Entanglement in fermionic systems

M.C. Bañuls, J.I. Cirac, M.M. Wolf

MAX-PLANCK-INSTITUT FÜR QUANTENOPTIK GARCHING

MPQ

Goal

Definition of entanglement

in a system of fermions

given the presence of superselection rules

that affect the concept of locality

Contents

- Basic ingredients
- Definitions
- Characterization and measures
- Many copies

Fermionic Systems

- Indistinguishability
  - Physical states restricted to totally (anti)symmetric part of Hilbert space
    - No tensor product structure
  - Second quantization language
    - Entanglement between modes
- Other SSR affect locality
  - Physical states have even or odd number of fermions
  - Physical operators do not change the parity

Fermionic Systems

- System of $m=m_A+m_B$ fermionic modes
  - partition $A=\{1, 2, \ldots m_A\}$
  - partition $B=\{m_A+1, \ldots m_A+m_B\}$

Ex. 1x1:

$A=\{1\}; B=\{2\}$

- Basic objects: creation and annihilation operators
  - canonical anticommutation relations $\left\{a_i, a_j^\dagger\right\}=\delta_{ij}$
  - $\left\{a_1, a_1^\dagger, \ldots a_{m_A}, a_{m_A}^\dagger\right\}$ and their products, generate operators on $\mathcal{A}$ ($\mathcal{B}$)
  - products of even number commute with parity

A by $\left\{a_1, a_1^\dagger\right\}$

B by $\left\{a_2, a_2^\dagger\right\}$

$a_1^\dagger a_1$ in A

$a_2^\dagger a_2$ in B

Fermionic Systems

- Fock representation, in terms of occupation number of each mode

$$\left|n_1 n_2 \ldots n_m\right\rangle = \left(a_1^\dagger\right)^{n_1}\left(a_2^\dagger\right)^{n_2}\ldots\left(a_m^\dagger\right)^{n_m}\left|0\right\rangle$$

$$\left|00\right\rangle = \left|0\right\rangle$$
$$\left|01\right\rangle = a_2^\dagger\left|0\right\rangle$$
$$\left|10\right\rangle = a_1^\dagger\left|0\right\rangle$$
$$\left|11\right\rangle = a_1^\dagger a_2^\dagger\left|0\right\rangle$$

  - isomorphic to m-qubit space
  - action of fermionic operators is not local
    - e.g. for $m_A=m_B=1$

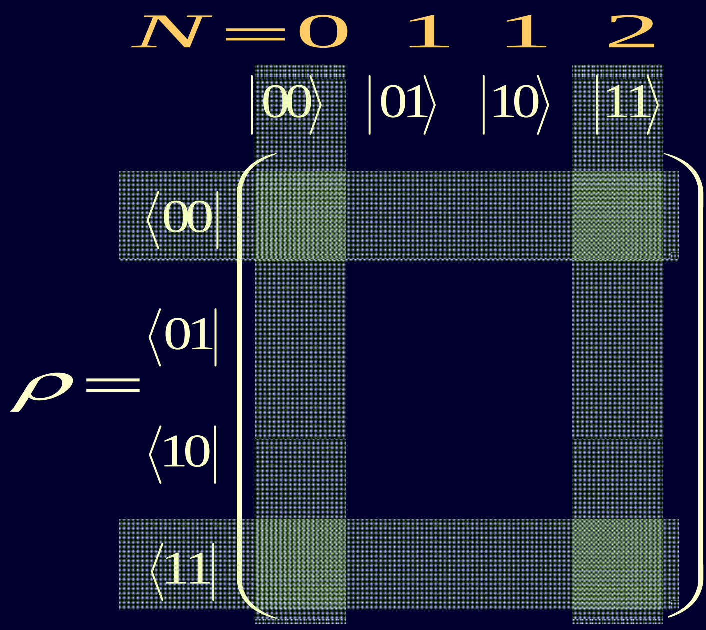
$$a_2^\dagger\left|00\right\rangle = \left|01\right\rangle$$
$$a_2^\dagger\left|10\right\rangle = -\left|11\right\rangle$$

