$N^3LO + N^3LL'$ from analyticity

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HXSWG ggf task force, November 18, 2014

Based on:

Richard Ball, M.B., Stefano Forte, Simone Marzani, Giovanni Ridolfi

arXiv:1303.3590 arXiv:1404.3204

and **M.B., Simone Marzani**

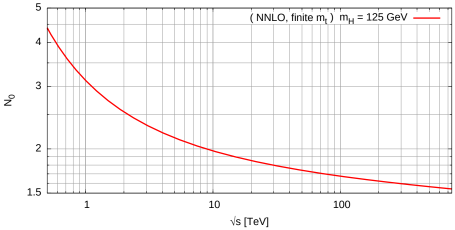
arXiv:1405.3654

XS σ is dominated by a single Mellin moment

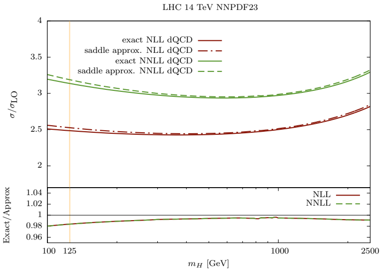
$$\sigma_{gg}(\tau) \propto \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i}\, \tau^{-N} f_g(N)\, f_g(N)\, C_{gg}(N,\alpha_s), \qquad \tau = \frac{m_{\rm H}^2}{s}$$

Largely dominated by the region close to the saddle point $N = N_0(\tau)$

[MB, Forte, Ridolfi 1204.5473]



LHC: $N_0 \sim 2.1 - 1.9$



$$\sigma_{gg}(\tau) \approx C_{gg}(N_0(\tau), \alpha_s)$$

[MB, Forte, Ridolfi, Rottoli 1409.0864]→

Also noticed (from different arguments) in [deFlorian,Mazzitelli,Moch,Vogt 1408.6277]

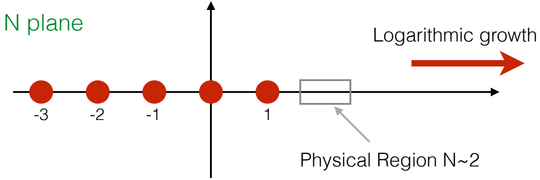
Analytic structure of the coefficient function

Mellin space: $$C_{gg}(N,\alpha_s) = \int_0^1 dz\, z^{N-1}\, C_{gg}(z,\alpha_s)$$

(ordinary funct) (distribution)

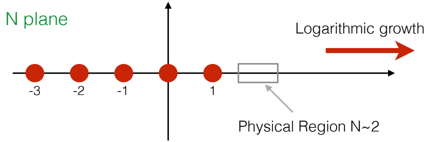
$C_{gg}(N,\alpha_s)$ is a meromorphic function in the complex N plane, with a known analytic structure (at finite perturbative order):

- logarithmic growth at large $\mathrm{Re}\,(N)$ [Sudakov factorisation]
- isolated poles in $N = 1, 0, -1, -2, \ldots$ [Regge theory, BFKL]
- no other singularities



Reconstruction of the coefficient function

A meromorphic function can be reconstructed from its singularities [Liouville]



What do we know?

- sing at $N \to \infty$:

$$C_{gg}(N, \alpha_s) \sim \alpha_s^n \log^k N, \qquad 0 < k < 2n$$

  from soft-gluon (Sudakov) resummation

- sing in $N = 1$:

$$C_{gg}(N, \alpha_s) \sim \alpha_s^n \frac{1}{(N-1)^k}, \qquad 1 < k < n$$

  from high-energy (BFKL) resummation

These are the most important for the physical region $N \sim 2$!

Ingredients of our predictions

We construct an **analytic approximation** according to

$$C_{gg}(N,\alpha_s) \simeq C_{\text{soft}}(N,\alpha_s) + C_{\text{high-energy}}(N,\alpha_s)$$

$$N \to \infty \qquad\qquad N \to 1$$

$$z \to 1 \qquad\qquad z \to 0$$

Both ingredients must respect the analytic structure of $C_{gg}(N,\alpha_s)$

current status	fixed order (α_s^3)	resummed level
soft part	✓	✓
high-energy part	✓	not yet

We include finite top mass effects in all pieces

We consider gg channel only ($\sim 97\%$ of the full NNLO)

High-energy part

LL poles of $C_{gg}(N, \alpha_s)$ in $N = 1$: $\frac{\alpha_s^k}{(N-1)^k} \quad \leftrightarrow \quad \alpha_s^k \frac{\log^{k-1} z}{z}$

In the effective $m_t \to \infty$ theory double poles: $\frac{\alpha_s^k}{(N-1)^{2k}}$. Totally wrong!!

Difference important at large collider energy $\sqrt{s}$

LL behaviour described by BFKL resummation:

[Marzani, Ball, Del Duca, Forte, Vicini 0801.2544]

$$C_{\text{high-energy}}(N, \alpha_s) = \sum_{k_1,k_2 \geq 0} c_{k_1,k_2}\left(\tfrac{m_H}{m_t}\right)\left[\gamma_+{}^{k_1}\right]\left[\gamma_+{}^{k_2}\right]$$

$\gamma_+(N)$: DGLAP anomalous dimension (largest eigenvalue)

c_{k_1,k_2}: coefficients determined from LO xs with off-shell gluons

- resummation more complicated (work in progress)
- momentum conservation: $C_{\text{high-energy}}(N = 2, \alpha_s) = 0$
- we remove spurious large-N growth (interfering with soft part)

[Ball,MB,Forte,Marzani,Ridolfi 1303.3590]

Soft part: traditional construction: N-soft

In the soft $N \to \infty$ limit soft-gluon (threshold) resummation gives

[Sterman 1987] [Catani, Trentadue 1989] [Forte, Ridolfi 2003]

$$C_{N\text{-soft}}(N, \alpha_s) = g_0\left(\alpha_s, \tfrac{m_{\rm H}}{m_t}\right) \times \exp \mathcal{S}(\alpha_s, \ln N)$$

$$\mathcal{S}(\alpha_s, \ln N) = \frac{1}{\alpha_s} g_1(\alpha_s \ln N) + g_2(\alpha_s \ln N) + \alpha_s g_3(\alpha_s \ln N) + \alpha_s^2 g_4(\alpha_s \ln N) + \dots$$

Current HXSWG recommendation: NNLO + NNLL' [deFlorian,Grazzini 1206.4133]

Now available up to N^{3}LL' [MB,Marzani 1405.3654]

Soft part: analyticity improvement

$$C_{N\text{-soft}}(N,\alpha_s) = g_0\left(\alpha_s, \frac{m_{\rm H}}{m_t}\right) \times \exp \mathcal{S}(\alpha_s, \ln N) = \sum_{n=0}^{\infty} \alpha_s^n \sum_{k=0}^{2n} c_{n,k} \ln^k N$$

has a cut in $N \leq 0$, not compatible with the analytic structure of $C_{gg}(N,\alpha_s)$!

The N-soft resummation/approximation is not acceptable!

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One step backward:

$$C_{gg}(N,\alpha_s) \overset{N\to\infty}{=} \bar{g}_0\left(\alpha_s, \frac{m_{\rm H}}{m_t}\right) \times \exp \bar{\mathcal{S}}(\alpha_s, N)$$

$$\bar{\mathcal{S}}(\alpha_s, N) = \int_0^1 dz\, \frac{z^{N-1}-1}{1-z} \left\{ \int_{\mu_{\rm F}^2}^{M^2(1-z)^2} \frac{d\mu^2}{\mu^2} 2A(\alpha_s(\mu^2)) + D\left(\alpha_s\left(M^2(1-z)^2\right)\right) \right\}$$

$$= \sum_{n=1}^{\infty} \alpha_s^n \sum_{k=0}^{n} b_{n,k} \mathcal{D}_k(N) \qquad\qquad \mathcal{D}_k(z) = \left(\frac{\log^k(1-z)}{1-z}\right)_+$$

$\mathcal{D}_k(N) = \int_0^1 dz\, z^{N-1} \mathcal{D}_k(z)$ is a sum of polygamma functions $\psi_j(N)$, which have poles in $N = 0, -1, -2, \ldots \to$ correct analytic structure ✓

Soft part: kinematic and collinear improvements

Single gluon emission: $$P_{gg}^{(0)}(z)\int_{\mu_\mathrm{F}^2}^{M^2\frac{(1-z)^2}{z}}\frac{dk_\mathrm{T}^2}{k_\mathrm{T}^2}=\frac{A_g^{(0)}(z)}{1-z}\ln\frac{M^2(1-z)^2}{\mu_\mathrm{F}^2 z}$$

Regular terms in $z=1$ usually dropped: $A_g^{(0)}(z)\to A_g^{(0)}(1)$, $\frac{(1-z)^2}{z}\to(1-z)^2$

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Regular terms in $z = 1$ usually dropped: $A_g^{(0)}(z) \to A_g^{(0)}(1)$, $\frac{(1-z)^2}{z} \to (1-z)^2$

We keep them! Effectively:

- we replace $\left(\frac{\log^k(1-z)}{1-z}\right)_+ \quad \to \quad \left(\frac{\log^k \frac{1-z}{\sqrt{z}}}{1-z}\right)_+$
- we include collinear contributions

$$A_g^{(0)}(z) = A_g^{(0)}(1) \frac{1-2z+3z^2-2z^3+z^4}{z}$$

predicts LL next-to-soft terms $(1-z)^p \alpha_s^k \ln^{2k-1}(1-z)$ to all orders

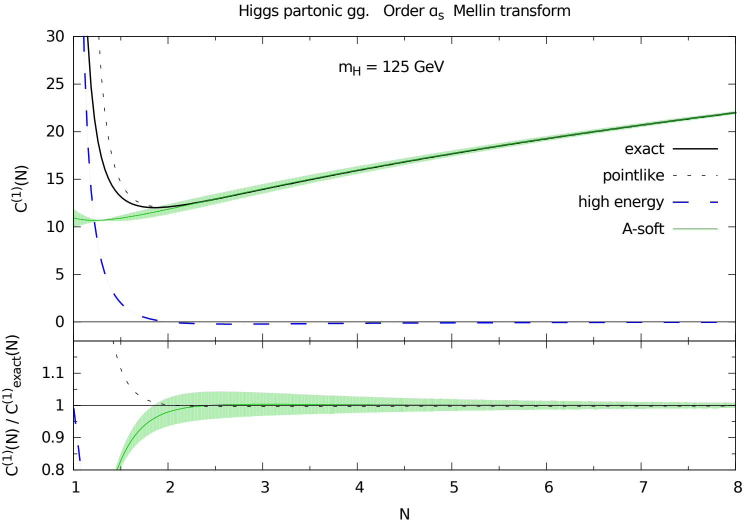
[Krämer, Laenen, Spira 1997]

- we call it A-soft

The $1/z$ in $A_g(z)$ would produce spurious $N=1$ poles, ruining the high-energy behaviour; we then expand $A_g(z)$ in powers of $(1-z)$, using different orders as a measure of the uncertainty.

