Computational aspects for three-loop DIS calculations

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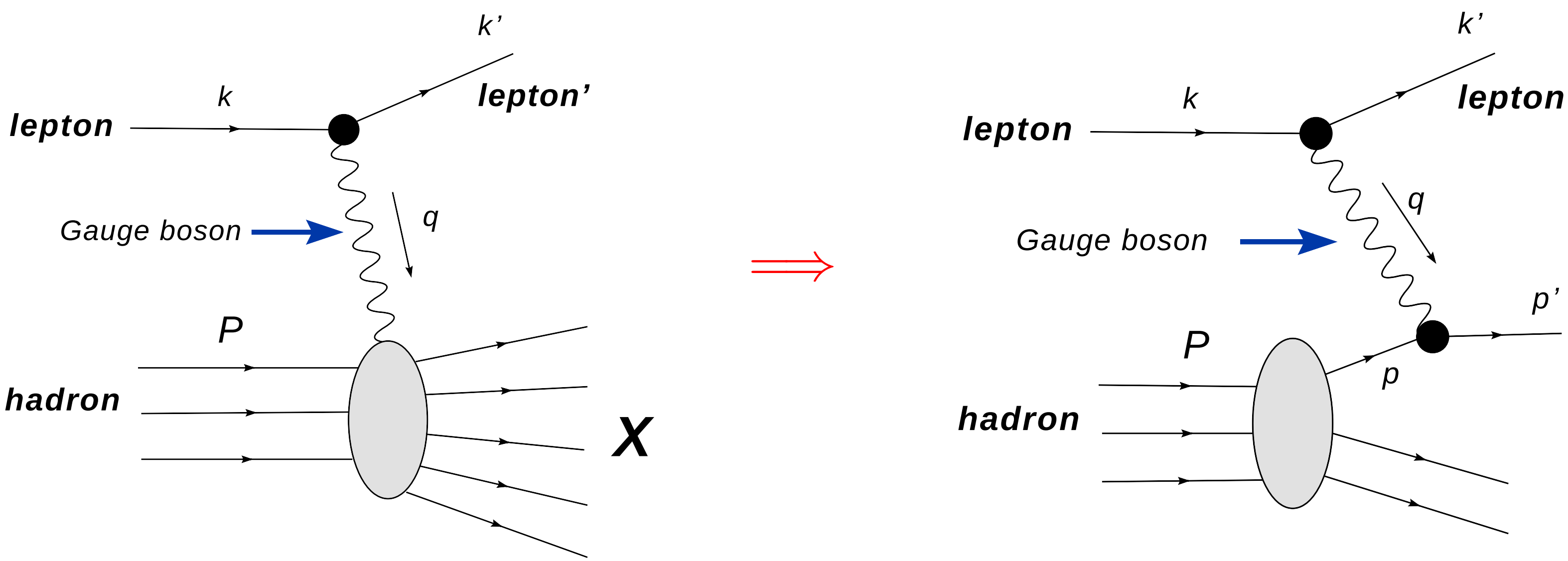
Introduction

- Deep-inelastic lepton-hadron scattering ($e^{\pm}p$, $e^{\pm}n$, νp, $\bar{\nu}p$, ... - collisions)



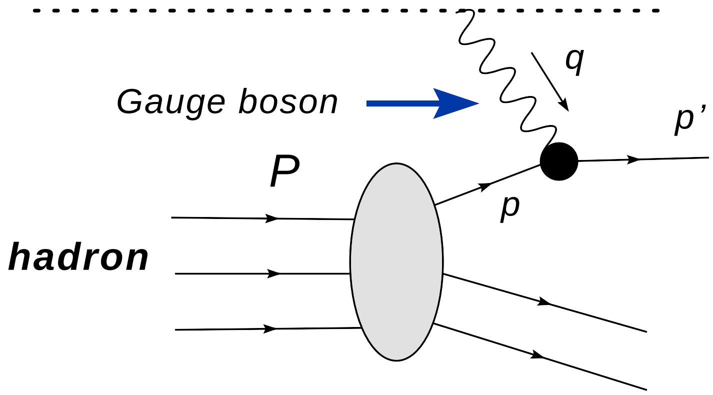
- *Gauge boson*:

