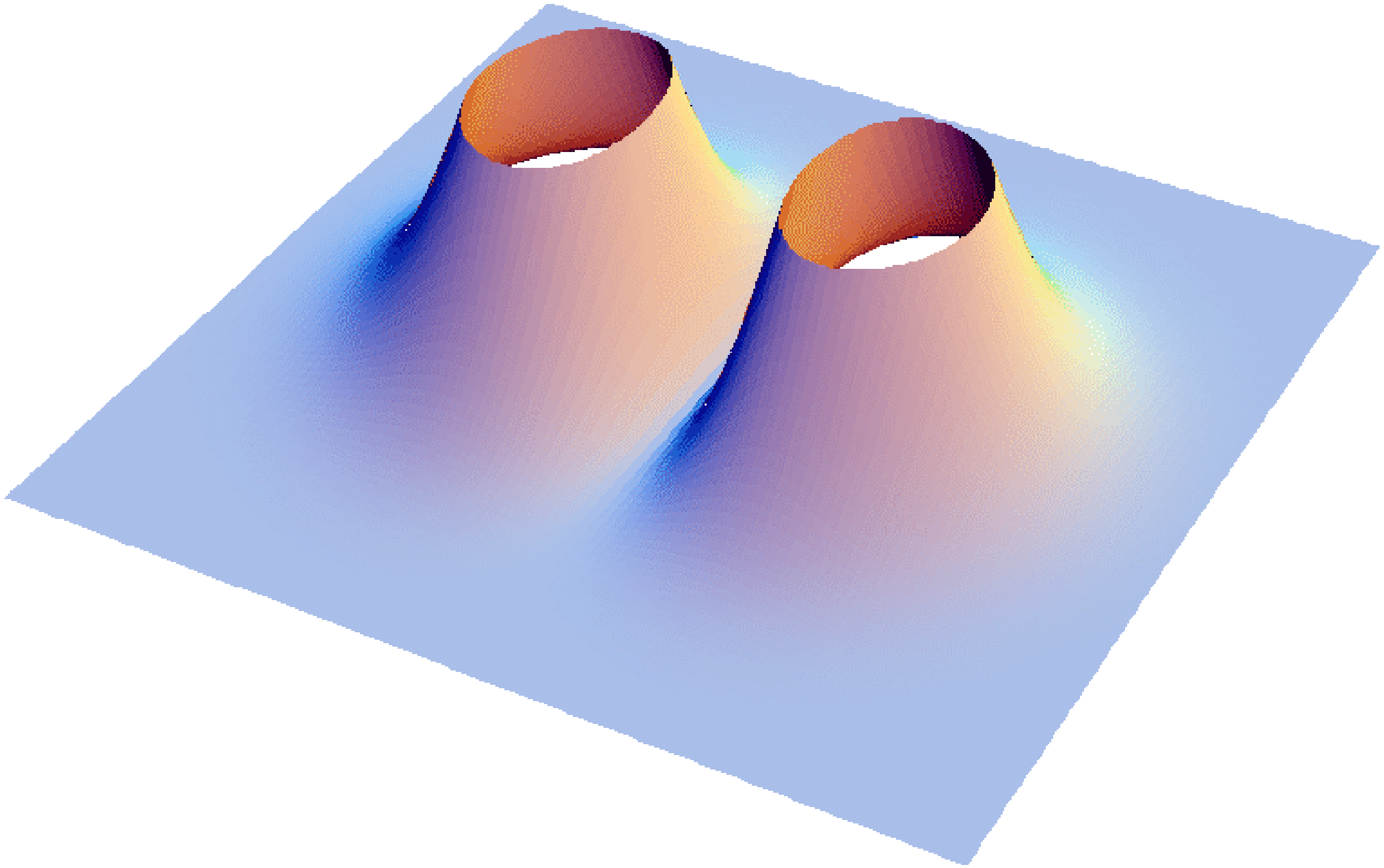
Initial state: |2,+> + |2,->

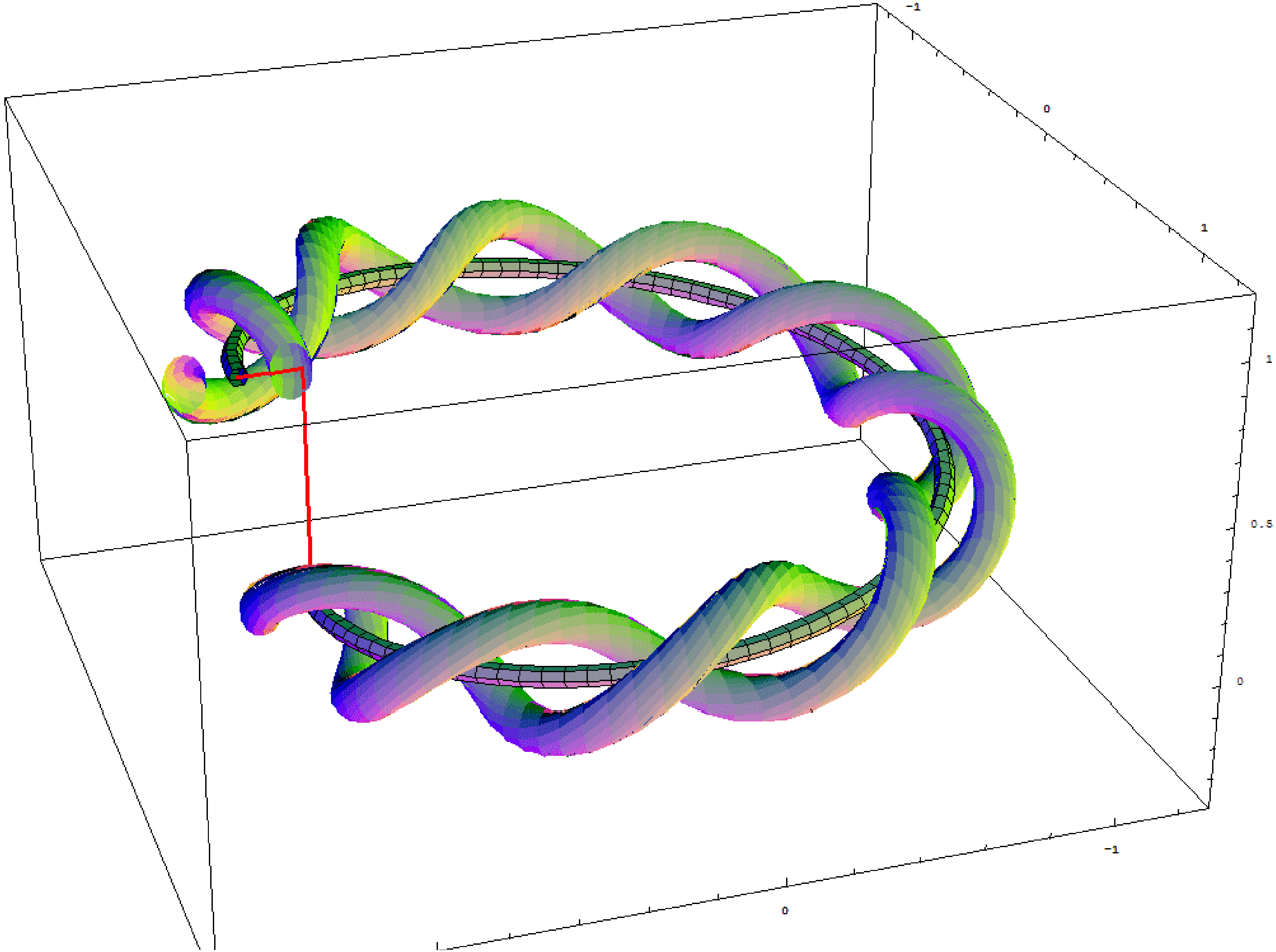