Full approximation: A-soft + high-energy

$$C_{gg}(N, \alpha_s) = 1 + \alpha_s\, C^{(1)}_{gg}(N) + \alpha_s^2\, C^{(2)}_{gg}(N) + \dots$$



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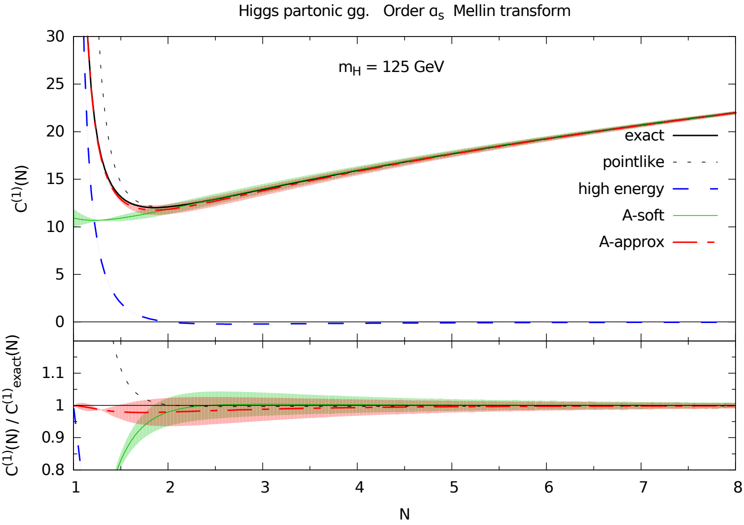

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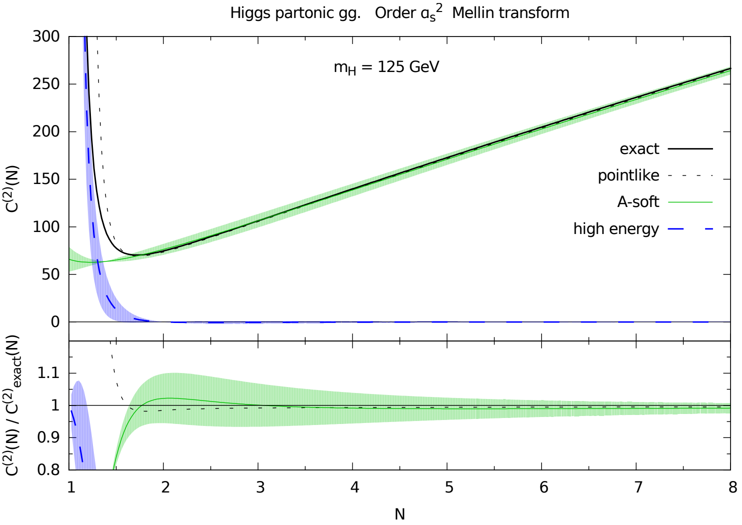

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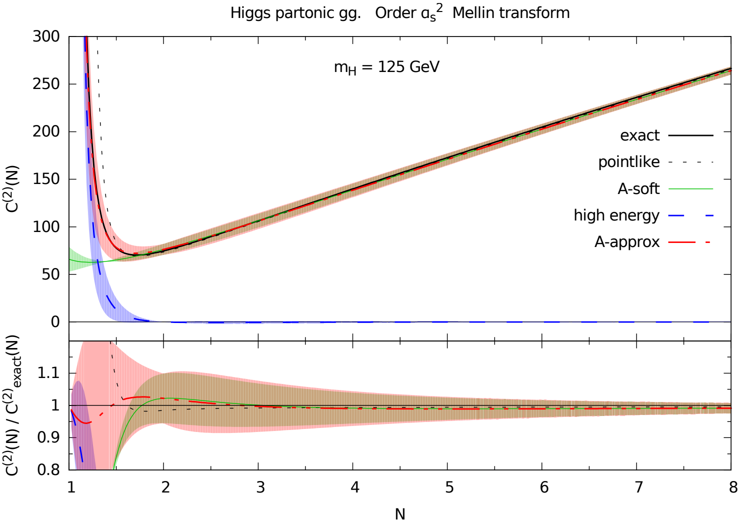
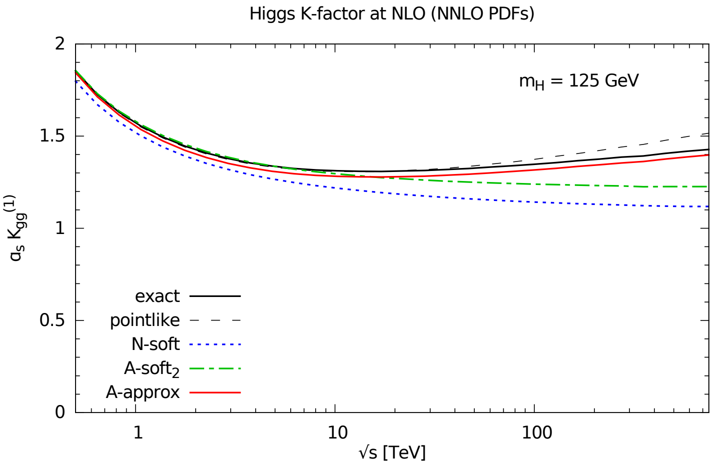


Parton level to hadron level: K-factors

$$\frac{\sigma_{gg}}{\sigma_{\rm LO}} = 1 + \alpha_s\, K_{gg}^{(1)} + \alpha_s^2\, K_{gg}^{(2)} + \ldots$$



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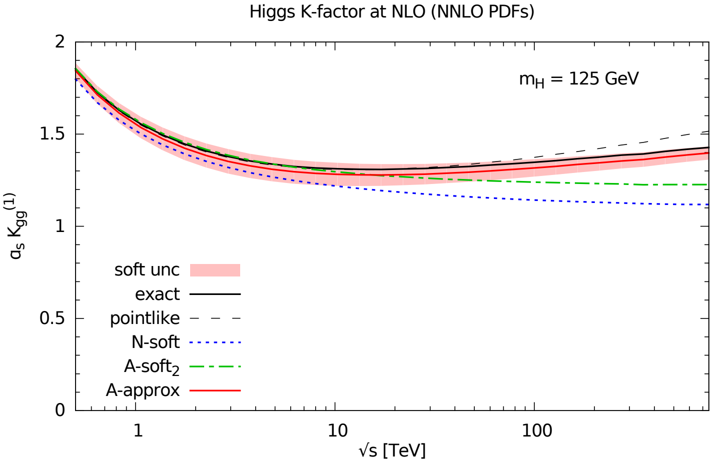

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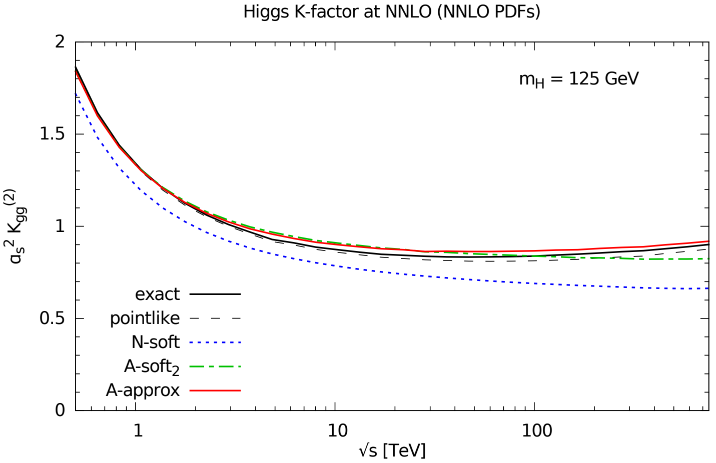

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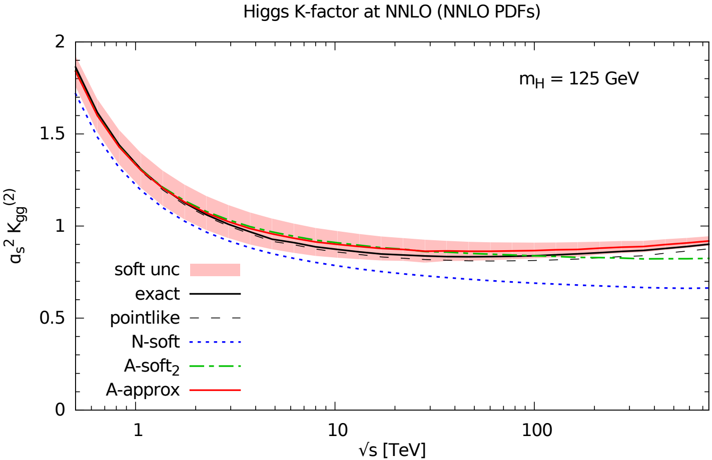
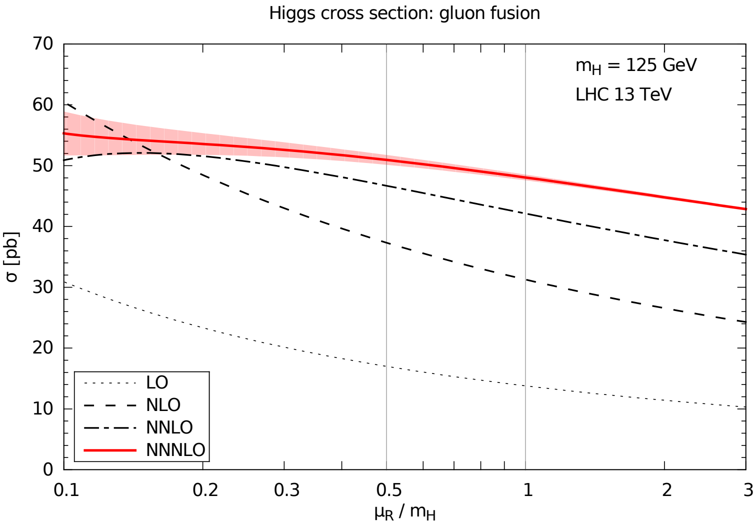


We can ignore h.e. uncertainty and rely on soft uncertainty only

N^3LO approximation at LHC



At N^3LO, μ_R scale dependence is exact with all channels

All-order A-soft resummation

Extension of A-soft to all orders trivial (though numerically challenging).

