Mathematical model of magnetically interacting rigid bodies

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Introduction

This presentation describes new results of investigating the dynamics of the magnetic interaction of rigid bodies and continues the cycle of papers devoted to the investigation of contact-free equilibrium of rigid bodies in magnetostatics. These papers show that the magnetic interaction for a wide class of magnetic bodies such as permanent magnets, inductance coils (superconductive and with direct current) and their different combinations ("mixed"type systems) can be described through potential energy of their interaction received from the Lagrangian formalism of electromechanical analogy. It was also shown, that there are such magnetic configurations of rigid bodies, including superconductive elements, that potential energy has the minimum. Such systems with stable magnetic equilibrium are called "Magnetic Potential Well"(S.S.Zub,V.M.Rashkovan [1,2]).

Introduction

To investigate not only quasistatic models but also dynamic stable configurations, as well as to consider a larger number of tasks (confinement, scattering, orbital motion) an adequate mathematical apparatus for investigating the dynamics of such systems is required. Such mathematical apparatus is Hamiltonian formalism, presented below.

In this presentation this formalism is given for two-body problem with magnetic (i.e. non-central force) interaction. Having the available Lagrangian formalism for magnetic interaction of bodies (S.S.Zub [3,2]) and the trivial expansion of the Poisson structures given in this paper it is easy to generalize this approach for describing the dynamics of $N > 2$ rigid bodies (symmetric top model).

Introduction

It should be noted that the Poisson structures were used when trying to classically describe the magnetic interaction of spins (Mario Feingold, Asher Peres; E. Magyari, H. Thomas, R. Weber, C. Kaufman and G. Miiller; N. Srivastava, C. Kaufman, G. Miiller, E. Magyari, H. Thomas, R. Weber [4,5,6,7]). These papers do not consider spatial motion of bodies, i.e. spatial variables are absent. The energy of interaction given in these papers cannot be used even for describing classic magnetic dipoles.

A more realistic description of magnetic interaction of two magnets was given by V.V. Kozoriz based on Lagrangian formalism (V.V. Kozoriz [8]). But the mathematical apparatus he uses does not give any description of "mixed type"systems. Moreover, it is well known that the generalized coordinates used (Euler's angles) cannot correctly map all orientations of a rigid body, which becomes apparent in the peculiarities of the coefficients of the differential equations of motion.

Introduction

We developed a formalism which results in the coordinate-free, i.e. the vector form of motion equations for the system of magnetically interactive bodies (including "mixed type"systems).

All Poisson brackets, expressions for motion equations, as well as the law of total momentum conservation of the system were tested by means of the analytical possibilities of the Maple system, but for the computational modeling of the orbital motion of two magnets MatLab is more preferable.

Hamiltonian formalism for two magnetic-interacting bodies

- Definition of the Poisson structure.

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- Definition of the Poisson structure.
- Poisson structures related with the dynamics of a rigid body.

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- Poisson structure for two magnetic-interacting bodies.

Definition of the Poisson structure

Definition

A Poisson bracket (or a Poisson structure) on a manifold M is a bilinear operation $\{,\}$ on $\mathscr{F}_M \equiv C^\infty(M)$: $F, G \in \mathscr{F}_M \Rightarrow \{F, G\} \in \mathscr{F}_M$.
Following Dirac the elements $\mathscr{F}_M$ will be referred to as dynamic variables.

Poisson bracket satisfies the next elemental rules [9,10]:

$$\begin{cases} \{\alpha_1 F_1 + \alpha_2 F_2, G\} = \alpha_1\{F_1, G\} + \alpha_2\{F_2, G\}; \quad \alpha_1, \alpha_2 = const; \\ \{F, G\} = -\{G, F\}; \\ \{G, F_1 F_2\} = \{G, F_1\}F_2 + F_1\{G, F_2\}; \\ \{F_1\{F_2, F_3\}\} + \{F_2\{F_3, F_1\}\} + \{F_3\{F_1, F_2\}\} = 0; \end{cases}$$

The last of these formulas is the Jacobi identity for Poisson brackets. Satisfying this identity jointly with the 1st and 2nd properties gives the space of dynamic variables a structure of Lee algebra with regard to

Definition of the Poisson structure

Definition

$G \in \mathscr{F}_M$ is named the Casimir function if $\{G, F\} = 0 \quad \forall F \in \mathscr{F}_M$ i.e. Poisson bracket of the dynamic variable G with any other dynamic variable vanish.

If $x^i, i = 1..dim(M)$ is a coordinate system on the Poisson manifold the following expression for Poisson brackets of all dynamic variables results from the properties of the Poisson structure:

$$\{F, G\} = \sum_{i,k} \{x^i, x^k\} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^k}; \tag{1}$$

Base Poisson brackets are structural functions of a Poisson manifold M with regard to a given (generally speaking, local)coordinate system. They form a structural tensor (matrix) J^{ik} with dimension $dim(M) \times dim(M)$:

$$J^{ik} = \{x^i, x^k\} \tag{2}$$

Definition of the Poisson structure

Thus formula (1) can be written like this

$$\{F, G\} = \sum_{i,k} J^{ik} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^k}; \tag{1a}$$

The structural tensor has the following properties: antisymmetry and

$$J^{ik} = -J^{ki}; \tag{3}$$

Jacobi identity

$$\sum_{r=1}^{dim(M)} J^{ir} \frac{\partial J^{jk}}{\partial x^r} + J^{jr} \frac{\partial J^{ki}}{\partial x^r} + J^{kr} \frac{\partial J^{ij}}{\partial x^r} = 0; \tag{3a}$$

Definition of the Poisson structure

For a classical phase space with *global* coordinates $q^i, p^i, i = 1..n$ the structural tensor has the form of (block) matrix,

$$J_{(q,p)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{4}$$

which results in the known expression for a classical Poisson bracket.

$$\{F, G\} = \sum_i^n \left(\frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial G}{\partial q^i} \right); \tag{5}$$

Any dynamic variable H generates a flow on the Poisson manifold by the formula

$$\frac{dx}{dt} = \{x, H\} \longrightarrow \frac{dF}{dt} = \{F, H\} \quad \forall F \in \mathscr{F}_M \tag{6}$$

Definition of the Poisson structure

Useful properties of Poisson brackets. If a dynamic variable G depends on generators (i.e. on x^i) as a composite function of dynamic variables h^k, similarly

$$\{F, G(h_1, \ldots, h^m)\} = J^{ik} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial h^r} \frac{\partial F}{\partial x^k} = \frac{\partial G}{\partial h^r} \left(J^{ik} \frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^k} \right) = \frac{\partial G}{\partial h^r} \{F, h^r\}$$

i.e.

$$\{F, G(h_1, \ldots, h^m)\} = \frac{\partial G}{\partial h^r} \{F, h^r\} \tag{7}$$

similarly

$$\{F(h_1, \ldots, h^m), G(h_1, \ldots, h^m)\} = \{h^r, h^s\} \frac{\partial F}{\partial h^r} \frac{\partial G}{\partial h^s} \tag{7a}$$

For a physical system these relations can help to turn to a smaller number of generators.