Parity SSR

- Physical states and observables commute with parity operator

1x1 modes

Fock representation



physical states

Parity SSR

- Physical states and observables commute with parity operator

1x1 modes

Fock representation

$$
\begin{array}{c|cccc}
N = & 0 & 1 & 1 & 2 \\
 & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\
\hline
\langle 00| & \bullet & 0 & 0 & \bullet \\
\langle 01| & 0 & & & 0 \\
\langle 10| & 0 & & & 0 \\
\langle 11| & \bullet & 0 & 0 & \bullet
\end{array}
$$

$$
\rho = \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & & & 0 \\ 0 & & & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix}
$$

physical states

Parity SSR

- Physical states and observables commute with parity operator

1x1 modes

Fock representation

$$
\begin{array}{cc|cccc}
 & N = & 0 & 1 & 1 & 2 \\
 & & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\
\hline
 & \langle 00| & \bullet & 0 & 0 & \bullet \\
\rho = & \langle 01| & 0 & \bullet & \bullet & 0 \\
 & \langle 10| & 0 & \bullet & \bullet & 0 \\
 & \langle 11| & \bullet & 0 & 0 & \bullet
\end{array}
$$

physical states

Parity SSR

- Physical states and observables commute with parity operator

1x1 modes

Fock representation

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\begin{array}{cccccc}
 & N= & 0 & 1 & 1 & 2 \\
 & & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\
 & \langle 00| & \bullet & 0 & 0 & \bullet \\
\rho = & \langle 01| & 0 & \bullet & \bullet & 0 \\
 & \langle 10| & 0 & \bullet & \bullet & 0 \\
 & \langle 11| & \bullet & 0 & 0 & \bullet
\end{array}
$$

$$\rho = \Pi_{even}\,\rho\,\Pi_{even} + \Pi_{odd}\,\rho\,\Pi_{odd}$$

physical states

Parity SSR

- Physical states and observables commute with parity operator

$m_A x m_B$ modes

Fock representation

$$\rho = \begin{array}{c} \\ \langle ee| \\ \langle eo| \\ \langle oe| \\ \langle oo| \end{array} \begin{array}{c} \begin{array}{cccc} |ee\rangle & |eo\rangle & |oe\rangle & |oo\rangle \end{array} \\ \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix} \end{array}$$

$$\rho = \Pi_{even}\,\rho\,\Pi_{even} + \Pi_{odd}\,\rho\,\Pi_{odd}$$

physical states

Parity SSR

- Physical states and observables commute with parity operator

$m_A x m_B$ modes

Fock representation

$$A_\pi \otimes B_\pi = \begin{array}{c} \\ \langle ee| \\ \langle eo| \\ \langle oe| \\ \langle oo| \end{array} \begin{array}{c} \begin{array}{cccc} |ee\rangle & |eo\rangle & |oe\rangle & |oo\rangle \end{array} \\ \begin{pmatrix} \bullet & 0 & 0 & 0 \\ 0 & \bullet & 0 & 0 \\ 0 & 0 & \bullet & 0 \\ 0 & 0 & 0 & \bullet \end{pmatrix} \end{array}$$

$$O = A_\pi \otimes B_\pi =$$

$$\Pi^A_{even}\Pi^B_{even} O \Pi^A_{even}\Pi^B_{even}$$
$$+ \Pi^A_{even}\Pi^B_{odd} O \Pi^A_{even}\Pi^B_{odd}$$
$$+ \Pi^A_{odd}\Pi^B_{even} O \Pi^A_{odd}\Pi^B_{even}$$
$$+ \Pi^A_{odd}\Pi^B_{odd} O \Pi^A_{odd}\Pi^B_{odd}$$

local physical observables

How to define entangled states?