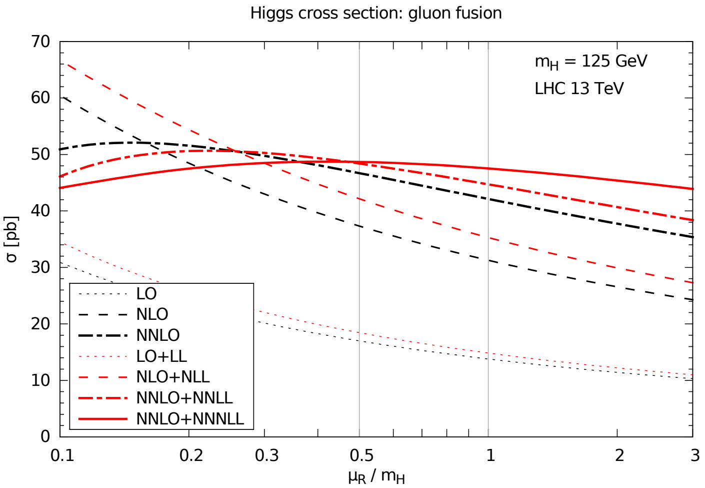
At resummed level, we also exponentiate $\bar{g}_0 = \exp \bar{h}_0$ [MB,Marzani 1405.3654]

pert. order	$\delta(1-z)$ coeff.	g_0	h_0	$\bar{g}_0$	$\bar{h}_0$
1	4.9374	8.7153	8.7153	4.9374	4.9374
2	8.94	40.10	2.122	10.92	-1.269
3	44.45	116.7	-12.1	2.0	-11.8

$\bar{g}_0$, $\log \bar{g}_0$ and $\log g_0$ have a much better behaviour than g_0

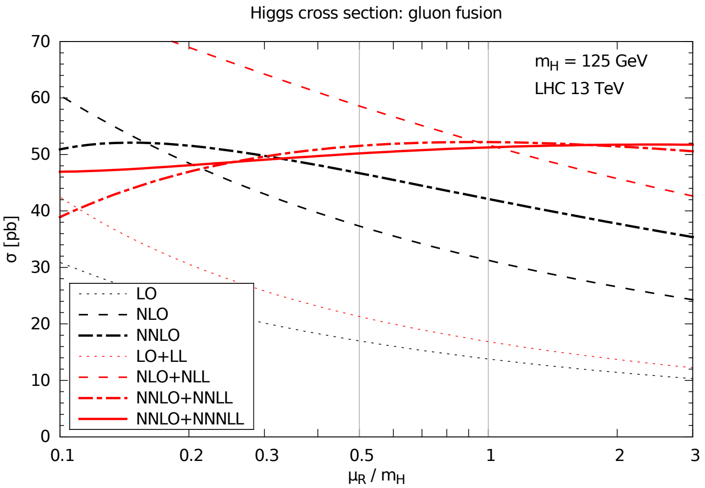
Similar to π^2 resummation [Ahrens,Becher,Neubert,Yang 0808.3008]

NNLO+N^{3}LL': comparison to N-soft



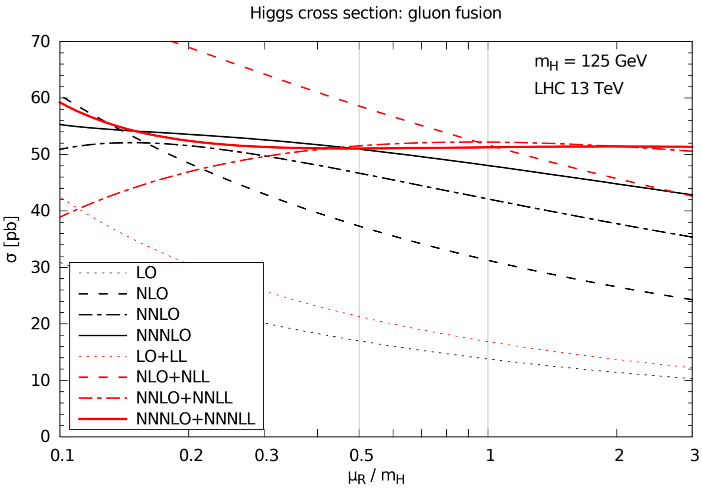
N-soft: poor convergence

NNLO+N^{3}LL': comparison to N-soft



A-soft: much better convergence!

N^3LO+N^3LL': effect of N^3LO



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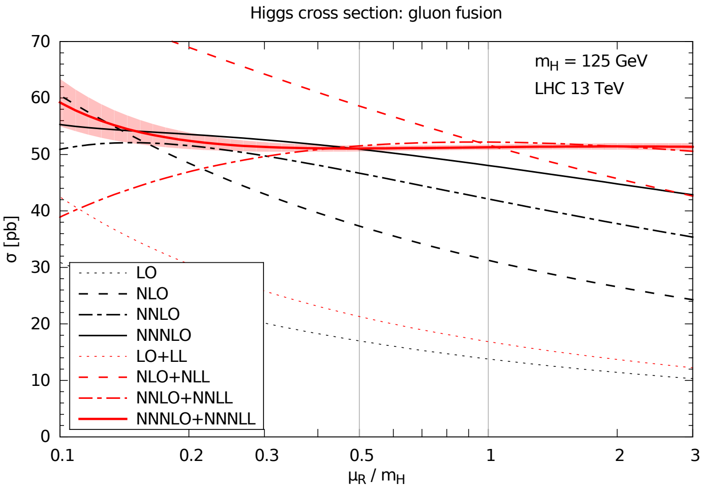

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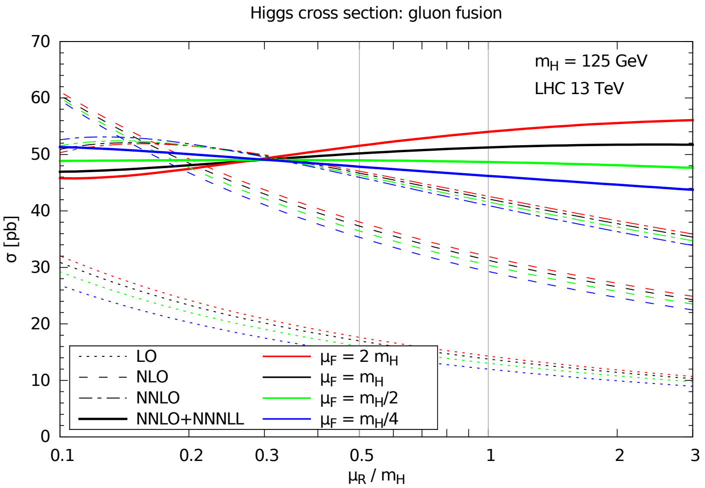
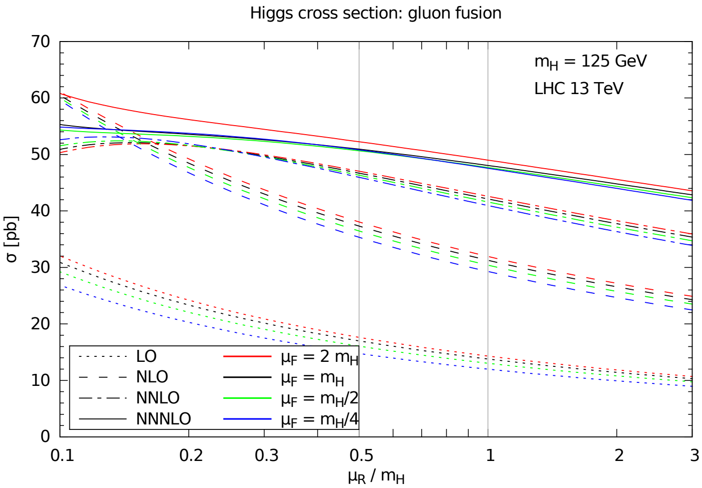
Approximation uncertainty as small as (or smaller than) scale dependence

Scale dependence

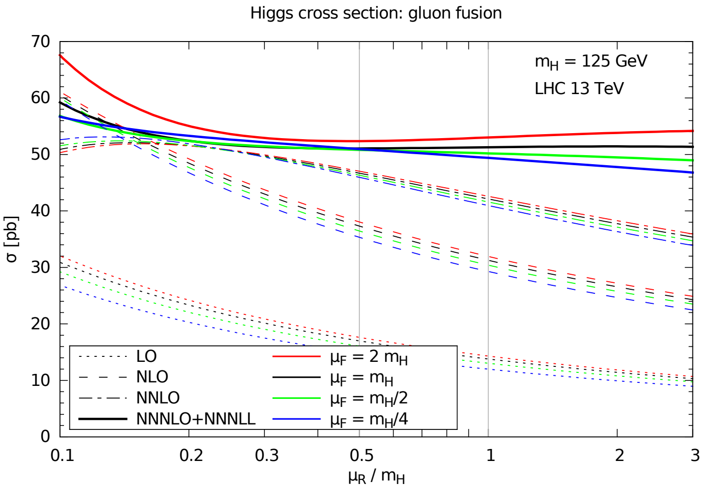


Scale dependence



At N^3LO, μ_R and μ_F scale dependence is exact with all channels

Scale dependence



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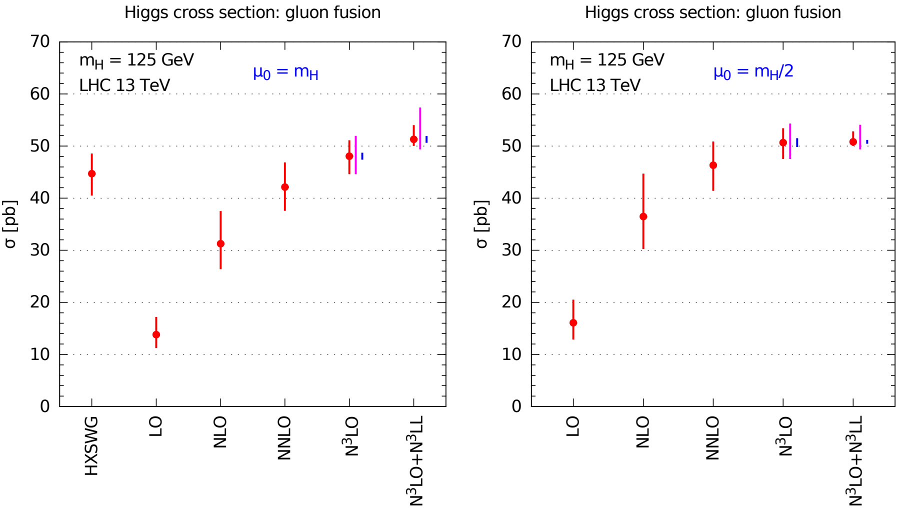
Final results



red: canonical scale variation (CSV)

purple: envelope of CSV for each soft variation

blue: soft variation at central scale only (to be summed to red)