Poisson structures related with the dynamics of a rigid body

Definition

If two main moments of inertia are equal, $I_1 = I_2 = I_\perp$, the rigid body is called a symmetric top.

The kinetic energy of the rotation of a body around its centre of inertia is expressed in the following way by means of the angular rate (or the moment $\vec{m}$) in the coordinate system relating to the principal axes of inertia.

$$T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right) = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right) \tag{8}$$

For the symmetric top as it will be shown below M_ν is the Casimir function, and the constant item in energy can be truncated (Appendix A):

$$T = \frac{1}{2I_\perp}m^2 + \frac{I_\perp - I_\nu}{2I_\nu I_\perp}M_\nu^2 \quad \text{equivalent to} \quad T = \frac{1}{2I_\perp}m^2 = \frac{\alpha}{2}m^2. \tag{9}$$

Poisson structures related with the dynamics of a rigid body

Usually

Components of the moment about the body are usually used as the inertia tensor in a body related system is not time-dependent.

In our case

However in the case of a symmetric top as is shown above it is convenient to use components of the moment about a *fixed* coordinate system and, correspondingly, direction cosines of the symmetry axis $\vec{\nu}$ also with regard to a *fixed* coordinate system.

Our model of rigid body

Henceforth when building a model a symmetric top fully described by the following vectors will only be used as a rigid body: $\vec{\nu}, \vec{m}$ – symmetry axis and body momentum.

Poisson structures related with the dynamics of a rigid body

Thus, the following Poisson structure can be defined

$$\{\nu_i, \nu_j\} = 0; \quad \{m_i, \nu_j\} = \varepsilon_{ijk}\nu_k; \quad \{m_i, m_j\} = \varepsilon_{ijk} m_k; \tag{10}$$

The corresponding structural tensor has the following form

$$J_{(\nu,m)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \nu_3 & -\nu_2 \\ 0 & 0 & 0 & -\nu_3 & 0 & \nu_1 \\ 0 & 0 & 0 & \nu_2 & -\nu_1 & 0 \\ 0 & \nu_3 & -\nu_2 & 0 & m_3 & -m_2 \\ -\nu_3 & 0 & \nu_1 & -m_3 & 0 & m_1 \\ \nu_2 & -\nu_1 & 0 & m_2 & -m_1 & 0 \end{bmatrix} \tag{11}$$

Poisson structures related with the dynamics of a rigid body

The progressive motion of a body is described by the generators $\vec{r_1}, \vec{p_1}$ and $\vec{r_2}, \vec{p_2}$. We change the variables of the type (Appendix B):

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}; \qquad \vec{r} = \vec{r}_2 - \vec{r}_1; \tag{12}$$

The kinetic energy of the progressive motion in the centre-of-mass system has the form

$$T = \frac{m_1 + m_2}{2}\dot{\vec{R}}^2 + \frac{m}{2}\dot{\vec{r}}^2 = \frac{1}{2m}\vec{p}^2; \qquad m = \frac{m_1 m_2}{m_1 + m_2}; \tag{13}$$

as $\dot{\vec{R}} = 0$ and m is the reduced mass, $\vec{r}$ is the orbital radius-vector.

Poisson structures related with the dynamics of a rigid body

Consequently the kinetic energy of two bodies is as follows

(14) $$T(p^2, \vec{m}'^2, \vec{m}''^2) = \frac{1}{2m}p^2 + \frac{\alpha'}{2}\vec{m}'^2 + \frac{\alpha''}{2}\vec{m}''^2,$$

where $\vec{r}, \vec{p}$ – orbital coordinates and impulses; $\vec{\nu}', \vec{m}'$ и $\vec{\nu}'', \vec{m}''$ – are symmetry axes and momenta of the 1st and 2nd bodies, respectively. Therefore the Poisson structure for a system of two magnetically interacting bodies has generators $\vec{r}, \vec{p}, \vec{\nu}', \vec{m}', \vec{\nu}'', \vec{m}''$. Each of 3 groups describes independent degrees of freedom; therefore the structural tensor has the following block form

(15) $$J = \begin{bmatrix} J_{(r,p)} & 0 & 0 \\ 0 & J_{(\nu',m')} & 0 \\ 0 & 0 & J_{(\nu'',m'')} \end{bmatrix}$$

The following dynamic variables are Casimir functions for this Poisson structure:

$\vec{\nu}'^2 = 1,\ \vec{\nu}''^2 = 1; (\vec{\nu}', \vec{m}') = M_3' = const_1,\ (\vec{\nu}'', \vec{m}'') = M_3'' = const_2;$

Poisson structures related with the dynamics of a rigid body

Proposition

As it will be shown below the potential energy of the type $U(r, c^{'}, c^{''}, c^{'''})$ describes interaction for a rather wide class of magnetic bodies. Where

$$\begin{cases} r = |\vec{r}|; \quad \vec{e}_r = \vec{r}/r; \\ c^{'} = (\vec{e}_r, \vec{\nu}^{'}); \\ c^{''} = (\vec{e}_r, \vec{\nu}^{''}); \\ c^{'''} = (\vec{\nu}^{'}, \vec{\nu}^{''}); \end{cases}$$

Poisson structures related with the dynamics of a rigid body

Remark

There are many physical models for which it is known that the potential energy adequately describes interaction and has a reduced form in the axisymmetric case:

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Poisson structures related with the dynamics of a rigid body

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Poisson structures related with the dynamics of a rigid body

Remark

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1. permanent magnets - in classical courses;
2. superconductive elements - in the monograph on electromechanics by White-Woodson (within quasistationary approximation);
3. systems consisting of superconductive elements and constant magnets - in my Ph.D. thesis (also within quasistationary approximation);

Poisson structures related with the dynamics of a rigid body

Example

Let two "magnetic charges" spaced on a fixed distance be a "dumbbell". This system modulates a field of a long thin cylinder. The potential energy of this system is as follows:

$$U = \frac{\mu_0 \kappa' \kappa''}{4\pi} \sum_{\varepsilon', \varepsilon'' = \pm 1} \frac{\varepsilon' \varepsilon''}{R_{\varepsilon' \varepsilon''}} \tag{16}$$

where $R_{\varepsilon' \varepsilon''}(r, c', c'', c''')$
$= \sqrt{r^2 + l'^2 + l''^2 + 2r(\varepsilon'' l'' c'' - \varepsilon' l' c') - 2\varepsilon' \varepsilon'' l' l'' c'''}$

It is this system that we used to check the capability of the orbital motion in the system of two magnets.