  γ, Z^0 - NC
  $W^{\pm}$ - CC

- **Kinematic variables**
  - momentum transfer $Q^2 = -q^2 > 0$
  - Bjorken variable $x = Q^2/(2P \cdot q)$
  - Inelasticity $y = (P \cdot q)/(P \cdot k)$

■ Particles: POLARIZED or UNPOLARIZED

Cross section: $d\sigma \sim L^{\mu\nu} W_{\mu\nu}$



$\Rightarrow$ $W_{\mu\nu}$-hadronic tensor

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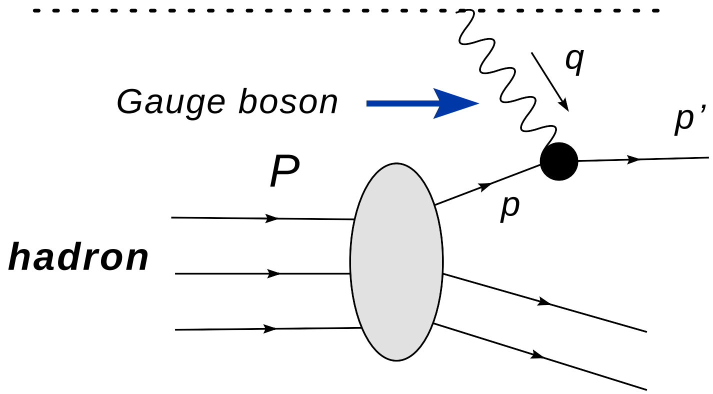


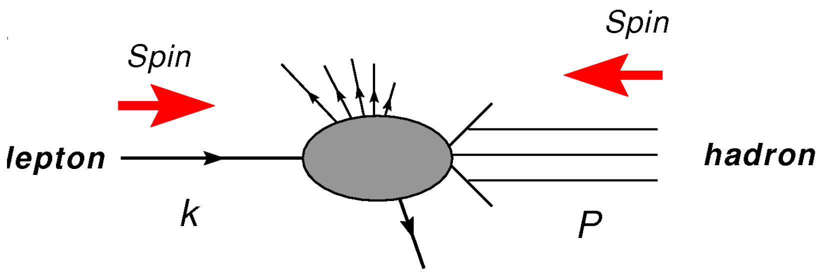
$\Rightarrow$ $W_{\mu\nu}$-hadronic tensor

For unpolarized DIS:

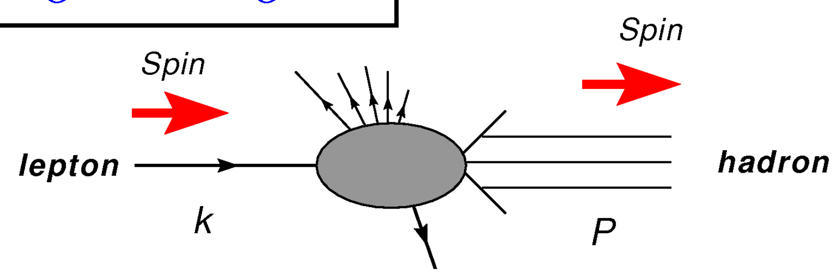
$$W_{\mu\nu} = e_{\mu\nu} \frac{1}{2x} F_L(x, Q^2) + d_{\mu\nu} \frac{1}{2x} F_2(x, Q^2) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta} \frac{P^\alpha q^\beta}{P \cdot q} F_3(x, Q^2)$$

- $e_{\mu\nu}$, $d_{\mu\nu}$ - tensors, depend on P, q. symmetric under $\mu \leftrightarrow \nu$
- main objects are $F_{2,L,3}$ structure functions

- Polarized DIS : lepton and hadron are polarized $\mapsto$



$$\Delta\sigma \equiv \sigma^{\rightrightarrows} - \sigma^{\leftrightarrows}$$



- Example: lepton-proton scattering, exchange via virtual photon γ

  ..., Zijlstra, van Neerven '93

$$\frac{d^2\Delta\sigma}{dx\,dy} = \frac{8\pi\alpha^2}{q^2}\left[\{2 - y - Mxy\}g_1(x,Q^2) - \frac{2Mx}{E}g_2(x,Q^2)\right]$$