Conclusions: central value

Our suggestion

$$\text{N}^3\text{LO}_{\text{approx}} + \text{N}^3\text{LL'}$$

- Fixed-order part at $\text{N}^3\text{LO}_{\text{approx}}$
  - up to NNLO known exactly (even with m_t corrections)
  - our approximate prediction is based on A-soft and LL high-energy logarithms
  - validated against known lower orders
  - finite m_t dependence almost complete (crucial at large $\sqrt{s}$)
- Resummed part at $\text{N}^3\text{LL'}$
  - one logarithmic order more than current recommendation (NNLL')
  - computed using A-soft: much better agreement than N-soft when expanded and compared against known orders, and faster convergence
  - finite m_t dependence almost complete
- Public codes `ggHiggs`, `ResHiggs`: `http://www.ge.infn.it/~bonvini/higgs/`
- Possible developments: high-energy LL resummation; other partonic channels

Conclusions: uncertainty

- Scale dependence

	$\mu_R = \mu_F = m_H$		$\mu_R = \mu_F = m_H/2$	
NNLO	$42.1^{+10.8\%}_{-10.4\%}$		$46.3^{+10.5\%}_{-10.2\%}$	
N^{3}LO	$48.1^{+6.0\%}_{-6.8\%}$	$(+14.3\%)$	$50.7^{+6.1\%}_{-5.9\%}$	$(+9.5\%)$
N^{3}LO+N^{3}LL'	$51.3^{+5.0\%}_{-2.1\%}$	$(+21.9\%)$	$50.8^{+4.6\%}_{-1.2\%}$	$(+9.7\%)$

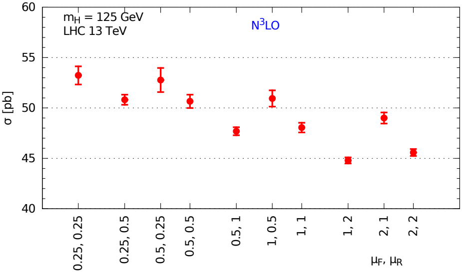
  - reduction of scale unc: $21\% > 13\% > 7\%$
  - stabilisation upon central scale choice: $10\% > 5\% > 1\%$
  - faster convergence and smaller scale dep at $\mu_R = \mu_F = m_H/2$

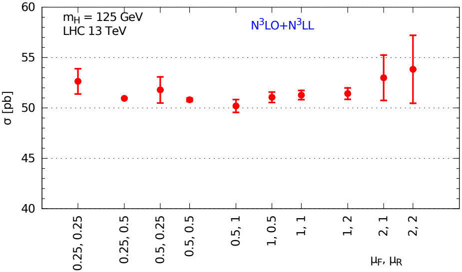
- Approximation uncertainty
  - at central scale: negligible compared to scale uncertainty
  - envelope at each scale point: few percent more at N^{3}LL'

- Quark channels uncertainty
  - we are assuming that quark channels are zero at N^{3}LO
  - if we don't, we get an additional 1-2% uncertainty
  - estimate from recent soft computation: -0.4%

Benchmark numbers (finite m_t)

μ_F/m_H	μ_R/m_H	LO	NLO	NNLO	N^{3}LO	N^{3}LO+N^{3}LL'
2	2	11.8	27.2	38.3	45.6 ± 0.3	53.8 ± 3.4
2	1	14.3	31.9	42.6	49.0 ± 0.6	53.0 ± 2.3
1	2	11.4	26.6	37.7	44.8 ± 0.3	51.4 ± 0.6
1	1	13.8	31.3	42.1	48.1 ± 0.5	51.3 ± 0.5
1	0.5	17.0	37.3	46.7	50.9 ± 0.8	51.1 ± 0.5
0.5	1	13.0	30.4	41.6	47.7 ± 0.4	50.2 ± 0.6
0.5	0.5	16.1	36.5	46.3	50.7 ± 0.7	50.8 ± 0.2
0.5	0.25	20.3	44.5	50.6	52.8 ± 1.2	51.8 ± 1.3
0.25	0.5	14.8	35.3	46.0	50.8 ± 0.5	51.0 ± 0.03
0.25	0.25	18.7	43.5	50.7	53.2 ± 0.9	52.6 ± 1.3





Benchmark numbers (infinite m_t effective theory)

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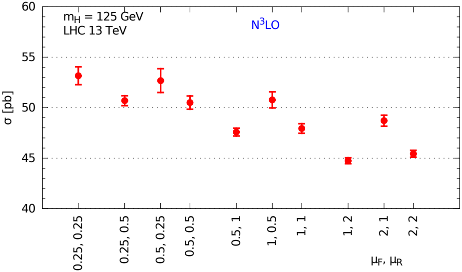

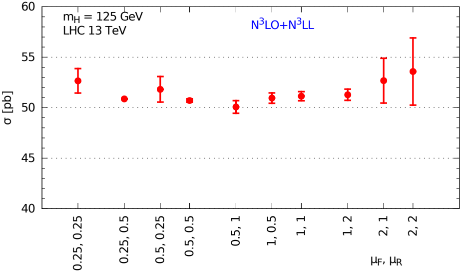

Comparisons

Comparison with other groups: soft expansion

$$L(z) = \log(1-z), \qquad \mathcal{D}_k(z) = \big(L^k(z)/(1-z)\big)_+$$

$$\begin{aligned} C_{gg}^{(3)}(z) &= a_{05}\mathcal{D}_5(z) + a_{04}\mathcal{D}_4(z) + a_{03}\mathcal{D}_3(z) + a_{02}\mathcal{D}_2(z) + a_{01}\mathcal{D}_1(z) + a_{00}\mathcal{D}_0(z) + a_{0\delta}\delta(1-z) \\ &+ a_{15}L^5(z) + a_{14}L^4(z) + a_{13}L^3(z) + a_{12}L^2(z) + a_{11}L(z) + a_{10} \\ &+ \Big[a_{25}L^5(z) + a_{24}L^4(z) + a_{23}L^3(z) + a_{22}L^2(z) + a_{21}L(z) + a_{20}\Big] \times (1-z) \\ &+ \ldots \end{aligned}$$

$$\begin{aligned} C_{gg}^{(3)}(N) &= c_{05}\ln^6 N + c_{04}\ln^5 N + c_{03}\ln^4 N + c_{02}\ln^3 N + c_{01}\ln^2 N + c_{00}\ln N + c_{0\delta} \\ &+ \Big[c_{15}\ln^5 N + c_{14}\ln^4 N + c_{13}\ln^3 N + c_{12}\ln^2 N + c_{11}\ln N + c_{10}\Big] \times \frac{1}{N} \\ &+ \Big[c_{25}\ln^5 N + c_{24}\ln^4 N + c_{23}\ln^3 N + c_{22}\ln^2 N + c_{21}\ln N + c_{20}\Big] \times \frac{1}{N^2} \\ &+ \ldots \end{aligned}$$

One-to-one correspondence between the two expansions.

To determine c_{pk}, you need all a_{ij} with $i \le p$, $j \ge k$.

Comparison with other groups: soft expansion

$$L(z) = \log(1-z), \qquad \mathcal{D}_k(z) = \big(L^k(z)/(1-z)\big)_+$$

$$\begin{aligned} C_{gg}^{(3)}(z) = {} & a_{05}\mathcal{D}_5(z) + a_{04}\mathcal{D}_4(z) + a_{03}\mathcal{D}_3(z) + a_{02}\mathcal{D}_2(z) + a_{01}\mathcal{D}_1(z) + a_{00}\mathcal{D}_0(z) + a_{0\delta}\delta(1-z) \\ & + a_{15}L^5(z) + a_{14}L^4(z) + a_{13}L^3(z) + a_{12}L^2(z) + a_{11}L(z) + a_{10} \\ & + \Big[a_{25}L^5(z) + a_{24}L^4(z) + a_{23}L^3(z) + a_{22}L^2(z) + a_{21}L(z) + a_{20}\Big] \times (1-z) \\ & + \dots \end{aligned}$$

$$\begin{aligned} C_{gg}^{(3)}(N) = {} & c_{05}\ln^6 N + c_{04}\ln^5 N + c_{03}\ln^4 N + c_{02}\ln^3 N + c_{01}\ln^2 N + c_{00}\ln N + c_{0\delta} \\ & + \Big[c_{15}\ln^5 N + c_{14}\ln^4 N + c_{13}\ln^3 N + c_{12}\ln^2 N + c_{11}\ln N + c_{10}\Big] \times \frac{1}{N} \\ & + \Big[c_{25}\ln^5 N + c_{24}\ln^4 N + c_{23}\ln^3 N + c_{22}\ln^2 N + c_{21}\ln N + c_{20}\Big] \times \frac{1}{N^2} \\ & + \dots \end{aligned}$$

Legend: correct, zero, non-zero by choice, non-zero by Mellin conversion

- Current HXSWG recommendation: NNLL' [deFlorian,Grazzini 1206.4133]

Comparison with other groups: soft expansion

$$L(z) = \log(1-z), \qquad \mathcal{D}_k(z) = \big(L^k(z)/(1-z)\big)_+$$

$$C_{gg}^{(3)}(z) = a_{05}\mathcal{D}_5(z) + a_{04}\mathcal{D}_4(z) + a_{03}\mathcal{D}_3(z) + a_{02}\mathcal{D}_2(z) + a_{01}\mathcal{D}_1(z) + a_{00}\mathcal{D}_0(z) + a_{0\delta}\delta(1-z)$$
$$+\ a_{15}L^5(z) + a_{14}L^4(z) + a_{13}L^3(z) + a_{12}L^2(z) + a_{11}L(z) + a_{10}$$
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$$+ \dots$$

$$C_{gg}^{(3)}(N) = c_{05}\ln^6 N + c_{04}\ln^5 N + c_{03}\ln^4 N + c_{02}\ln^3 N + c_{01}\ln^2 N + c_{00}\ln N + c_{0\delta}$$
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$$+\Big[c_{25}\ln^5 N + c_{24}\ln^4 N + c_{23}\ln^3 N + c_{22}\ln^2 N + c_{21}\ln N + c_{20}\Big] \times \frac{1}{N^2}$$
$$+ \dots$$

Legend: correct, zero, non-zero by choice, non-zero by Mellin conversion

- [Moch,Vogt hep-ph/0508265]

Comparison with other groups: soft expansion

$$L(z) = \log(1-z), \qquad \mathcal{D}_k(z) = \left(L^k(z)/(1-z)\right)_+$$

$$\begin{aligned}
C_{gg}^{(3)}(z) = {} & a_{05}\mathcal{D}_5(z) + a_{04}\mathcal{D}_4(z) + a_{03}\mathcal{D}_3(z) + a_{02}\mathcal{D}_2(z) + a_{01}\mathcal{D}_1(z) + a_{00}\mathcal{D}_0(z) + a_{0\delta}\delta(1-z) \\
& + a_{15}L^5(z) + a_{14}L^4(z) + a_{13}L^3(z) + a_{12}L^2(z) + a_{11}L(z) + a_{10} \\
& + \Big[a_{25}L^5(z) + a_{24}L^4(z) + a_{23}L^3(z) + a_{22}L^2(z) + a_{21}L(z) + a_{20}\Big] \times (1-z) \\
& + \dots
\end{aligned}$$

$$\begin{aligned}
C_{gg}^{(3)}(N) = {} & c_{05}\ln^6 N + c_{04}\ln^5 N + c_{03}\ln^4 N + c_{02}\ln^3 N + c_{01}\ln^2 N + c_{00}\ln N + c_{0\delta} \\
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& + \Big[c_{25}\ln^5 N + c_{24}\ln^4 N + c_{23}\ln^3 N + c_{22}\ln^2 N + c_{21}\ln N + c_{20}\Big] \times \frac{1}{N^2} \\
& + \dots
\end{aligned}$$