Poisson structures related with the dynamics of a rigid body

Cases

The potential energy of the magnetic interaction of 2 bodies has the same form for the cases: 2 "dumbbells"; 2 superconductive loops of a ring form; magnetic dipole - superconductive loop of a ring form.

Proposition

In the case of a permanent magnet having axisymmatric form, when the scalar magnetic potential outside the body $\psi = \psi(r, z) = \psi(r, rc')$, the potential energy of its interaction both with a magnetic dipole and a "dumbbell"has the form $U(r, c', c'', c''')$.

Here z – is the dipole coordinate in the coordinate system, the axis $\vec{z}$ of which coincides with the axis of the magnetic symmetry of the body;
r – the distance to the magnetic dipole from the datum point located on the axis $\vec{z}$.

Poisson structures related with the dynamics of a rigid body

Potential energy of the magnetic dipole and magnetic axisymmetric body:

$$U = \mu(\partial_r \psi(r, rc')c'' + \partial_z \psi(r, rc')c''') \tag{17}$$

Potential energy of the magnetic "dumbbell"and magnetic axisymmetric body:

$$U = \kappa \sum_{\varepsilon=\pm 1} \psi(R_\varepsilon(r, c''), z_\varepsilon(r, c', c'')) \tag{18}$$

where $R_\varepsilon^2 = r^2 + l^2 + 2\varepsilon lrc''$, $z_\varepsilon = rc' + \varepsilon lc'''$.

Poisson structure for two magnetic-interacting bodies

Hamiltonian of the system is given in the following expression

$$H = T(p^2, \vec{m}'^2, \vec{m}''^2) + U(r, c', c'', c''') \tag{19}$$

The motion equations for this Hamiltonian have the form

$$\begin{cases} \dot{\vec{r}} = \frac{1}{m}\vec{p}; \\ \dot{\vec{p}} = -\partial_r U\vec{e} - \frac{1}{r}(\partial_{c'} UP^e_\perp(\vec{\nu}') + \partial_{c''} UP^e_\perp(\vec{\nu}'')); \\ \dot{\vec{\nu}}' = \alpha'(\vec{m}' \times \vec{\nu}'); \\ \dot{\vec{m}}' = \partial_{c'} U(\vec{e} \times \vec{\nu}') - \partial_{c'''} U(\vec{\nu}' \times \vec{\nu}''); \\ \dot{\vec{\nu}}'' = \alpha''(\vec{m}'' \times \vec{\nu}''); \\ \dot{\vec{m}}'' = \partial_{c''} U(\vec{e} \times \vec{\nu}'') + \partial_{c'''} U(\vec{\nu}' \times \vec{\nu}''); \end{cases} \tag{20}$$

where operator $P^e_\perp$ – is the projector on the plane perpendicular to the vector $\vec{e}$, i.e. $P^e_\perp(\vec{\nu}') = \vec{\nu}' - c'\vec{e}$.

Poisson structure for two magnetic-interacting bodies

The components of the total momentum of the system are integrals of motion.

$$\vec{j} = \vec{l} + \vec{m}' + \vec{m}'' = const_3, \quad \vec{l} = \vec{x} \times \vec{p}; \tag{21}$$

This results from symmetry considerations, but, besides, this can be checked by direct calculation.

$$\frac{d\vec{j}}{dt} = \{\vec{j}, H\} = 0; \tag{22}$$

Poisson structure for two magnetic-interacting bodies

Thus, the system of equations (20) should be supplemented with the relations:

$$(23)\quad \begin{cases} \vec{\nu}\,'^2 = 1, \quad \vec{\nu}\,''^2 = 1; \\ (\vec{\nu}\,', \vec{m}') = M_3' = const_1, \quad (\vec{\nu}\,'', \vec{m}'') = M_3'' = const_2; \\ \vec{j} = \vec{l} + \vec{m}' + \vec{m}'' = const_3, \quad \vec{l} = \vec{x} \times \vec{p}; \end{cases}$$

These relations can be used to reduce the order of the system differential equations. In particular, it is very easy to exclude for example the dynamic variable $\vec{m}''$ using line 3 in (23). Using all relations (23) it is possible exclude $4 + 3 = 7$ dynamic variable and have in essence a system of the 11th order.

Application of Maple and MatLab systems

- Base procedures for computing Poisson brackets (Maple).

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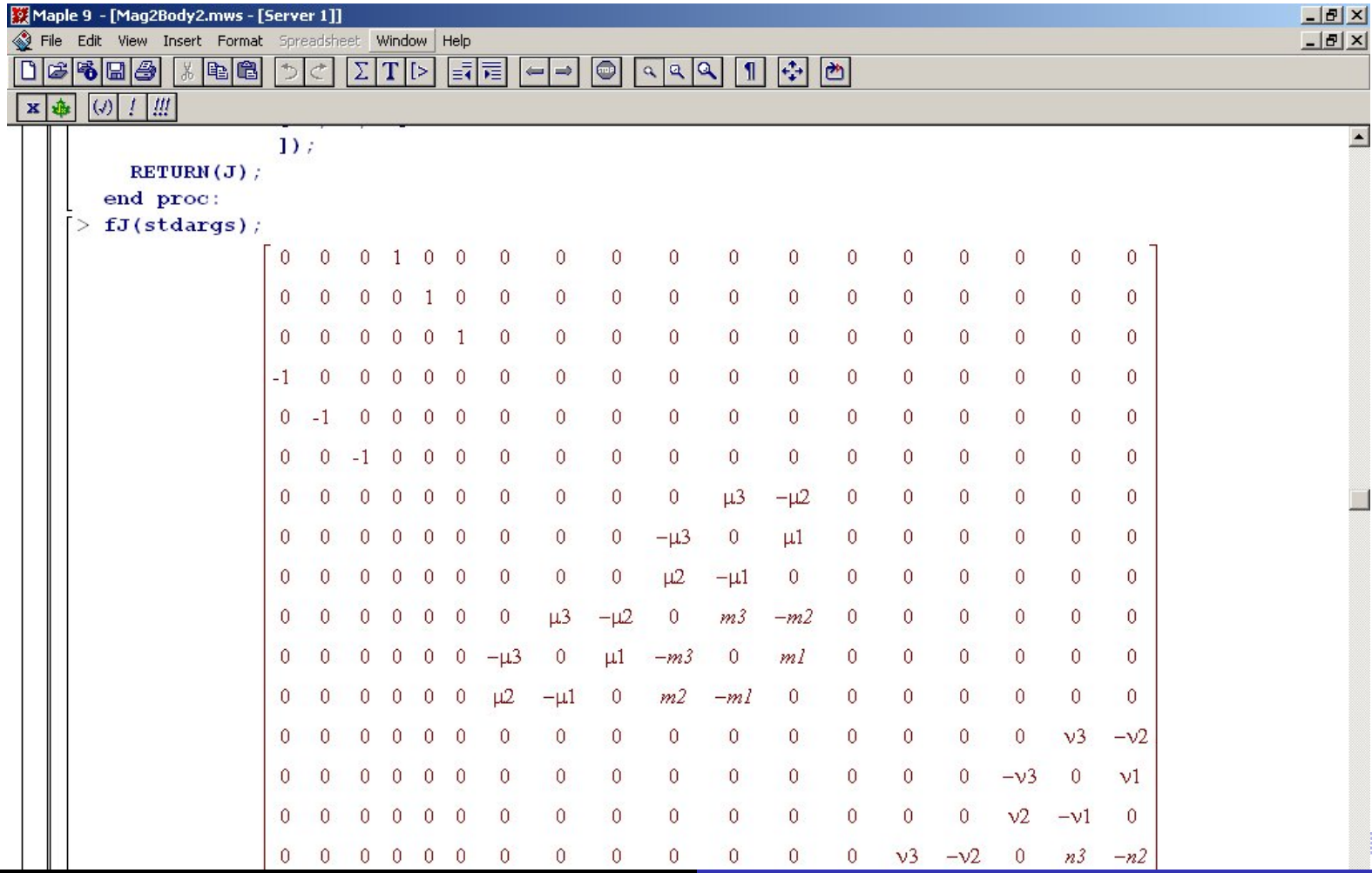
Base procedures for computing Poisson brackets