Cyclic change: x=cos t, y=sin t, z=2

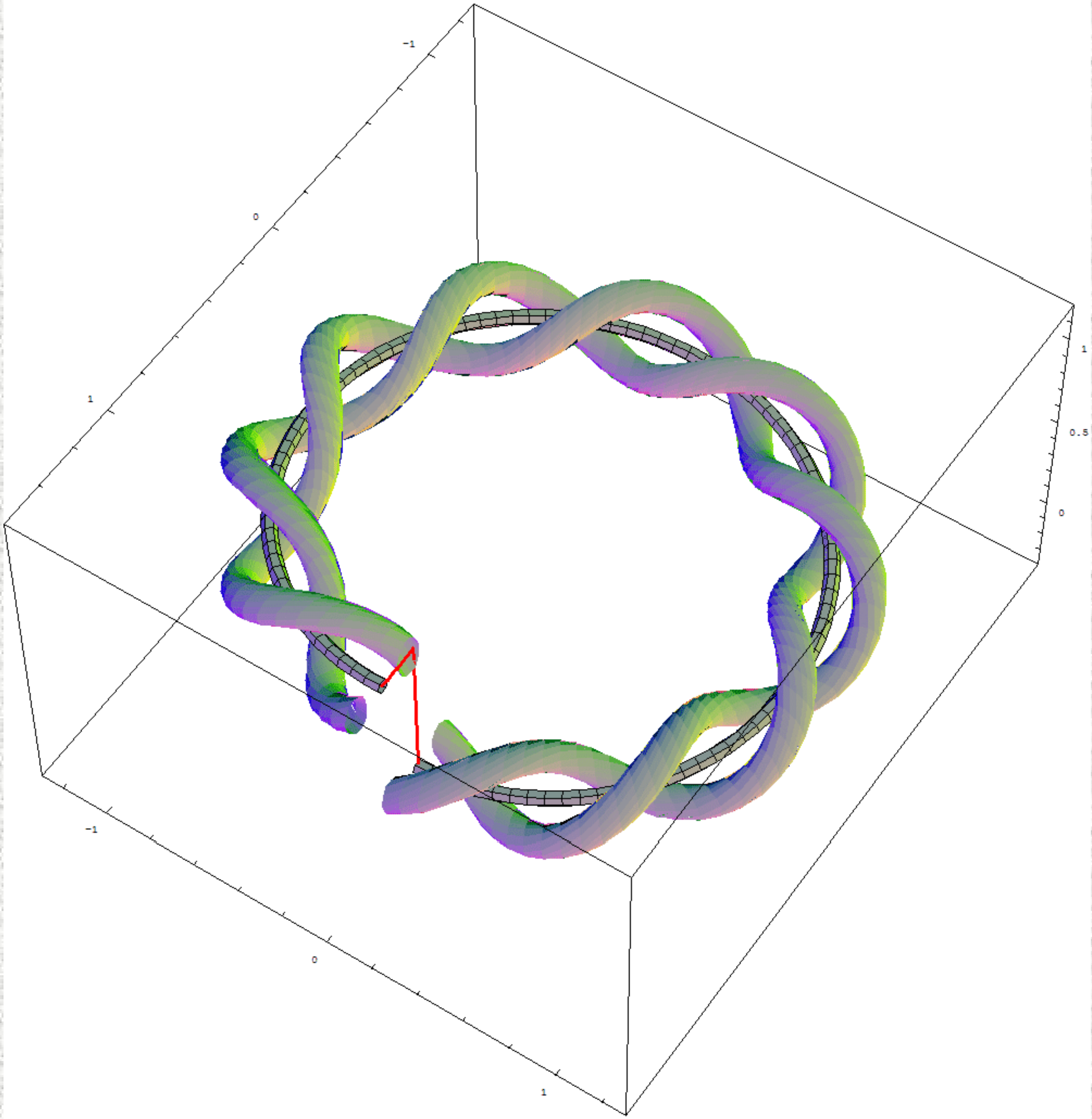
$$\begin{aligned} |+\rangle + |-\rangle = |1,0\rangle \quad &\longrightarrow \quad e^{i\xi}|+\rangle + e^{-i\xi}|-\rangle \\ &= \quad \cos(\xi)|1,0\rangle - \sin(\xi)|0,1\rangle \end{aligned}$$