Legend: correct, zero, non-zero by choice, non-zero by Mellin conversion

- N-soft: [Moch,Vogt hep-ph/0508265] + constant

Comparison with other groups: soft expansion

$$L(z) = \log(1-z), \qquad \mathcal{D}_k(z) = \big(L^k(z)/(1-z)\big)_+$$

$$C_{gg}^{(3)}(z) = a_{05}\mathcal{D}_5(z) + a_{04}\mathcal{D}_4(z) + a_{03}\mathcal{D}_3(z) + a_{02}\mathcal{D}_2(z) + a_{01}\mathcal{D}_1(z) + a_{00}\mathcal{D}_0(z) + a_{0\delta}\delta(1-z)$$
$$+ a_{15}L^5(z) + a_{14}L^4(z) + a_{13}L^3(z) + a_{12}L^2(z) + a_{11}L(z) + a_{10}$$
$$+\Big[a_{25}L^5(z) + a_{24}L^4(z) + a_{23}L^3(z) + a_{22}L^2(z) + a_{21}L(z) + a_{20}\Big] \times (1-z)$$
$$+ \dots$$

$$C_{gg}^{(3)}(N) = c_{05}\ln^6 N + c_{04}\ln^5 N + c_{03}\ln^4 N + c_{02}\ln^3 N + c_{01}\ln^2 N + c_{00}\ln N + c_{0\delta}$$
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$$+ \dots$$

Legend: correct, zero, non-zero by choice, non-zero by Mellin conversion

- z-soft: [Anastasiou,Duhr,Dulat,Furlan,Gehrmann,Herzog,Mistlberger 1403.4616]

Comparison with other groups: soft expansion

$$L(z) = \log(1-z), \qquad \mathcal{D}_k(z) = \big(L^k(z)/(1-z)\big)_+$$

$$\begin{aligned} C_{gg}^{(3)}(z) &= a_{05}\mathcal{D}_5(z) + a_{04}\mathcal{D}_4(z) + a_{03}\mathcal{D}_3(z) + a_{02}\mathcal{D}_2(z) + a_{01}\mathcal{D}_1(z) + a_{00}\mathcal{D}_0(z) + a_{0\delta}\delta(1-z) \\ &+ a_{15}L^5(z) + a_{14}L^4(z) + a_{13}L^3(z) + a_{12}L^2(z) + a_{11}L(z) + a_{10} \\ &+ \Big[a_{25}L^5(z) + a_{24}L^4(z) + a_{23}L^3(z) + a_{22}L^2(z) + a_{21}L(z) + a_{20}\Big] \times (1-z) \\ &+ \ldots \end{aligned}$$

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Legend: correct, zero, non-zero by choice, non-zero by Mellin conversion

- z-soft-expansion: next-to-soft
  [Anastasiou,Duhr,Dulat,Furlan,Gehrmann,Herzog,Mistlberger 1411.3584]
  also [Bonocore, Laenen, Magnea, Vernazza, White 1410.6406]

Comparison with other groups: soft expansion

$$L(z) = \log(1-z), \qquad \mathcal{D}_k(z) = \big(L^k(z)/(1-z)\big)_+$$

$$\begin{aligned}
C_{gg}^{(3)}(z) &= a_{05}\mathcal{D}_5(z) + a_{04}\mathcal{D}_4(z) + a_{03}\mathcal{D}_3(z) + a_{02}\mathcal{D}_2(z) + a_{01}\mathcal{D}_1(z) + a_{00}\mathcal{D}_0(z) + a_{0\delta}\delta(1-z) \\
&+ a_{15}L^5(z) + a_{14}L^4(z) + a_{13}L^3(z) + a_{12}L^2(z) + a_{11}L(z) + a_{10} \\
&+ \Big[a_{25}L^5(z) + a_{24}L^4(z) + a_{23}L^3(z) + a_{22}L^2(z) + a_{21}L(z) + a_{20}\Big] \times (1-z) \\
&+ \dots
\end{aligned}$$

$$\begin{aligned}
C_{gg}^{(3)}(N) &= c_{05}\ln^6 N + c_{04}\ln^5 N + c_{03}\ln^4 N + c_{02}\ln^3 N + c_{01}\ln^2 N + c_{00}\ln N + c_{0\delta} \\
&+ \Big[c_{15}\ln^5 N + c_{14}\ln^4 N + c_{13}\ln^3 N + c_{12}\ln^2 N + c_{11}\ln N + c_{10}\Big] \times \frac{1}{N} \\
&+ \Big[c_{25}\ln^5 N + c_{24}\ln^4 N + c_{23}\ln^3 N + c_{22}\ln^2 N + c_{21}\ln N + c_{20}\Big] \times \frac{1}{N^2} \\
&+ \dots
\end{aligned}$$

Legend: correct, zero, non-zero by choice, non-zero by Mellin conversion

- A-soft: [Ball,MB,Forte,Marzani,Ridolfi 1404.3204] [MB,Marzani 1405.3654]

Comparison with other groups: soft expansion

$$L(z) = \log(1-z), \qquad \mathcal{D}_k(z) = \big(L^k(z)/(1-z)\big)_+$$

$$C_{gg}^{(3)}(z) = a_{05}\mathcal{D}_5(z) + a_{04}\mathcal{D}_4(z) + a_{03}\mathcal{D}_3(z) + a_{02}\mathcal{D}_2(z) + a_{01}\mathcal{D}_1(z) + a_{00}\mathcal{D}_0(z) + a_{0\delta}\delta(1-z)$$
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$$C_{gg}^{(3)}(N) = c_{05}\ln^6 N + c_{04}\ln^5 N + c_{03}\ln^4 N + c_{02}\ln^3 N + c_{01}\ln^2 N + c_{00}\ln N + c_{0\delta}$$
$$+\Big[c_{15}\ln^5 N + c_{14}\ln^4 N + c_{13}\ln^3 N + c_{12}\ln^2 N + c_{11}\ln N + c_{10}\Big] \times \frac{1}{N}$$
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$$+\ \ldots$$

Legend: correct, zero, non-zero by choice, non-zero by Mellin conversion

- dFMMV: [deFlorian,Mazzitelli,Moch,Vogt 1408.6277]

Comparison at next-to-soft

soft expansion: [Anastasiou,Duhr,Dulat,Furlan,Gehrmann,Herzog,Mistlberger 1411.3584]

$$C_{gg}^{(3)}(N) \sim \frac{1}{N}\left[\ln^5 N + 5.701 \ln^4 N + 18.86 \ln^3 N + 30.6 \ln^2 N + 32.0 \ln N + 6.5\right]$$

dFMMV: [deFlorian,Mazzitelli,Moch,Vogt 1408.6277]

$$C_{gg}^{(3)}(N) \sim \frac{1}{N}\left[\ln^5 N + 5.701 \ln^4 N + 18.9 \ln^3 N + 46 \ln^2 N + 18 \ln N + 9\right]$$

$$+50\% \qquad -44\% \qquad +38\%$$

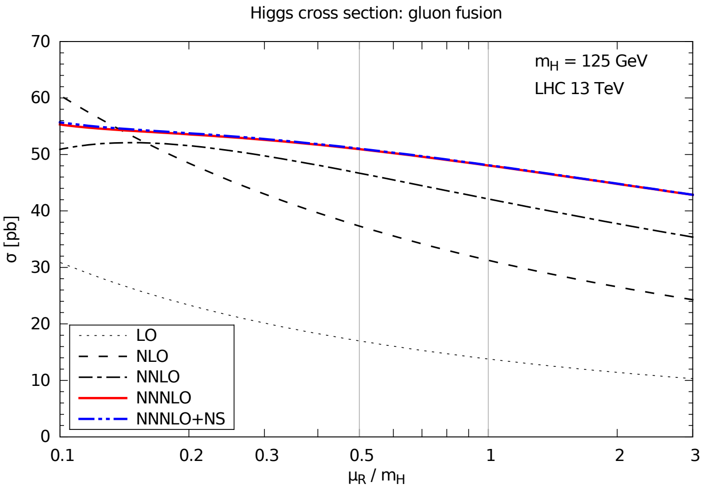
A-soft: [Ball,MB,Forte,Marzani,Ridolfi 1404.3204]

$$C_{gg}^{(3)}(N) \sim \frac{1}{N}\left[\ln^5 N + 3.951 \ln^4 N + 13.8 \ln^3 N + 19.5 \ln^2 N + 24.8 \ln N + 8.8\right]$$

$$-31\% \qquad -27\% \qquad -36\% \qquad -22\% \qquad +35\%$$

Remember: saddle point $N_0 \sim 2$ $\Rightarrow$ $\ln^5 N < \ln^4 N < \ldots < \ln^0 N$

Inclusion of exact next-to-soft



Hardly makes a difference...