M - mass of the proton, E - energy of the lepton

$M = 0 \Rightarrow$ ONLY $g_1(x,Q^2)$ polarized structure function

Structure functions in QCD improved parton model

- Wilson coefficient functions $C_{i,parton}$

$$F_i(x,Q^2) \;=\; \sum_{parton} \int_x^1 \frac{d\xi}{\xi}\, PDF_{parton}(\xi,\mu^2)\; C_{i,parton}\left(\frac{x}{\xi},\alpha_s(\mu^2),\frac{\mu^2}{Q^2}\right)$$

- Parton distribution functions PDF_{parton}
  - NOT calculable in perturbative QCD, extracted from experiment data
  - evolution of $PDF's$ via *DGLAP* evolution equation with help of splitting functions $P_{\mathrm{parton},\mathrm{parton}'}$

$$\frac{Q^2}{\partial Q^2}\begin{bmatrix} q \\ g \end{bmatrix} = \begin{bmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{bmatrix} \otimes \begin{bmatrix} q \\ g \end{bmatrix}$$

- The same features for polarized structure function $g_1(x,Q^2)$

Chain of calculations

- $P_{\mathrm{parton,parton'}}$, $C_{i,parton}$: are in hadronic tensor $W_{\mu\nu}$ and also !!! in its analogue – partonic tensor $w_{\mu\nu}$

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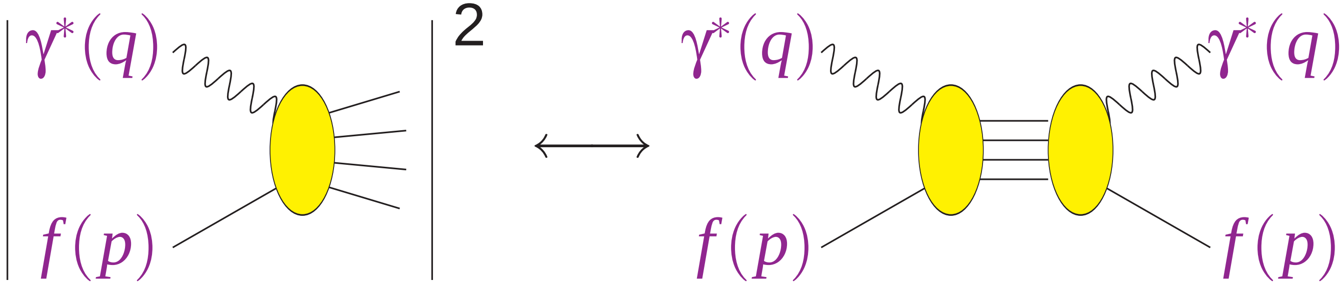
- For $w_{\mu\nu}$ use of *OPTICAL THEOREM*: $w_{\mu\nu}(p,q) \propto Im\, t_{\mu\nu}(p,q)$

- The partonic tensor $w_{\mu\nu}(p,q)$ is related to the imaginary part of the partonic forward Compton scattering amplitude $t_{\mu\nu}(p,q)$
  - α_s^3 calculation in DIS with help of loop technology



$$w_{\mu\nu}(p,q) = \frac{1}{2\pi} Im\, t_{\mu\nu}(p,q)$$

▲Dispersion relation in x : coefficient of $(2p\cdot q)^N \leftrightarrow N$-th moment