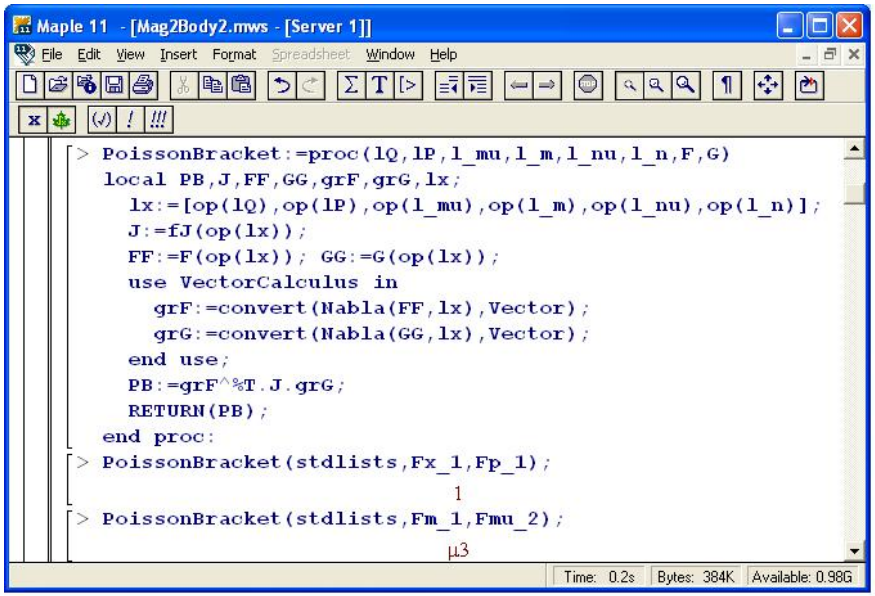
Total structural tensor for a two-body system:



$$\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu 3 & -\mu 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu 3 & 0 & \mu 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu 2 & -\mu 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu 3 & -\mu 2 & 0 & m3 & -m2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\mu 3 & 0 & \mu 1 & -m3 & 0 & m1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu 2 & -\mu 1 & 0 & m2 & -m1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu 3 & -\nu 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\nu 3 & 0 & \nu 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu 2 & -\nu 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu 3 & -\nu 2 & 0 & n3 & -n2
\end{bmatrix}$$

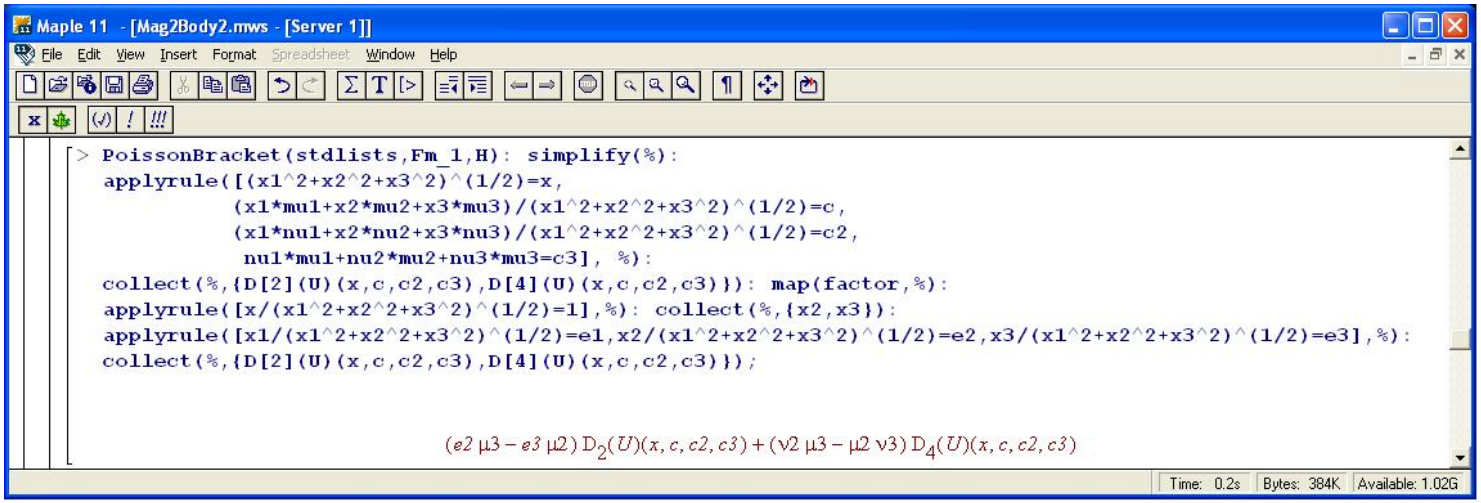
Base procedures for computing Poisson brackets

Procedure for computing Poisson brackets.