- Entangled states = are not separable
- Separable states = convex combinations of product states
- Define product states…

Product states

P1

$$\rho(A_\pi \cdot B_\pi) = \rho(A_\pi)\rho(B_\pi) \quad \forall A_\pi \in \mathcal{A}, B_\pi \in \mathcal{B}$$

P2

$$\rho = \rho_A \otimes \rho_B \quad \text{in Fock space representation}$$

P3

$$\rho(A \cdot B) = \rho(A)\rho(B) \quad \forall A \in \mathcal{A}, B \in \mathcal{B}$$

$$\text{P3, P2} \subset \text{P1}$$

Product states

P1

$$\rho(A_\pi \cdot B_\pi) = \rho(A_\pi)\rho(B_\pi) \quad \forall A_\pi \in \mathcal{A}, B_\pi \in \mathcal{B}$$

P2

$$\rho = \rho_A \otimes \rho_B \quad \text{in Fock space representation}$$

P3

$$\rho(A \cdot B) = \rho(A)\rho(B) \quad \forall A \in \mathcal{A}, B \in \mathcal{B}$$

BUT when restricted to physical states (ρ commuting with parity)

$$\mathrm{P3}_\pi = \mathrm{P2}_\pi \subset \mathrm{P1}_\pi$$

Product states

Example:

1x1 modes

P1

$$\rho = \frac{1}{16}\begin{pmatrix} 9 & 0 & 0 & -i \\ 0 & 3 & -i & 0 \\ 0 & i & 3 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix} \otimes \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$$

in P1 not in P2

$$\rho \neq \rho_A \otimes \rho_B$$

P2

$$\rho = \begin{pmatrix} a & 0 \\ 0 & 1-a \end{pmatrix} \otimes \begin{pmatrix} b & 0 \\ 0 & 1-b \end{pmatrix}$$

We have…

…two different sets of product states

$$P3_{\pi} = P2_{\pi} \subset P1_{\pi}$$

For pure states, they are
all the same!!

Use them to construct separable states

Separable states

S1

$$\rho = \sum \lambda_k \rho_k, \quad \rho_k(A_\pi \cdot B_\pi) = \rho_k(A_\pi)\rho_k(B_\pi)$$

S2’

$$\rho = \sum \lambda_i \rho_{iA} \otimes \rho_{iB}$$

S2=S3

$$\rho = \sum \lambda_i \rho_i^{P_A} \otimes \rho_i^{P_B}$$

For physical states (ρ commuting with parity)

$$S3 = S2 \subset S2' \subset S1$$

Separable states

Local measurements cannot distinguish states that produce the same expectation values for all physical local operators

- define equivalent states

$$\left\langle A_{\pi} B_{\pi} \right\rangle_{\rho_1} = \left\langle A_{\pi} B_{\pi} \right\rangle_{\rho_2}$$

- define separability as equivalence to separable state

Separable states

[S2]

$$\langle A_\pi \cdot B_\pi \rangle_\rho = \langle A_\pi \cdot B_\pi \rangle_{\tilde\rho}, \quad \tilde\rho \in S2$$

S1

$$\rho = \sum \lambda_k \rho_k, \quad \rho_k(A_\pi \cdot B_\pi) = \rho_k(A_\pi)\rho_k(B_\pi)$$

S2'

$$\rho = \sum \lambda_i \rho_{iA} \otimes \rho_{iB}$$

S2=S3

$$\rho = \sum \lambda_i \rho_i^{P_A} \otimes \rho_i^{P_B}$$

For physical states (ρ commuting with parity)

$$S3 = S2 \subset S2' \subset S1 \subset [S2]$$

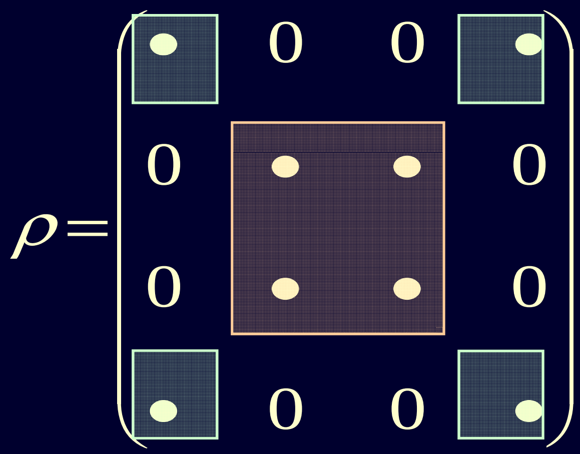
Separable states

Example:

1x1 modes

S2

$$\rho = \begin{pmatrix} 2/7 & 0 & 0 & 0 \\ 0 & 1/7 & 0 & 0 \\ 0 & 0 & 1/7 & 0 \\ 0 & 0 & 0 & 3/7 \end{pmatrix}$$



Separable states

Example:

1x1 modes

$$\rho = \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix}$$

S2

$$\rho = \begin{pmatrix} 2/7 & 0 & 0 & 0 \\ 0 & 1/7 & 0 & 0 \\ 0 & 0 & 1/7 & 0 \\ 0 & 0 & 0 & 3/7 \end{pmatrix}$$

S2’

$$\rho = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}$$

Separable states

Example:

1x1 modes

$$\rho=\begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix}$$

S2

$$\rho=\begin{pmatrix} 2/7 & 0 & 0 & 0 \\ 0 & 1/7 & 0 & 0 \\ 0 & 0 & 1/7 & 0 \\ 0 & 0 & 0 & 3/7 \end{pmatrix}$$

S1= S2’

$$\rho=\begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}$$

[S2]

$$\rho=\begin{pmatrix} 1/3 & 0 & 0 & 2/3\sqrt{5} \\ 0 & 1/5 & 1/5 & 0 \\ 0 & 1/5 & 1/5 & 0 \\ 2/3\sqrt{5} & 0 & 0 & 4/15 \end{pmatrix}$$

There are…

…four different sets of separable states

$$S3 = S2 \subset S2' \subset S1 \subset [S2]$$

They correspond to four classes of states

- different capabilities for preparation and measurement

S2 → Preparable by local operations and classical communication, restricted by parity

S2' → Convex combination of products in the Fock representation

S1 → Convex combination of states s.t. locally measurable observables factorize

[S2] → All measurable correlations can be produced by one of the above

Characterization

Criteria in terms of usual separability

$$\rho = \begin{array}{c} \\ \langle ee| \\ \langle eo| \\ \langle oe| \\ \langle oo| \end{array} \!\!\! \begin{array}{c} \begin{array}{cccc} |ee\rangle & |eo\rangle & |oe\rangle & |oo\rangle \end{array} \\ \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix} \end{array}$$

Characterization

Criteria in terms of usual separability

[S2]

$$\rho = \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix}$$

S1

convex combination of products

$$\rho = \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix}$$

S2'

$$\rho = \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix}$$

S2

$$\rho = \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix}$$

Characterization

Criteria in terms of usual separability

[S2]

$$\rho = \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix}$$

S1

convex combination of products

$$\rho = \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix}$$

S2'

$$\rho = \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix}$$

S2

$$\rho = \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix} \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix}$$

Measures of entanglement

- For S2' and S2, the entanglement of formation can be defined

$$\mathrm{EoF}_{S2'} = \mathrm{EoF}(\rho) = \min_{\{\lambda_k, |\psi_k\rangle\}} \sum \lambda_k \, \mathrm{E}(|\psi_k\rangle)$$

$$\mathrm{EoF}_{S2} = \omega \mathrm{EoF}(\rho_+) + (1-\omega)\, \mathrm{EoF}(\rho_-)$$

  - For 1x1 modes, EoF in terms of the elements of the density matrix

Multiple Copies

- Not all the definitions of separability are stable under taking several copies of the state

$$\rho^{\otimes 2} \in [S2] \Rightarrow \rho \in [S2]$$

$$\rho^{\otimes 2} \in S1 \Rightarrow \rho \in S1$$

$$\rho^{\otimes 2} \in S2' \Leftrightarrow \rho \in S2'$$

$$\rho^{\otimes 2} \in S2 \Leftrightarrow \rho \in S2$$

$$\rho^{\otimes 2} \in [S2] \Rightarrow \rho \; \mathrm{PPT}$$

$\Rightarrow$distillable states not in [S2]