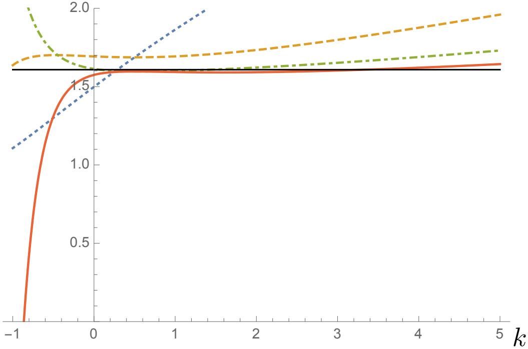
How important are next-to-next-to-soft corrections?

$$C(z) = z^k \left[\frac{C(z)}{z^k}\right]_{\text{soft-expanded}}$$

vary k to study the convergence of the soft expansion

NLO

Expansion up to order
0 (dotted),
1 (dashed),
2 (dot-dash),
3 (solid),
∞ (solid)



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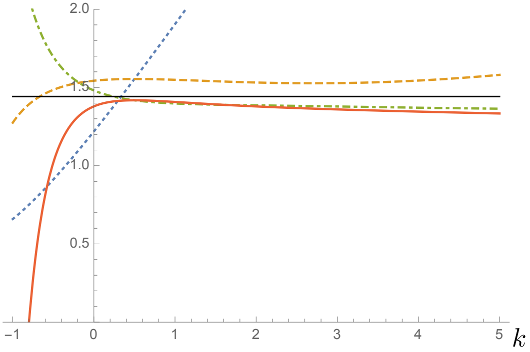

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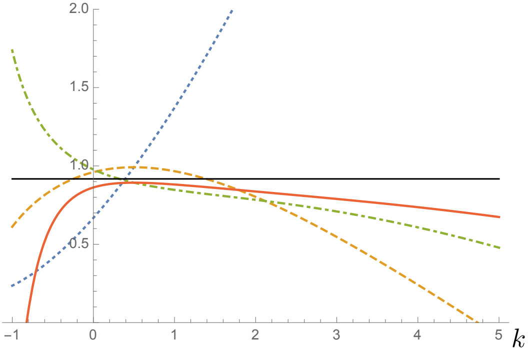


At N^3LO, at least 3-4 orders in a soft expansion are required to reach stability.
Faster convergence possible [some tests with F.Herzog]

Backup slides

High-energy part: construction

LL behaviour described by BFKL resummation:

[Marzani, Ball, Del Duca, Forte, Vicini 0801.2544]

$$C_{\text{high-energy}}(N, \alpha_s) = \sum_{k_1,k_2\geq 0} c_{k_1,k_2}\big(\tfrac{m_H}{m_t}\big)\big[\gamma_+{}^{k_1}\big]\big[\gamma_+{}^{k_2}\big]$$

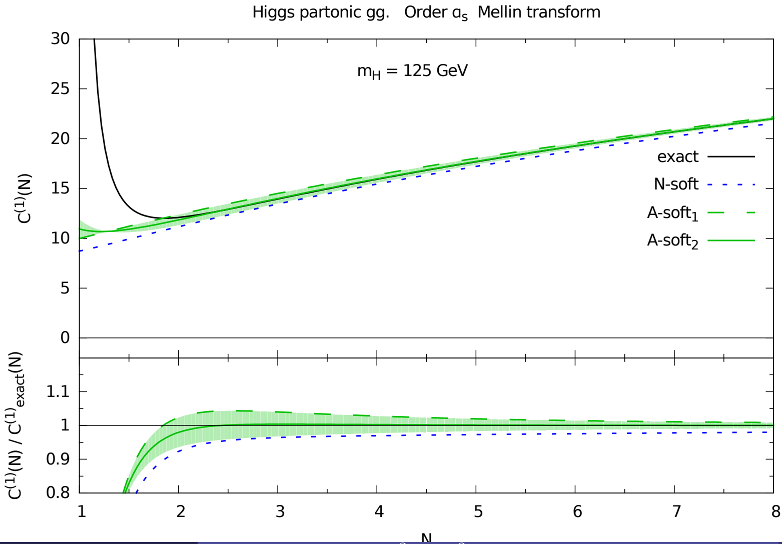
$\gamma_+(N)$: DGLAP anomalous dimension (largest eigenvalue)
c_{k_1,k_2}: coefficients determined from LO xs with off-shell gluons

- formally accurate only at LL (with running coupling effects)
- resummation more complicated (work in progress)
- momentum conservation $C_{\text{high-energy}}(N = 2, \alpha_s) = 0$, but grows at $N \to \infty$
  Halt! The $N \to \infty$ behaviour is jurisdiction of the soft part!
- we use an expansion of $\gamma_+(N)$ about $N = 1$ to NLL:
  removes the growth ✓, but we loose momentum conservation
- we enforce mom. cons. by adding subdominant terms (poles at $N \leq 0$)
- we vary subleading terms as a measure of the uncertainty

[Ball,MB,Forte,Marzani,Ridolfi 1303.3590]

Soft part: validation

$$C_{gg}(N,\alpha_s) = 1 + \alpha_s\, C^{(1)}_{gg}(N) + \alpha_s^2\, C^{(2)}_{gg}(N) + \dots$$



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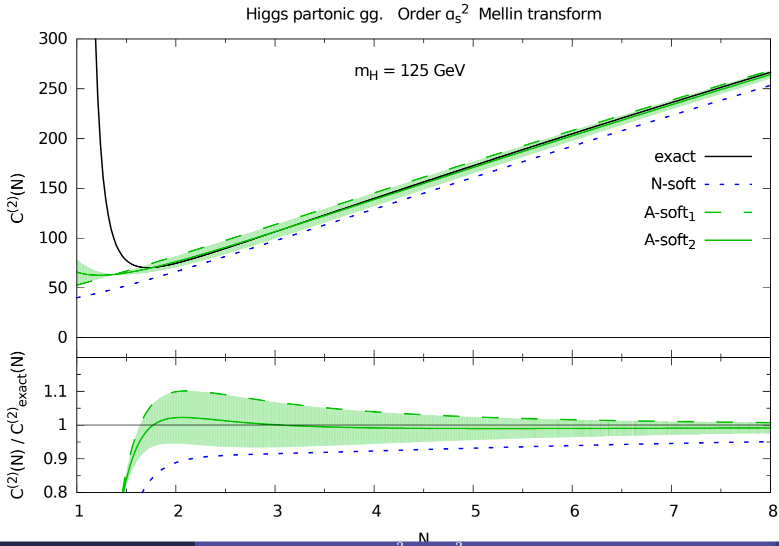
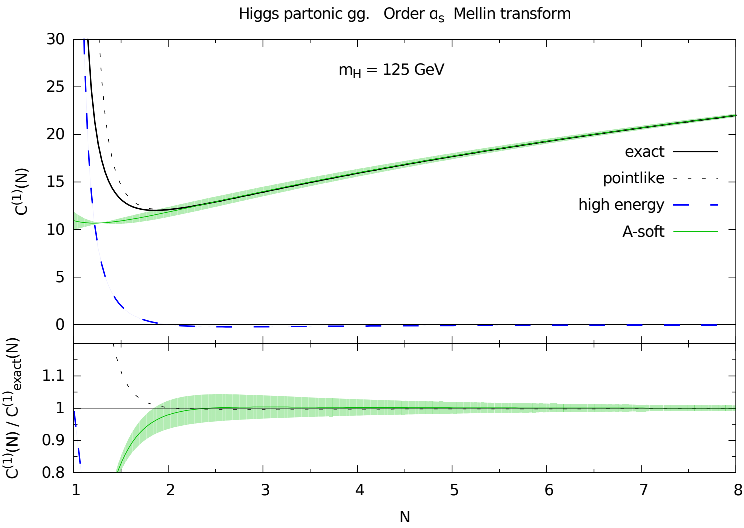


Full approximation: A-soft + high-energy

$$C_{gg}(N,\alpha_s) = 1 + \alpha_s\, C^{(1)}_{gg}(N) + \alpha_s^2\, C^{(2)}_{gg}(N) + \alpha_s^3\, C^{(3)}_{gg}(N) + \dots$$



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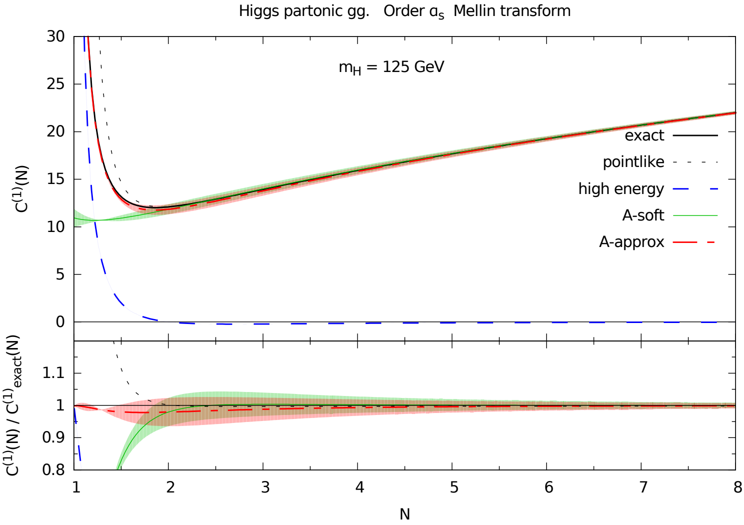

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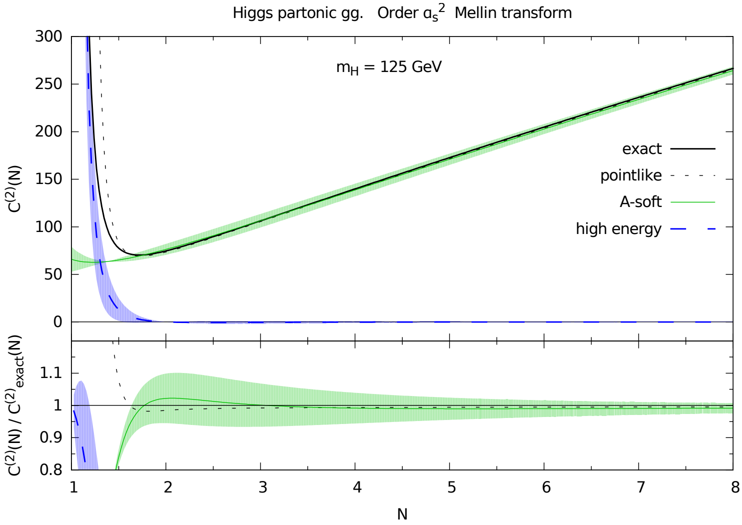

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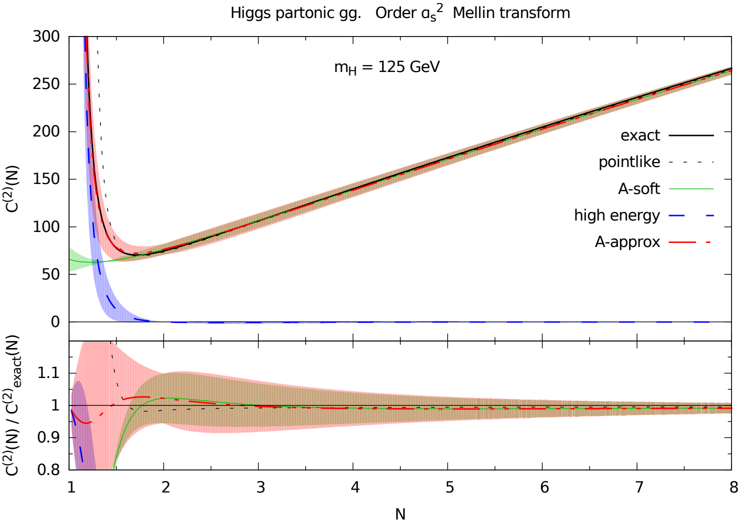