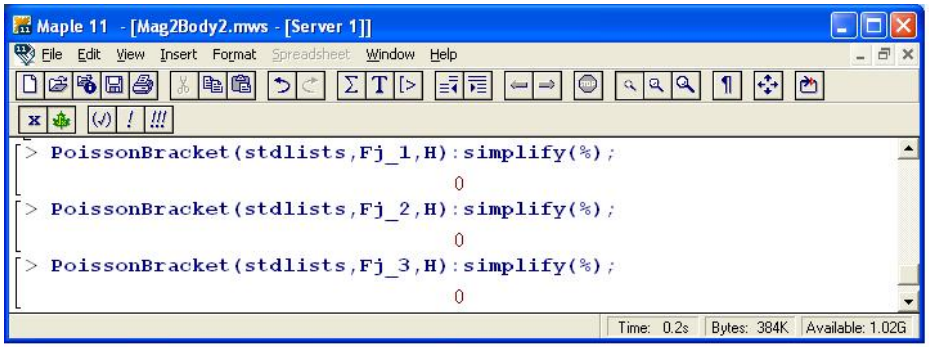
Base procedures for computing Poisson brackets

$$\dot{\vec{m}}' = \{m_i', H\} = \{m_i', U(r, c', c'', c''')\} = \partial_{c'} U(\vec{e} \times \vec{\nu}') - \partial_{c'''} U(\vec{\nu}' \times \vec{\nu}'')$$



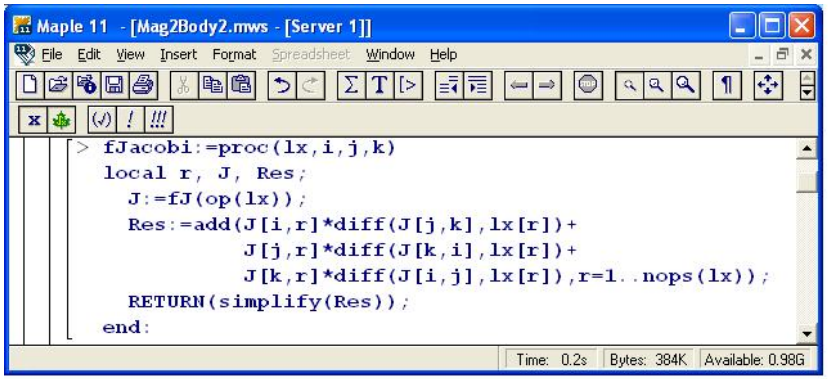
Base procedures for computing Poisson brackets

Total moment of momentum conservation law: $\{j_i, H\} = 0 \quad \forall i$



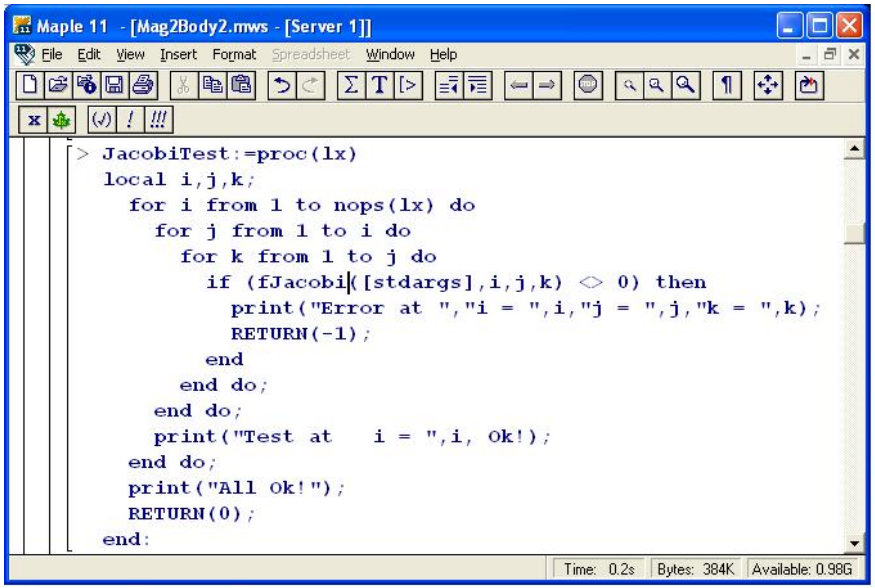
Base procedures for computing Poisson brackets

Jacobi identity:



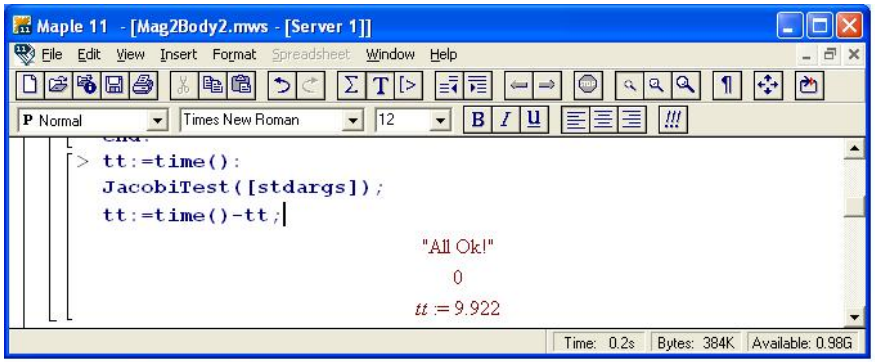
Base procedures for computing Poisson brackets

Jacobi identity testing procedure.



Base procedures for computing Poisson brackets

Test of Jacobi identity for structural tensor.