- S2 and S2’ asymptotically equivalent
  - 1x1 modes: all of them equivalent in the limit of large N

To conclude…

Different definitions of entanglement between fermionic modes are possible

They are related to different physical situations, different abilities to prepare, measure the state

Different measures of entanglement

Different behaviour for several copies

More details: Phys. Rev. A 76, 022311 (2007)

Application to a particular case

- Fermionic Hamiltonian

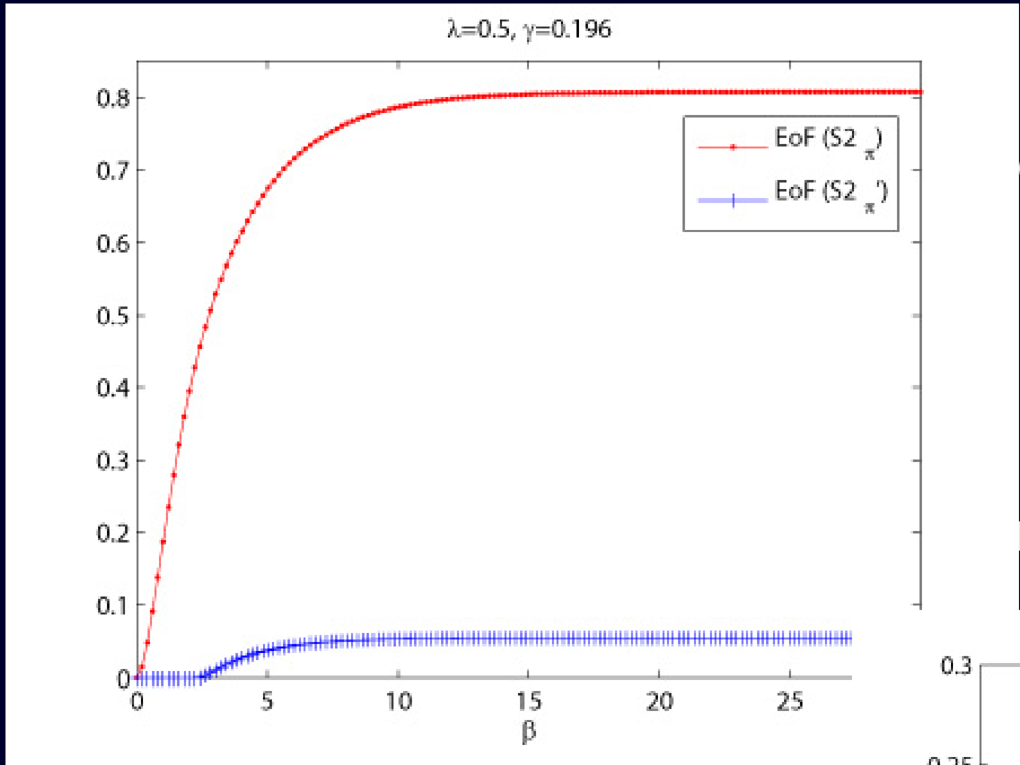
$$H = -\frac{1}{2}\sum_j \left(a_j^\dagger a_{j+1} + \mathrm{h.c.}\right) - \lambda \sum_j a_j^\dagger a_j - \gamma \sum_j \left(a_j^\dagger a_{j+1}^\dagger + \mathrm{h.c.}\right)$$

- Reduced 2-mode density matrix calculated from

$$\rho = \frac{\mathrm{e}^{-\beta H}}{\mathrm{tr}\left(\mathrm{e}^{-\beta H}\right)}$$