$$A^N \equiv \int_0^1 dx\, x^{N-1} A(x)$$

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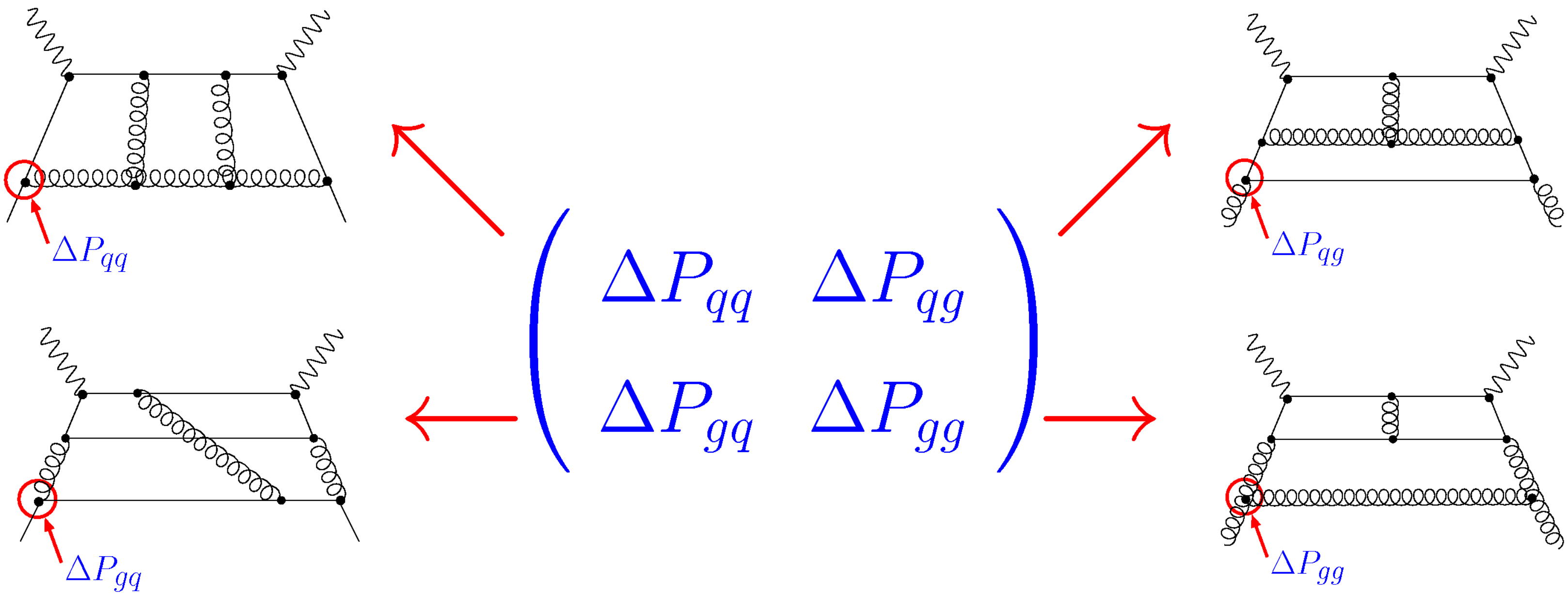
- $t_{\mu\nu}(p,q)$ - generation of all possible diagrams up to 3 loops $\rightarrow$ QGRAF Nogueira '93

- Example: diagrams for the polarized splitting functions



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- Depending on NC/CC, polarized/unpolarized use of symmetries to reduce number of diagrams: *procedure of* Moch, Vermaseren Vogt '05

▲ Symbolic manipulations $\mapsto$ FORM and TFORM Tentyukov, Vermaseren '07

Use of symmetries

- Number of diagrams for the polarized DIS after treatment by *procedure of* Moch, Vermaseren Vogt '05

legs	tree	1-loop	2-loop	3-loop
$q\gamma q\gamma$	1	3	25	359
$g\gamma g\gamma$	0	4	46	900
$q\varphi q\psi$	0	4	131	3890
$g\varphi g\psi$	2	31	924	29383 !!!

Original QGRAF *output has up to 4 times more!*

Chain of calculations

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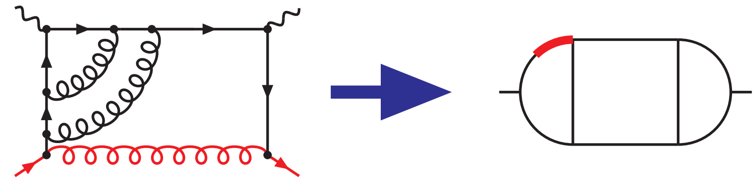
$$\Downarrow$$

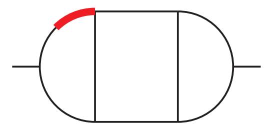
- Calculation of diagrams $\mapsto$ done in Mellin-N space
  - Method of projection Gorishnii, Larin, Tkachev '83: Gorishnii, Larin '87
  - fixed Mellin-moments $\mapsto$ MINCER in FORM Larin, Tkachev, Vermaseren '91
  - symbolic N calc. and inverse Mellin transf. to Bjorken-x space $\mapsto$ *recipe of* Moch, Vermaseren, Vogt '04