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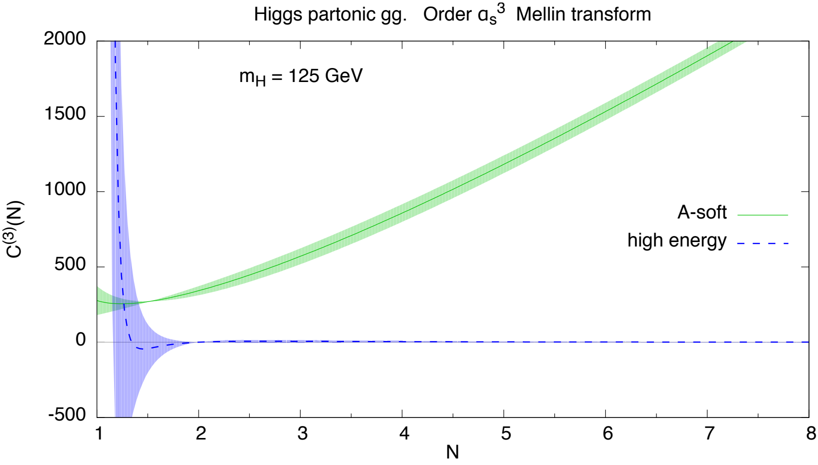

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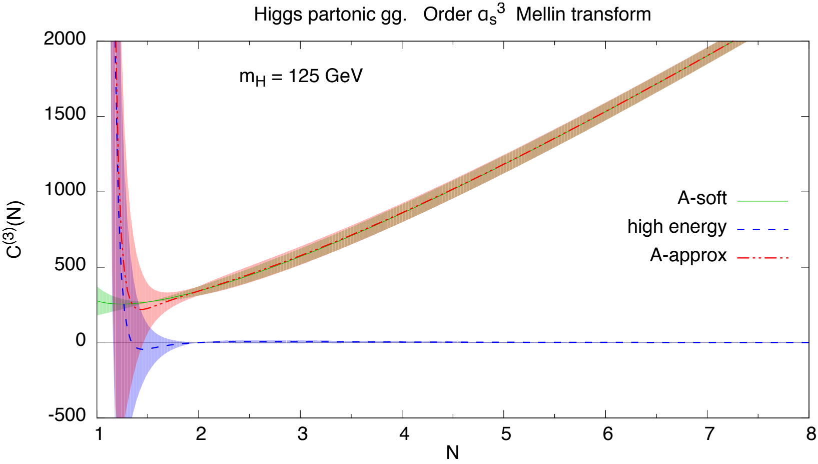
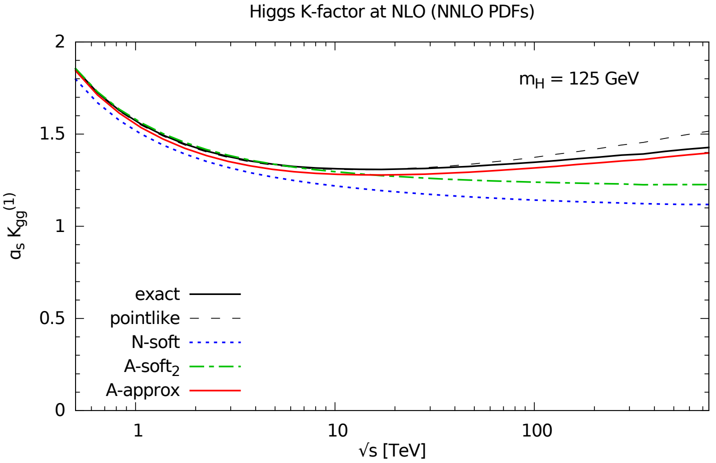


Parton level to hadron level: K-factors

$$\frac{\sigma_{gg}}{\sigma_{\rm LO}} = 1 + \alpha_s\, K_{gg}^{(1)} + \alpha_s^2\, K_{gg}^{(2)} + \alpha_s^3\, K_{gg}^{(3)} + \ldots$$



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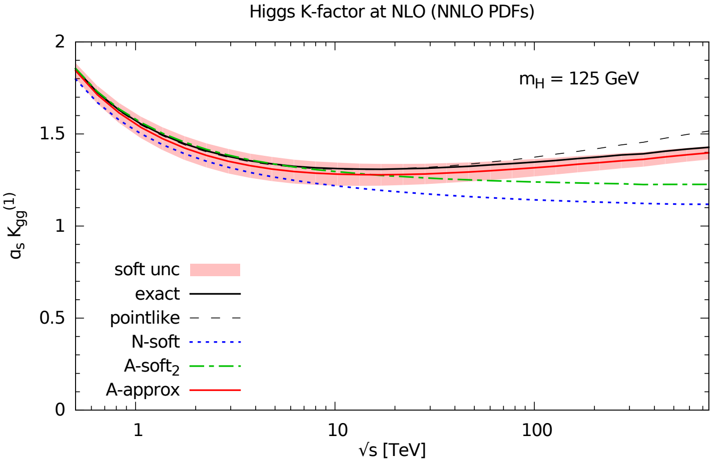

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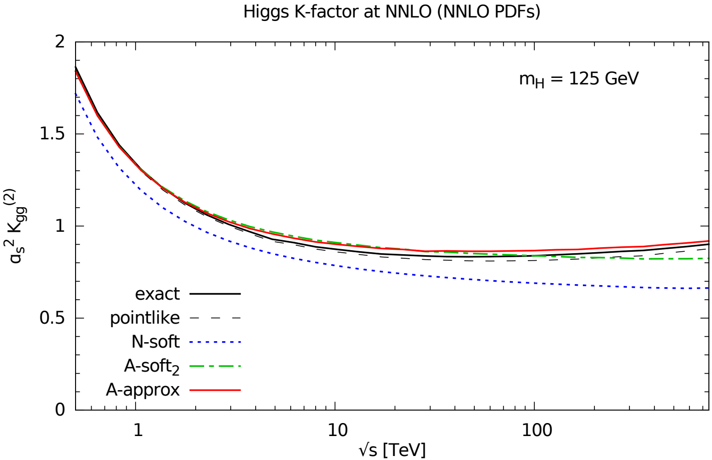
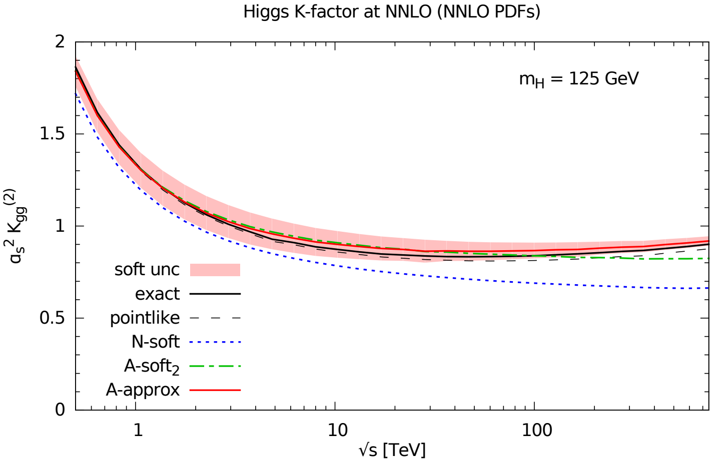


Parton level to hadron level: K-factors

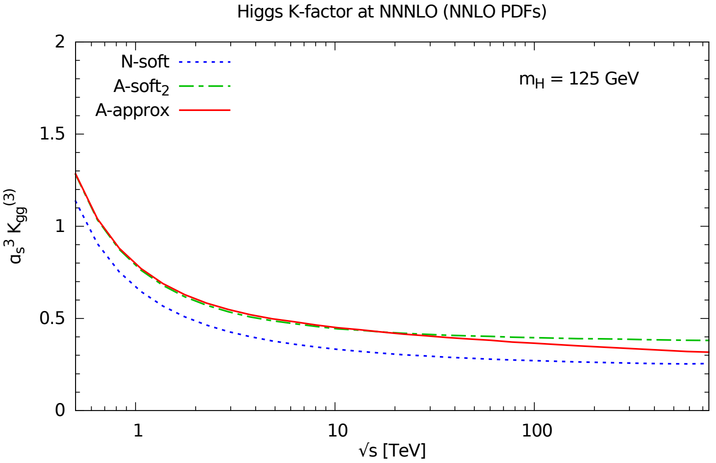
$$\frac{\sigma_{gg}}{\sigma_{\rm LO}} = 1 + \alpha_s\, K^{(1)}_{gg} + \alpha_s^2\, K^{(2)}_{gg} + \alpha_s^3\, K^{(3)}_{gg} + \dots$$



We can ignore h.e. uncertainty and rely on soft uncertainty only

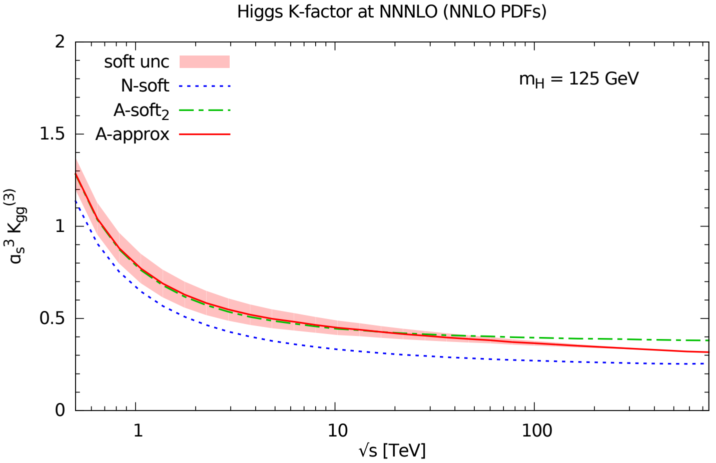
Parton level to hadron level: K-factors

$$\frac{\sigma_{gg}}{\sigma_{\mathrm{LO}}} = 1 + \alpha_s\, K^{(1)}_{gg} + \alpha_s^2\, K^{(2)}_{gg} + \alpha_s^3\, K^{(3)}_{gg} + \dots$$