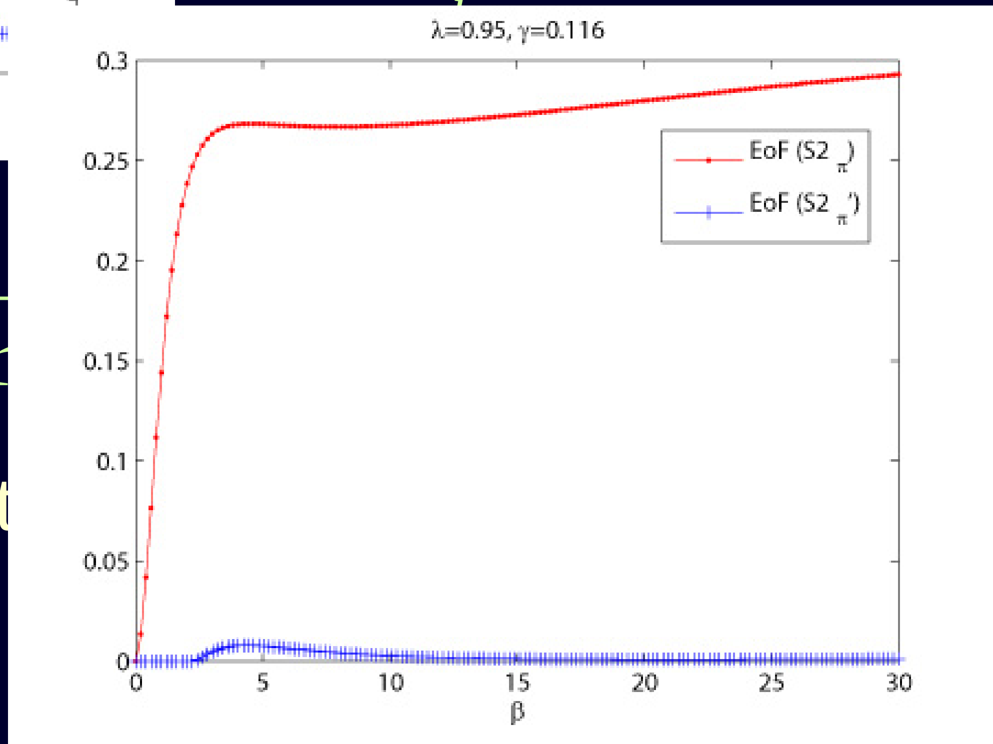
- Regions of separability as a function of β, λ, γ
- EoF for S2, S2'

articular case



$$a_j^\dagger a_j - \gamma \sum_j \left(a_j^\dagger a_{j+1}^\dagger + \mathrm{h.c.} \right)$$