Cables with quantum memory VI





-1
0
1
1
0.5
0
-1
0



-1
0
1
-1
0
1
0
0.5
1

Back reaction I

Example: Spin coupled to position

$$H = \frac{P^2}{2M} + X \cdot \sigma$$

Adiabatic approximation: $\Psi = \psi(X)|\hat{n}(X)\rangle$

Effective hamiltonian:

$$H_{\text{eff}} = \langle \hat{n}(X)|H|\hat{n}(X)\rangle = \frac{\tilde{P}^2}{2M} + |X| + V(X)\,, \qquad \tilde{P} \equiv P - iA(X)$$

Noncommuting momenta

$$[\tilde{P}_i, \tilde{P}_j] = K_{ij}$$

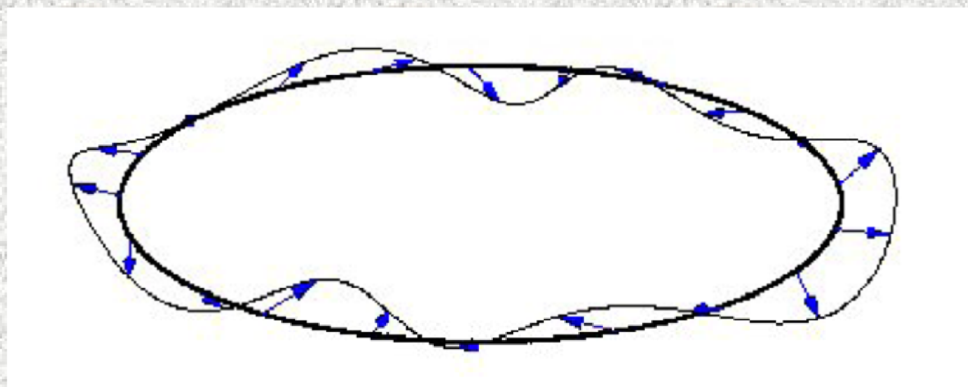
Back reaction II

What if $|\hat{n}(X)\rangle \rightarrow |\hat{n}(X,P)\rangle$?

$$H(X,P) \rightarrow H_{\text{eff}}(\tilde{X},\tilde{P})$$

Apply to wire quantization:

$$\vec{X}\,(s,t) = \vec{R}\,(s) + \epsilon\,\vec{\varphi}\,(s,t)$$



Back reaction III

Lagrangian density

$$\mathcal{L}(\varphi,\dot{\varphi},\varphi',\varphi'') = \frac{1}{2}(\dot{\varphi}^i)^2 - \frac{\mu}{2}\left((\varphi^{n''}+\varphi^n)^2 + \frac{1}{\lambda}(\varphi^{t'}-\varphi^n)^2\right)$$

In terms of Fourier modes

$$L = \int ds\,\mathcal{L} = \sum_{k=-\infty}^{\infty} \frac{1}{2}\left[\,\dot{\phi}_k^{i*}\dot{\phi}_k^i - \mu(1-k^2)^2\,\phi_k^{n*}\phi_k^n - \frac{\mu}{\lambda}\left(k^2\phi_k^{t*}\phi_k^t + ik(\phi_k^t\phi_k^{n*} - \phi_k^{t*}\phi_k^n) + \phi_k^{n*}\phi_k^n\right)\right]$$

CCR's

$$\begin{aligned}
&[\varphi^i(s,t),\varphi^j(s',t)] = 0,\\
&[\varphi^i(s,t),\pi_j(s',t)] = i\delta^i_j\delta(s-s')\\
&[\pi_i(s,t),\pi_j(s',t)] = 0,
\end{aligned}$$

$$\begin{aligned}
&[\phi^i_k(t),\phi^j_{k'}(t)] = 0,\\
&[\phi^i_k(t),\pi^{k'}_j(t)] = i\delta^i_j\delta^{k'}_{-k}\\
&[\pi^k_i(t),\pi^{k'}_j(t)] = 0;
\end{aligned}$$