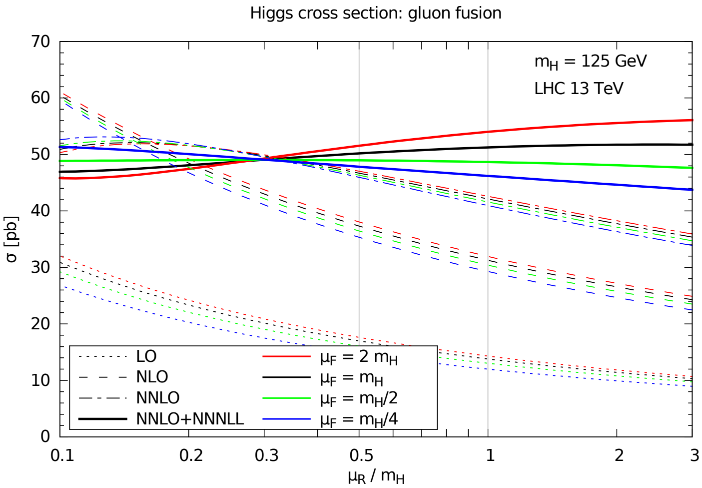


Parton level to hadron level: K-factors

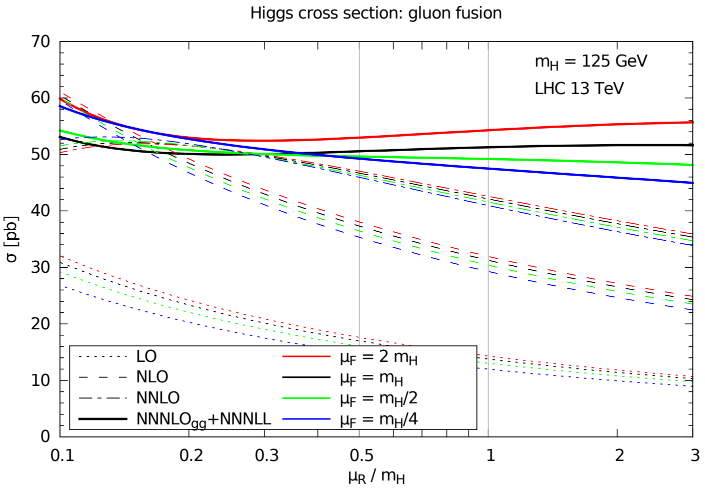
$$\frac{\sigma_{gg}}{\sigma_{\rm LO}} = 1 + \alpha_s\, K_{gg}^{(1)} + \alpha_s^2\, K_{gg}^{(2)} + \alpha_s^3\, K_{gg}^{(3)} + \dots$$



More on scale dependence

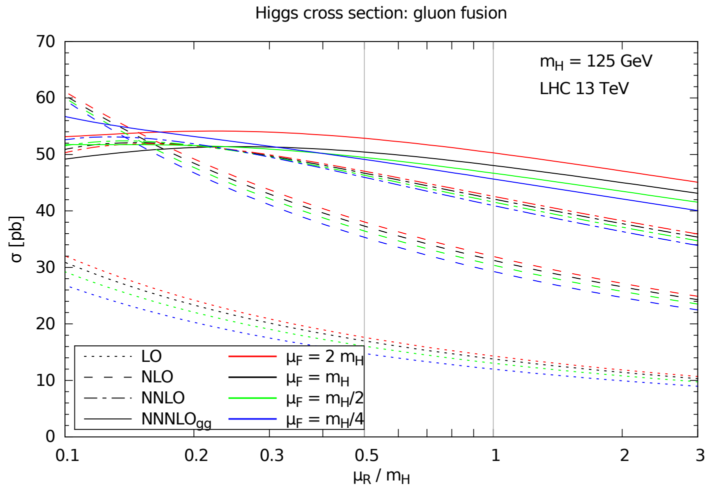


More on scale dependence



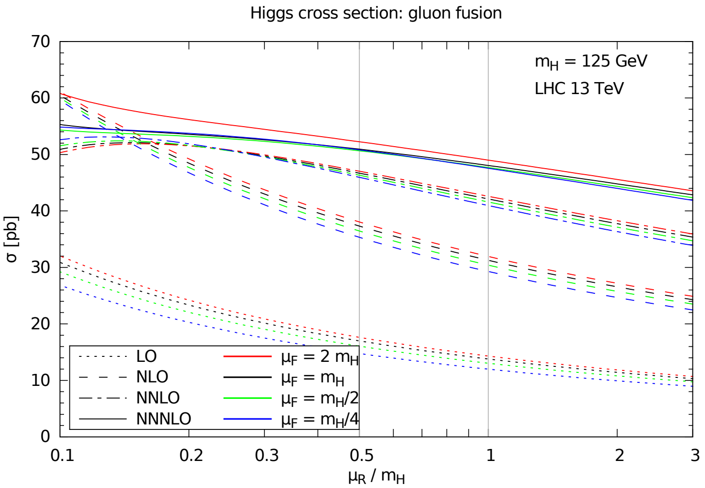
At N^3LO, $\mu_{\rm R}$ and $\mu_{\rm F}$ scale dependence is exact but **only gg channel**

More on scale dependence



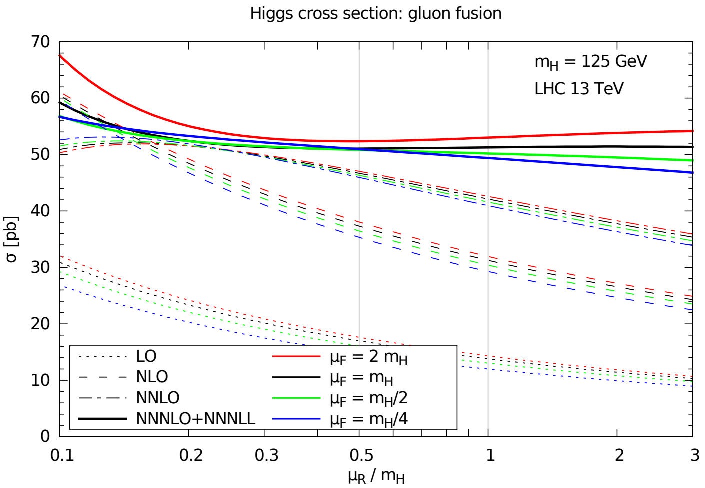
At N^3LO, μ_R and μ_F scale dependence is exact but **only** gg **channel**

More on scale dependence



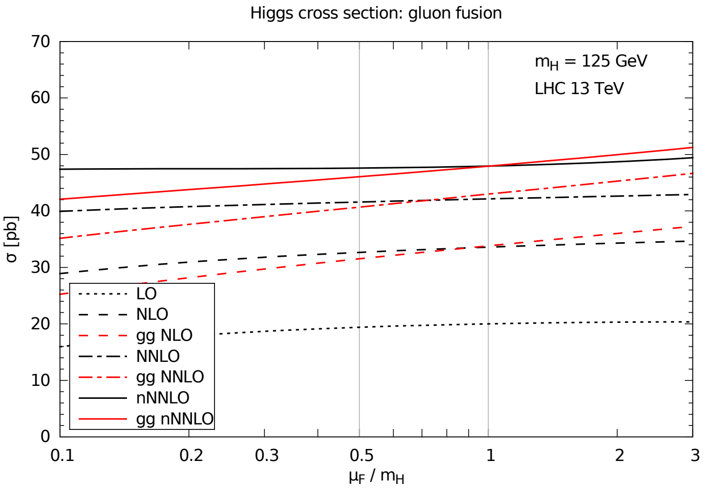
At N^3LO, $\mu_{\rm R}$ and $\mu_{\rm F}$ scale dependence is exact with **all channels**

More on scale dependence

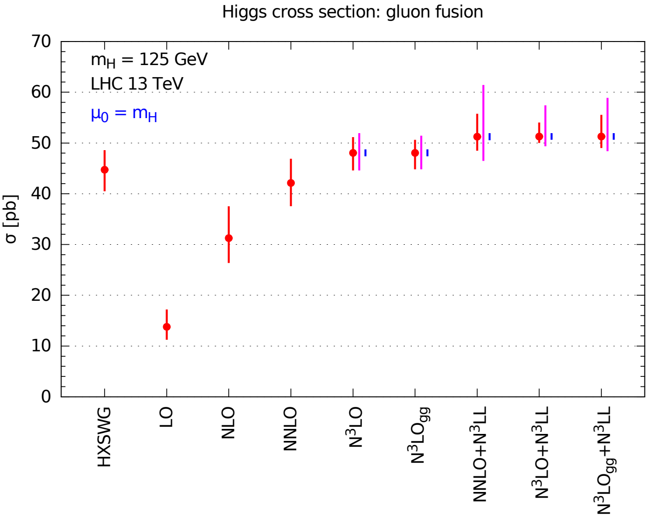


At N^3LO, μ_R and μ_F scale dependence is exact with **all channels**

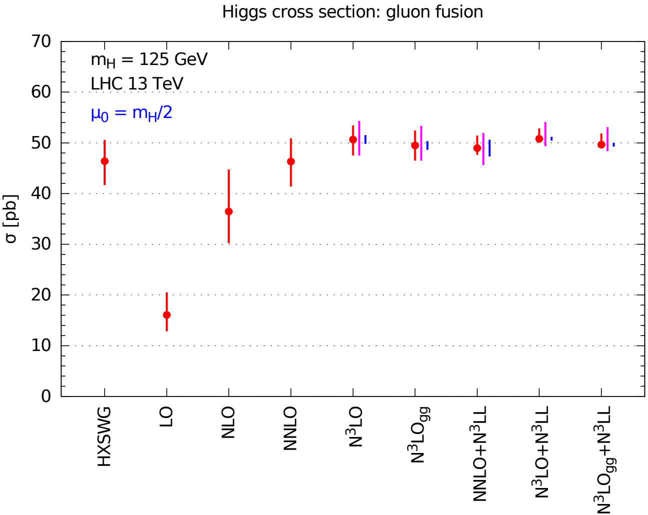
More on scale dependence



More on final results



More on final results



Threshold resummation: logarithmic counting

$$C(N,M^2) = 1 + \sum_{n=1}^{\infty} \alpha_s^n \sum_{k=0}^{2n} c_{nk} \ln^k N \qquad \ln C(N,M^2) = \sum_{n=1}^{\infty} \alpha_s^n \sum_{k=0}^{n+1} \hat{b}_{nk} \ln^k N$$

	$A(\alpha_s)$	$D(\alpha_s)$	$\bar{g}_0(\alpha_s)$	accuracy: c_{nk}	$\hat{b}_{nk}$
LL	1-loop	—	tree-level	$k = 2n$	$k = n+1$
NLL	2-loop	1-loop	tree-level	$2n-1 \le k \le 2n$	$n \le k \le n+1$
NLL'	2-loop	1-loop	1-loop	$2n-2 \le k \le 2n$	$n \le k \le n+1$
NNLL	3-loop	2-loop	1-loop	$2n-3 \le k \le 2n$	$n-1 \le k \le n+1$
NNLL'	3-loop	2-loop	2-loop	$2n-4 \le k \le 2n$	$n-1 \le k \le n+1$
NNNLL	4-loop	3-loop	2-loop	$2n-5 \le k \le 2n$	$n-2 \le k \le n+1$
NNNLL'	4-loop	3-loop	3-loop	$2n-6 \le k \le 2n$	$n-2 \le k \le n+1$

Un-primed counting: appropriate for $\ln C(N,M^2)$, assumes $\alpha_s \ln N \sim 1$
Primed counting: more appropriate for $C(N,M^2)$, assumes just $\alpha_s \ln^2 N \sim 1$

NNLL': [Catani, de Florian, Grazzini, Nason 2003] [de Florian, Grazzini 2012]
NNNLL: [Ahrens, Becher, Neubert, Yang 2008] (within SCET)
NNNLL': [MB, Marzani 2014]