Computational modeling in Maple

Motion equations of a two-dipole system

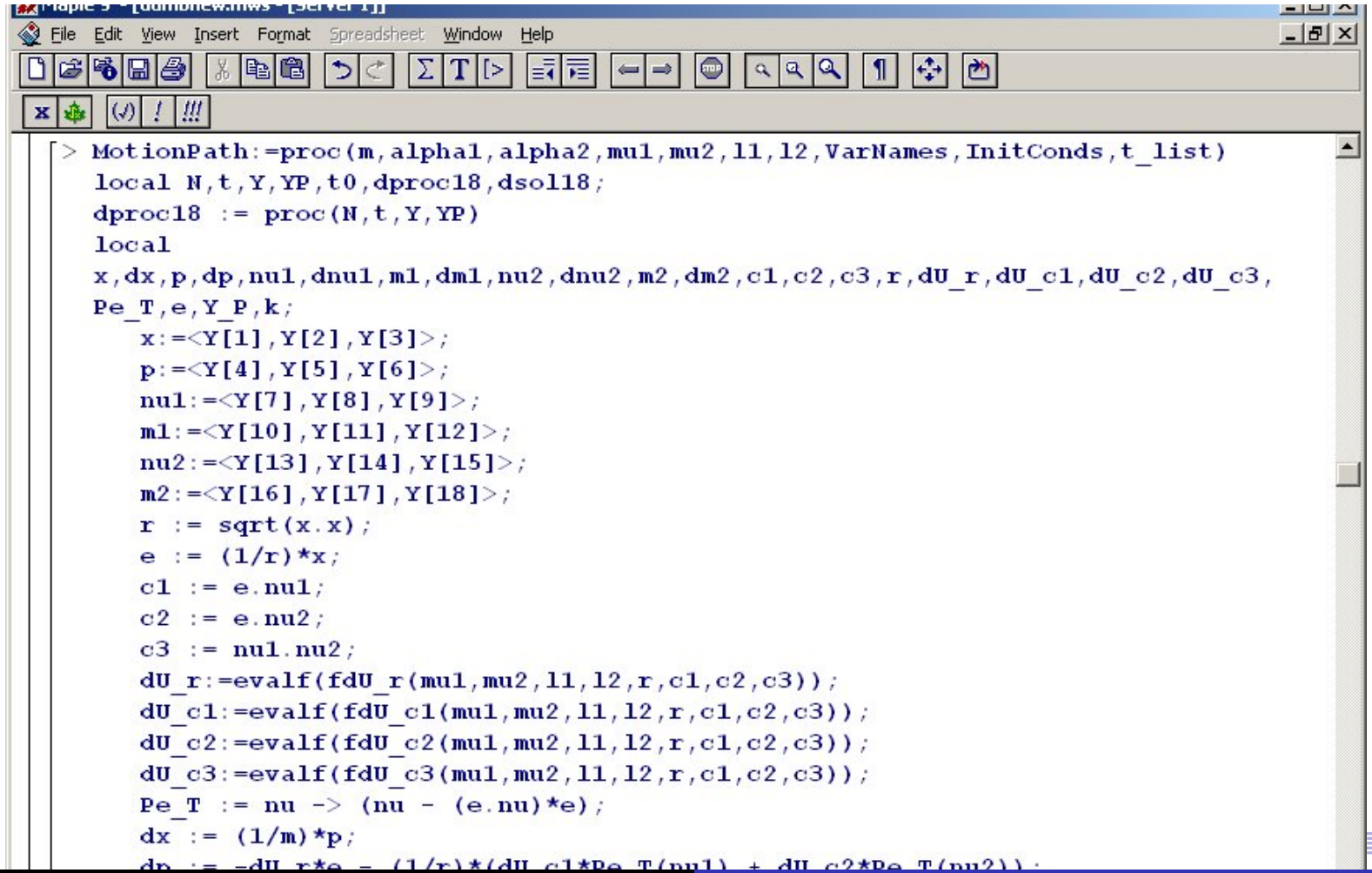
Maple 9 - [dipoles.mws - [Server 1]]

File Edit View Insert Format Spreadsheet Window Help

```
> MotionPath:=proc(m,alpha1,alpha2,mu1,mu2,VarNames,InitConds,t_list)
  local N,t,Y,YP,t0,dproc18,dsol18,k12;
  k12:=mu1*mu2;
  dproc18 := proc(N,t,Y,YP)
  local x,dx,p,dp,nu1,dnu1,m1,dm1,nu2,dnu2,m2,dm2,c1,c2,c3,r,dU_r,dU_c1,dU_c2,dU_c3,Pe_T,e,Y_P,k;
     x:=<Y[1],Y[2],Y[3]>;
     p:=<Y[4],Y[5],Y[6]>;
     nu1:=<Y[7],Y[8],Y[9]>;
     m1:=<Y[10],Y[11],Y[12]>;
     nu2:=<Y[13],Y[14],Y[15]>;
     m2:=<Y[16],Y[17],Y[18]>;
     r := sqrt(x.x);
     e := (1/r)*x;
     c1 := e.nu1;
     c2 := e.nu1;
     c3 := nu1.nu2;
     dU_r:=evalf(-3/4*k12*mu0/Pi*(c3-3*c1*c2)/r^4);
     dU_c1:=evalf(-3/4*k12*mu0/Pi*c2/r^3);
     dU_c2:=evalf(-3/4*k12*mu0/Pi*c1/r^3);
     dU_c3:=evalf(1/4*k12*mu0/Pi/r^3);
     Pe_T := nu -> (1/r)*(nu - c1*e);
     dx := (1/m)*p;
     dp := -dU_r*e - (1/r)*(dU_c1*Pe_T(nu1) + dU_c2*Pe_T(nu2));
     dnu1 := alpha1*(m1 &x nu1);
     dm1 := dU_c1*(e &x nu1) - dU_c3*(nu1 &x nu2);
     dnu2 := alpha2*(m2 &x nu2);
     dm2 := dU_c2*(e &x nu2) + dU_c3*(nu1 &x nu2);
     Y_P := convert(<dx,dp,dnu1,dm1,dnu2,dm2>,list);
     for k from 1 to N do YP[k] := Y_P[k] end do;
  end proc: # dproc18
    dsol18 := dsolve(numeric,number=nops(VarNames),procedure=dproc18,start=t_list[1],initial=InitConds,
```