Back reaction IV

Effective 1D particle hamiltonian

$$i\partial_t \psi_\sigma = \langle H_p \rangle_\perp \, \psi_\sigma$$

$$\begin{aligned}
\langle H_p \rangle_\perp &= \langle H_0 \rangle_\perp + \langle V \rangle_\perp , \\
\langle H_0 \rangle_\perp &= -\frac{1}{2}\partial_s^2 + i\sigma\tau\partial_s + \frac{1}{2}\left(i\sigma\tau' + \tau^2 - \frac{1}{4}\kappa^2 \right), \\
\langle V \rangle_\perp &= \epsilon \left(\dot\varphi^t \partial_s + \frac{1}{2}\dot\varphi^n + i\sigma\dot\varphi^{b''} \right),
\end{aligned}$$

$$\begin{aligned}
\langle H_0 \rangle_\perp &= -\frac{1}{2}\left(\partial_s^2 + \frac{1}{4}\right) + \epsilon\left(i\sigma(\varphi^{b'''} + \varphi^{b'})\partial_s + \frac{1}{2} i\sigma(\varphi^{b''''} + \varphi^{b''}) - \frac{1}{4}(\varphi^{n''} + \varphi^n) \right), \\
&= -\frac{1}{2}\left(\partial_s^2 + \frac{1}{4}\right) + \epsilon \sum_k \left[(k^2 - 1) e^{iks} \left(\frac{i\sigma k}{2}\varphi_k^b (k - 2i\partial_s) - \frac{1}{4}\varphi_k^n \right) \right].
\end{aligned}$$

$$\begin{aligned}
\langle V \rangle_\perp &= \epsilon \sum_k \left(\dot\varphi_k^t \partial_s + \frac{1}{2}\dot\varphi_k^n - i\sigma k^2 \dot\varphi_k^b \right), \\
&= \epsilon \sum_k \left(\pi_t^k \partial_s + \frac{1}{2}\pi_n^k - i\sigma k^2 \pi_b^k \right).
\end{aligned}$$

Back reaction V

1D wavefunction

$$\psi_\sigma^0 = \frac{1}{\sqrt{2\pi}} + \epsilon \sum_{p\neq 0} \frac{1}{p^2} \left[\frac{1-p^2}{2} \left(2i\sigma p^2 \varphi_p^b + \varphi_p^n\right) \frac{e^{ips}}{\sqrt{2\pi}} - \left(i\pi_n^p + 2\sigma\pi_b^p\right) \frac{e^{-ips}}{\sqrt{2\pi}} \right]$$

Berry's curvature

$$K_{n_p,b_q} = -\frac{\sigma}{p^2}(1-p^2)(1-q^2)\delta_{q,-p},$$

$$K_{n_p}^{n_q} = \frac{1-p^2}{q^2p^2}\delta_{q,p},$$

$$K_{b_p}^{b_q} = -\frac{4(1-p^2)}{q^2}\delta_{q,p},$$

$$K^{n_p,b_q} = \frac{4\sigma}{p^2q^2}\delta_{q,-p}.$$

Back reaction VI

Deformed CCR's:

$$[\tilde{\pi}_n(s,t), \tilde{\pi}_b(s',t)] = \frac{i\sigma}{2\pi}\sum_{k\neq 0}\frac{(1-k^2)^2}{k^2}\, e^{ik(s-s')},$$

$$[\tilde{\pi}_n(s,t), \tilde{\varphi}^n(s',t)] = i\delta(s-s') + \frac{i}{2\pi}\sum_{k\neq 0}\frac{1-k^2}{k^4}\, e^{ik(s-s')},$$

$$[\tilde{\pi}_b(s,t), \tilde{\varphi}^b(s',t)] = i\delta(s-s') - \frac{4i}{2\pi}\sum_{k\neq 0}\frac{1-k^2}{k^2}\, e^{ik(s-s')},$$

$$[\tilde{\varphi}^n(s,t), \tilde{\varphi}^b(s',t)] = \frac{4i\sigma}{2\pi}\sum_{k\neq 0}\frac{1}{k^4}\, e^{ik(s-s')}.$$