Method of projection in pictures

- Identify scalar topologies

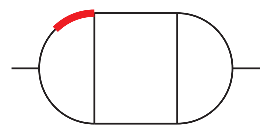


- Scalar diagram with external momenta P and Q



$$= \int \prod_{n}^{3} d^D l_n \, \frac{1}{(P-l_1)^2} \, \frac{1}{l_1^2 \ldots l_8^2}$$

- N-th moment: $\longrightarrow$ coefficient of $(2P \cdot Q)^N$



$$= \frac{(2\,P \cdot Q)^N}{(Q^2)^{N+\alpha}} \, C_N$$

- Taylor expansion

$$\frac{1}{(P-l_1)^2} = \sum_i \frac{(2P \cdot l_1)^i}{(l_1^2)^{i+1}} \quad \longrightarrow \quad \frac{(2P \cdot l_1)^N}{(l_1^2)^N}$$

- Feed scalar two-point functions in MINCER $\Rightarrow$

Mincer

- $\int dP \frac{\partial}{\partial P^\mu}\left[(P-l_j)^\mu \times I(l_1,..,P,...)\right] = 0$ - integration by part identities
  t'Hooft, Veltman'72; Chetyrkin , Tkachov '81
  Leibniz, Newton :-)

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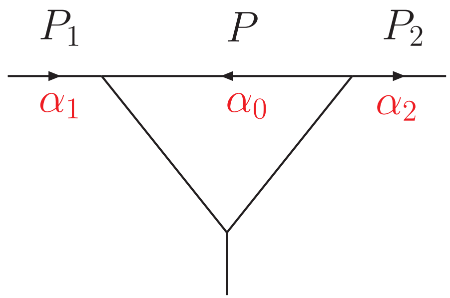

Triangle rule

Define

$$I(\alpha_0,\beta_1,\beta_2,\alpha_1,\alpha_2) = \int d^D P \frac{1}{(P^2)^{\alpha_0}((P+P_1)^2)^{\beta_1}(P_1^2)^{\alpha_1}((P+P_2)^2)^{\beta_2}(P_2^2)^{\alpha_2}}$$

and act the integrand with $\frac{\partial}{\partial P_\mu}P_\mu = D + P_\mu \frac{\partial}{\partial P_\mu}$. Result $\Rightarrow$

Recursion relation:

$$I(\alpha_0,\beta_1,\beta_2,\alpha_1,\alpha_2)\times(D-2\alpha_0-\beta_1-\beta_2) =$$
$$\beta_1(I(\alpha_0-1,\beta_1+1,\beta_2,\alpha_1,\alpha_2) - I(\alpha_0,\beta_1+1,\beta_2,\alpha_1-1,\alpha_2))$$
$$\beta_2(I(\alpha_0-1,\beta_1,\beta_2+1,\alpha_1,\alpha_2) - I(\alpha_0,\beta_1,\beta_2+1,\alpha_1,\alpha_2-1))$$

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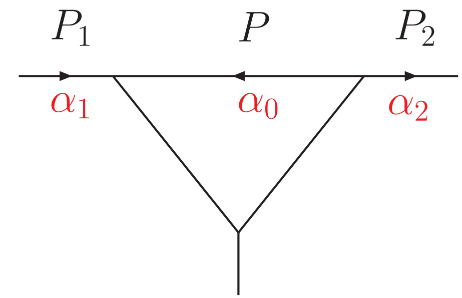