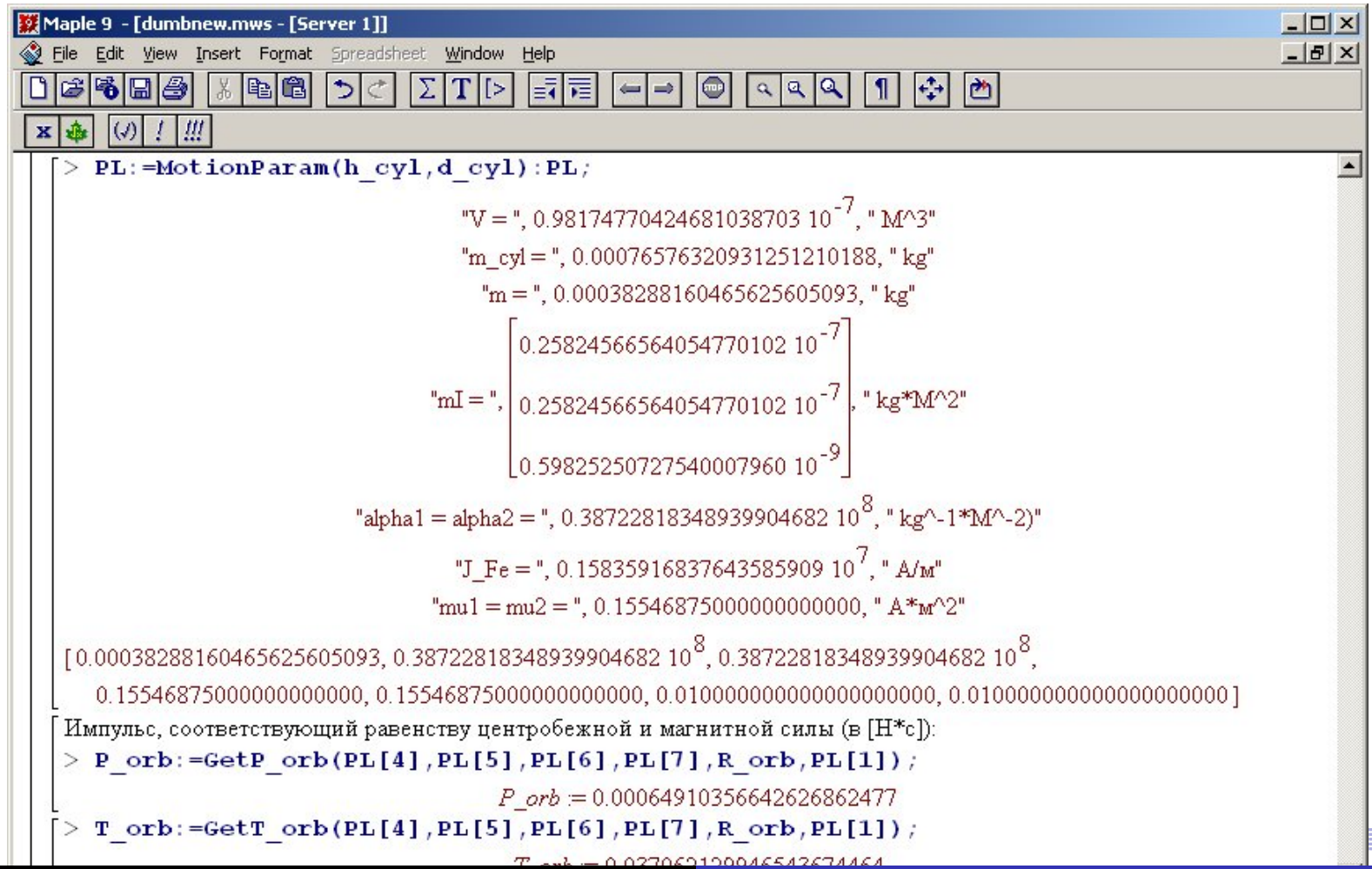
Computational modeling in Maple

Motion equations of a two-dipole system.



Computational modeling in Maple

Parameters of cylindrical magnets (dumb-bells).



Computational modeling in MatLab

Motion equations of a two-dumbbell system

Editor - D:\users\stah\ACAT_2008\matlab\sys17

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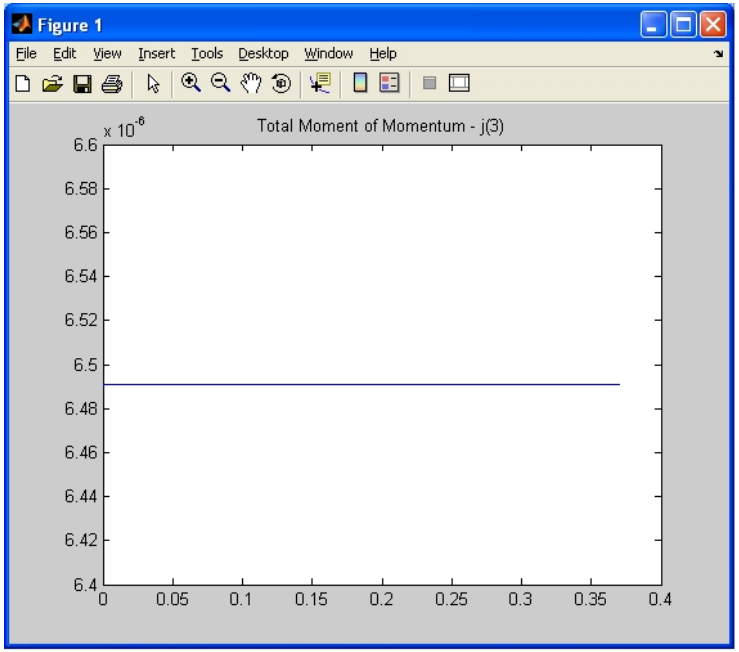
```
function dydt = sys17(t,y,m,alpha1,alpha2,mu1,mu2,l1,l2)
% Mnemonic reequipment variables
Y=y';
x=Y(1:3); p=Y(4:6); nu1=Y(7:9); m1=Y(10:12); nu2=Y(13:15); m2=Y(16:18);
% Ancillary variables
r=sqrt(x*x');
e=(1/r)*x;
% Directional cosines of two dumbbells system
c1=e*nu1';
c2=e*nu2';
c3=nu1*nu2';
% Derivatives of the two dumbbells magnetic interection potential energy
dUr=dU_r(r,c1,c2,c3,mu1,mu2,l1,l2);
dUc1=dU_c1(r,c1,c2,c3,mu1,mu2,l1,l2);
dUc2=dU_c2(r,c1,c2,c3,mu1,mu2,l1,l2);
dUc3=dU_c3(r,c1,c2,c3,mu1,mu2,l1,l2);
%Pe_TX Projection operator on the plane perpendicular to the vector (e)
Pe_T1=(nu1-(nu1*e')*e);
Pe_T2=(nu2-(nu2*e')*e);
% System (17) in the mnemonic form
dx=  (1/m)*p;
dp=  -dUr*e-(1/r)*(dUc1*Pe_T1+dUc2*Pe_T2);
dnu1= alpha1*cross(m1,nu1);
dm1=  dUc1*cross(e,nu1)-dUc3*cross(nu1,nu2);
dnu2= alpha2*cross(m2,nu2);
dm2=  dUc2*cross(e,nu2)+dUc3*cross(nu1,nu2);
dydt= [dx(1); dx(2); dx(3); ...
       dp(1); dp(2); dp(3); ...
       dnu1(1); dnu1(2); dnu1(3); ...
       dm1(1); dm1(2); dm1(3); ...
       dnu2(1); dnu2(2); dnu2(3); ...
       dm2(1); dm2(2); dm2(3)];
end % sys17
```