$$\rho = \frac{}{\mathrm{tr}}$$



- Regions of separabilit
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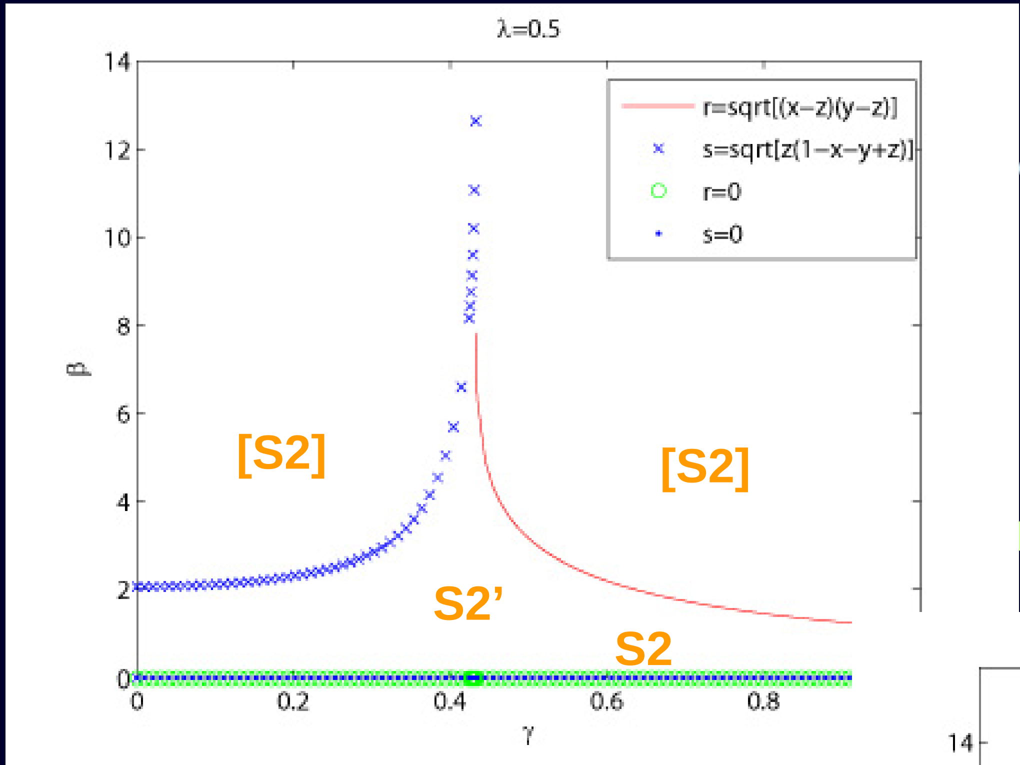
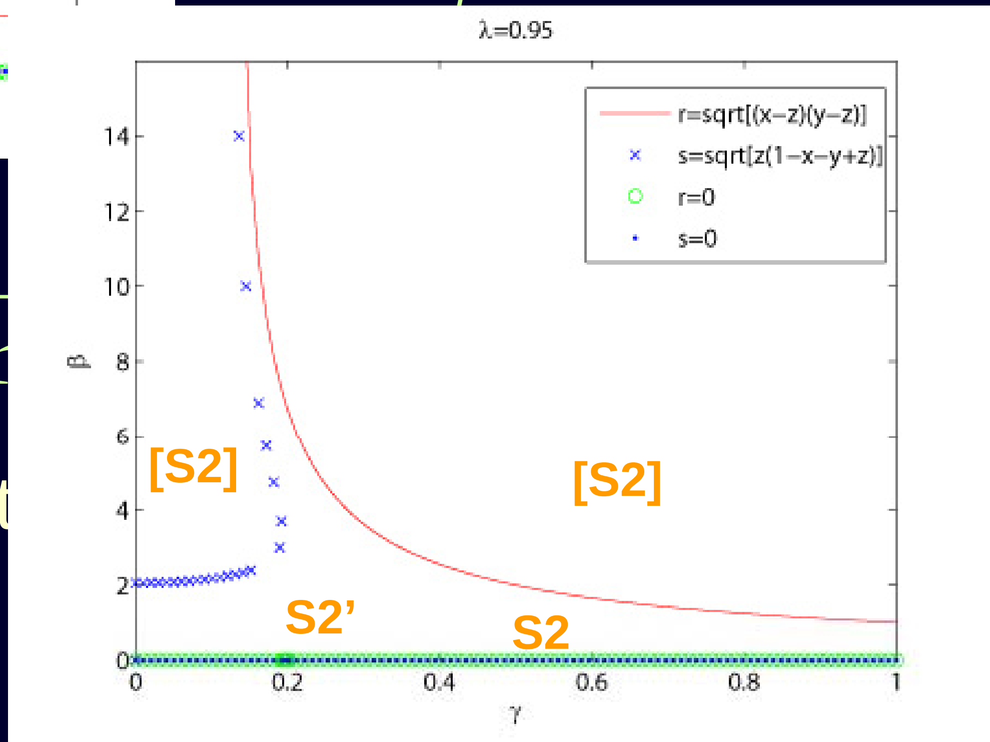
articular case

$$a_j^\dagger a_j - \gamma \sum_j \left(a_j^\dagger a_{j+1}^\dagger + \text{h.c.} \right)$$

$$\rho = \frac{\quad}{\mathrm{tr}}$$





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