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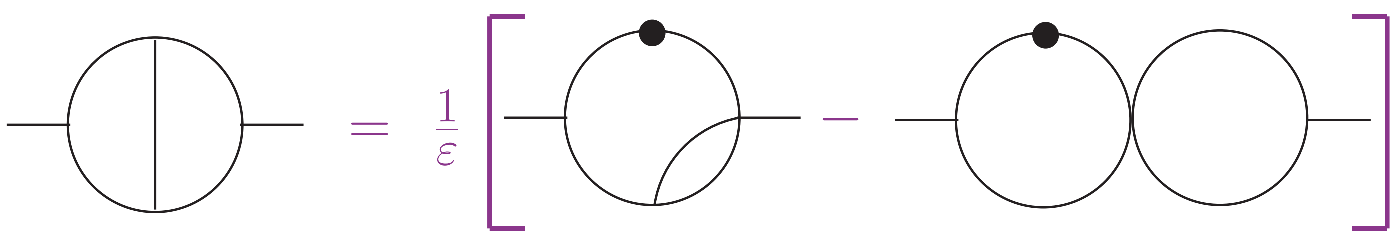
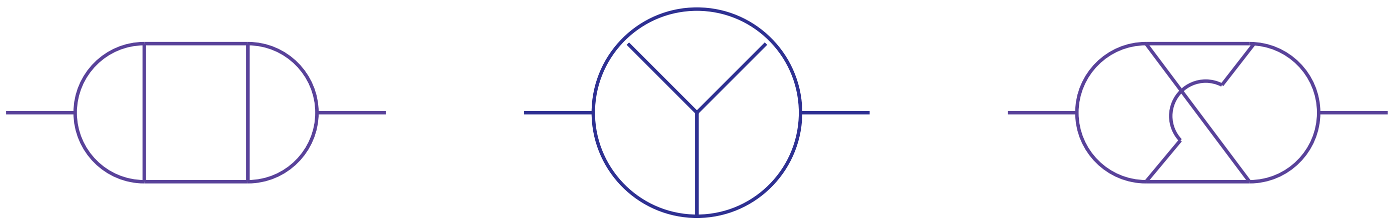
$$\beta_2(I(\alpha_0-1,\beta_1,\beta_2+1,\alpha_1,\alpha_2) - I(\alpha_0,\beta_1,\beta_2+1,\alpha_1,\alpha_2-1))$$

In pictures



Classification of loop integrals

- Classify according to topology of underlying two-point function
  - top-level topology types ladder, benz, non-planar $\Rightarrow$



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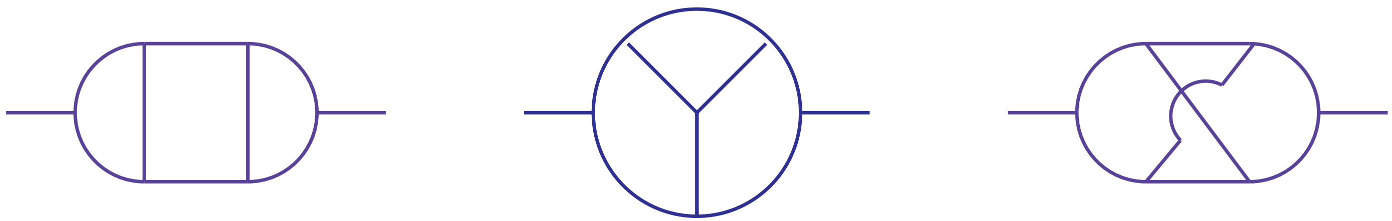
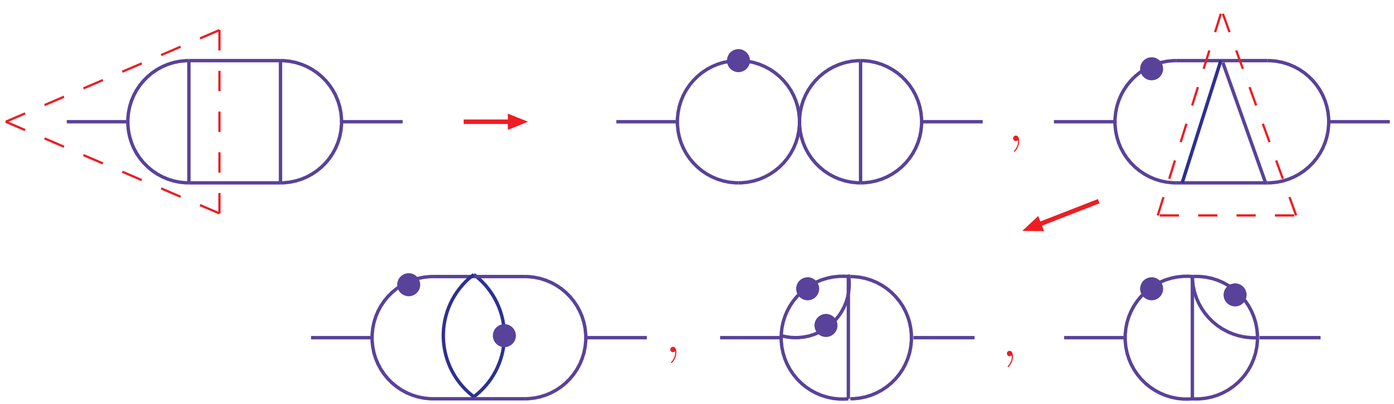