slide1.m slide2.m sys17

plain text file Ln 9 Col 11 OVR

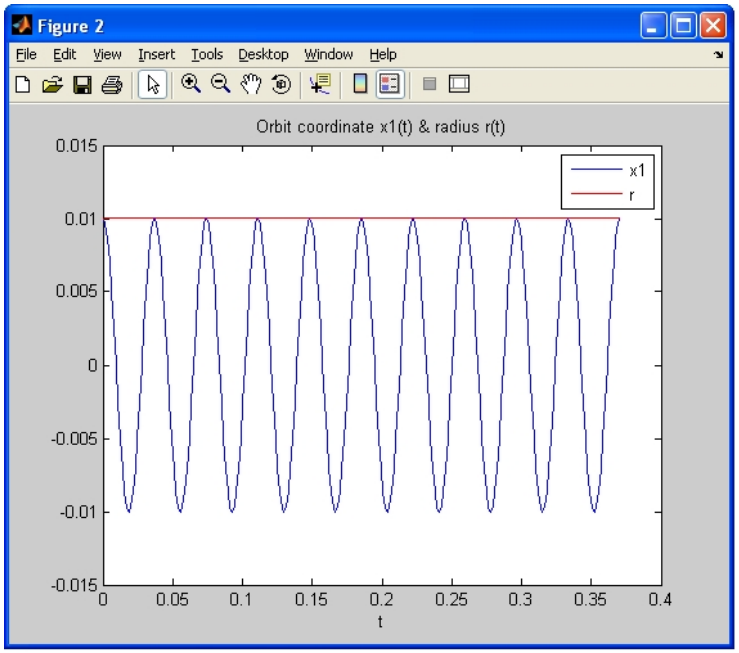
Computational modeling in MatLab

Total moment of momentum conservation law (z - component of vector).



Computational modeling in MatLab

Change of orbital radius on background of x - coordinate changing.



Summary

- The Hamiltonian formalism has been developed which results in a contact-free, i.e. vector, form of motion equations for a system of magnetically interacting bodies provided that the axial symmetry of distribution of the mass of a body and its magnetic properties is the same.
- In the Maple system of symbolic computation the procedure of calculating Poisson brackets has been programmed for a system of 2 magnetically interacting symmetric tops.

Summary

Using symbolic methods the following has been checked:

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Summary

Using symbolic methods the following has been checked:

1. the Jacobi identity for a structural tensor of the Poisson structure;
2. Poisson brackets between all dynamic variables which are of interest for our problem;
3. system motion equations;
4. conservation of the components of the total moment of momentum;

Summary

- The orbital motion of two magnets ("dumbbells"model) has been modeled numerically both in Maple and MatLab.
- During computational modeling the uniformity of Casimir functions and integrals of motions have been checked.
- The example demonstrates stability of the orbital motion of magnets with certain relations between their parameters.

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Calculation of the Principal Magnetic Properties of Two Current Circles with Arbitrary Relative Position.
Int. Symposium on Non-Linear Electromagnetic Systems, LNCS 985, pages 126. 1997.

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Appendix Outline

On the tensor of symmetric top inertia

Definition

If $I_{ik} = \sum m(x_i^2 \delta_{ik} - x_i x_k)$ is a symmetric tensor $(I_{ik} = I_{ki})$ then I_{ik} is called the tensor of inertia.

For the symmetric top inertia tensor

$$I = \sum_k I_k E_k \otimes E^k \tag{10}$$

where E_k – are eigenvectors of the inertia tensor, i.e

$$IE_k = I_k E_k \tag{11}$$

On the tensor of symmetric top inertia

Taking into account the symmetry of the top ($I_1 = I_2$), we will obtain

(12) $$I = I_1(E_1 \otimes E^1 + E_2 \otimes E^2) + I_3 E_3 \otimes E^3$$

or

(12a) $$I = I_\perp P_\perp + I_\nu P_\nu$$

where $P_\nu, P_\perp$ - projectors

(13) $$P_\nu = \nu \otimes \nu, \quad P_\perp = E - P_\nu$$

The main properties of projectors in this case are as follows

(14) $$P_\nu^2 = P_\nu, \quad P_\perp^2 = P_\perp, \quad P_\perp P_\nu = 0, \quad P_\perp + P_\nu = E;$$

Using these properties it is easy to obtain

(15) $$I = I_\perp E + (I_\nu - I_\perp) P_\nu$$

and

(16) $$I^{-1} = I_\perp^{-1} P_\perp + I_\nu^{-1} P_\nu$$

On the tensor of symmetric top inertia

In accordance with the inertia tensor we have

$$m_i = I_{ij}\omega^j \Rightarrow \omega_i = I_{ij}^{-1} m^j \tag{17}$$

The kinetic energy of the rotation of a body around its centre of inertia is expressed by means of the angular velocity (or moment) in a fixed coordinate system in the following way

$$T = \frac{1}{2} I_{ij}\omega^i\omega^j = \frac{1}{2} I_{ij}^{-1} m^i m^j \tag{18}$$

$$T = \frac{1}{2I_\perp} m^2 + \frac{I_\perp - I_\nu}{2I_\nu I_\perp} M_3 M_3 \approx \frac{1}{2I_\perp} m^2 = \frac{\alpha}{2} m^2, \tag{19}$$

as $M_3 = M_\nu$ is the Casimir function, α is a quantity reciprocal to the moment of body inertia, namely: $\alpha = 1/I_\perp$.

Transfer to the inertial reference frame

Let us consider a Lagrangian system of 2 material particles the interaction of which is not necessarily of central character

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_1 \dot{\vec{r}}_1^2 - U(\vec{r}_1, \vec{r}_2) \tag{20}$$

Let us change the variables of the type

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}; \qquad \vec{r} = \vec{r}_2 - \vec{r}_1; \tag{21}$$

This change of variables is always possible as the reverse conversion is possible as well.

$$\vec{r}_1 = \vec{R} - \frac{m_2 \vec{r}}{m_1 + m_2}; \qquad \vec{r}_2 = \vec{R} + \frac{m_1 \vec{r}}{m_1 + m_2}; \tag{21a}$$

However, this change will have neither physical sense, nor mathematical n benefit if the system is not invariant with regard to spatial translations.

Transfer to the inertial reference frame

If the system is invariant with regard to translations which is the case when particles interact only with one another rather than with outer objects, then, firstly, the potential energy of interaction depends only on the difference of the coordinates of particles rather than on their absolute values, and, secondly, the impulse of the system as a single whole is maintained and the motion of the centre of masses is uniform and linear. This means in particular that the coordinates of the centre of masses $\vec{R}$ can be chosen as the datum point for the inertial system and the relative motion of particles in this new reference system can be considered. Then, if we turn to the centre-of-mass system $\dot{\vec{R}} = 0$ and m are reduced masses, $\vec{r}$ is orbital radius-vector.

$$L = \frac{1}{2} m \dot{\vec{r}}^2 - U(\vec{r}); \qquad m = \frac{m_1 m_2}{m_1 + m_2}; \tag{22}$$

If a two-body system also has other degrees of freedom, for example, rotational degrees, this transformation does not affect them.