- Using **IBP** identities more complicated topologies are reduced to simpler topologies

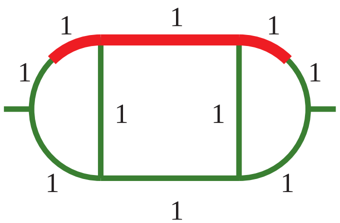


- Combine identities: integration by parts, scaling, Passarino-Veltman type

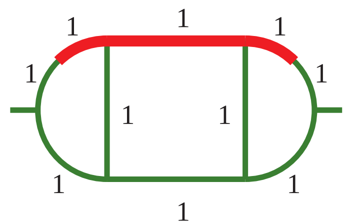
$\Rightarrow$ Difference equations for $I(N)$ [recall: coefficient of $(2p{\cdot}q)^N$]

$$a_0(N)I(N) - \cdots - a_n(N)I(N-n) = I_0(N)$$

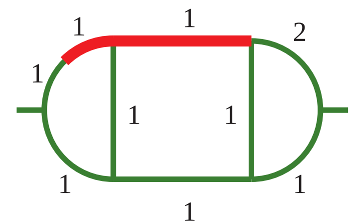
Simple scalar example [red line: flow of massless parton momentum p]



$+ \; \dfrac{N+3+3\epsilon}{N+2}\,\dfrac{2p{\cdot}q}{q^2}$



$= \; \dfrac{2}{N+2}$



- Successive reduction to simpler (lower topologies or ‘less red’) integrals

Examples of results

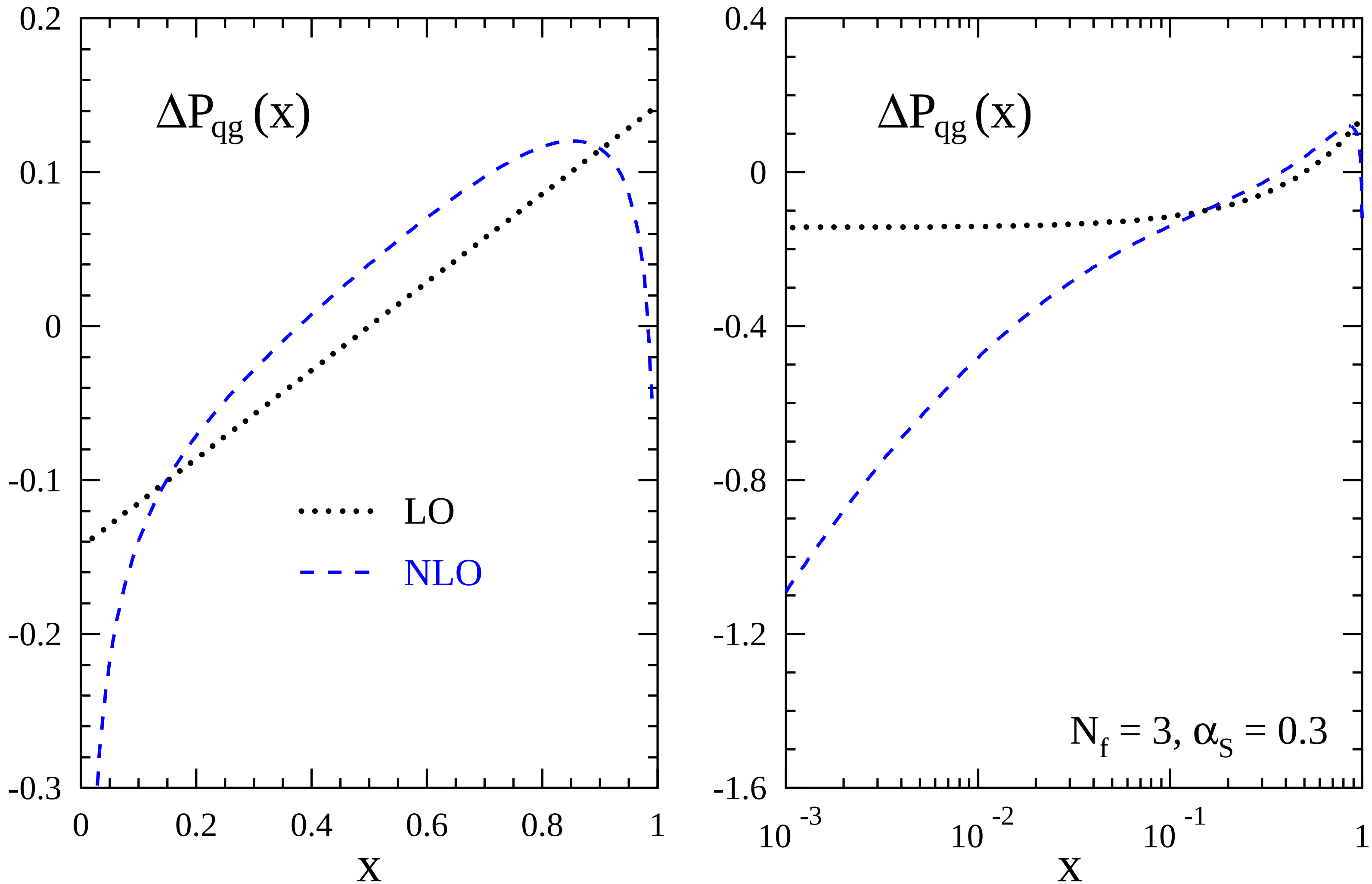
CC DIS: 10'th Mellin of F_3 structure function

Moch, Rogal '07

$$
\begin{aligned}
C^{\rm ns}_{3,10} \;=\;& 1 + a_s C_F \frac{1953379}{138600} + a_s^2 C_F n_f \left(-\frac{537659500957277}{15975002736000} \right) + a_s^2 {C_F}^2 \left(\frac{5973994463755 24589}{1476090252806400 0} \right. \\
& \left. + \frac{7202}{105} \zeta_3 \right) + a_s^2 C_A C_F \left(\frac{583260205812226 7}{290454595200 00} - \frac{99886}{1155} \zeta_3 \right) \\
& + a_s^3 C_F {n_f}^2 \left(\frac{513397566731946 17191}{996360920644320000} + \frac{48220}{18711} \zeta_3 \right) \\
& + a_s^3 {C_F}^2 n_f \left(-\frac{1254838179460551213 51353}{2092357933353072000 00} - \frac{59829376}{3274425} \zeta_3 + \frac{24110}{693} \zeta_4 \right) \\
& + a_s^3 {C_F}^3 \left(-\frac{744474223606695878525401307}{70889086782002079360000 00} + \frac{28630985464358}{24960941775} \zeta_3 \right. \\
& \qquad \left. + \frac{151796299}{8004150} \zeta_4 - \frac{53708}{99} \zeta_5 \right) \\
& + a_s^3 C_A C_F n_f \left(-\frac{1852213500455074877 53}{226445663782800000} + \frac{8071097}{39690} \zeta_3 - \frac{24110}{693} \zeta_4 \right) \\
& + a_s^3 C_A {C_F}^2 \left(\frac{19770078729338607732075449}{836943173341228800000 0} - \frac{619383700181}{5546875950} \zeta_3 \right. \\
& \qquad \left. - \frac{151796299}{5336100} \zeta_4 - \frac{37322}{99} \zeta_5 \right) \\
& + a_s^3 {C_A}^2 C_F \left(\frac{937987196390566481 25143}{36231306205248000000} - \frac{43202630363}{20582100} \zeta_3 \right. \\
& \qquad \left. + \frac{151796299}{16008300} \zeta_4 + \frac{195422}{231} \zeta_5 \right) .
\end{aligned}
$$

Polarized DIS: The spin splitting function ΔP_{qg} to NNLO

Vogt, Moch, Rogal, Vermaseren '08



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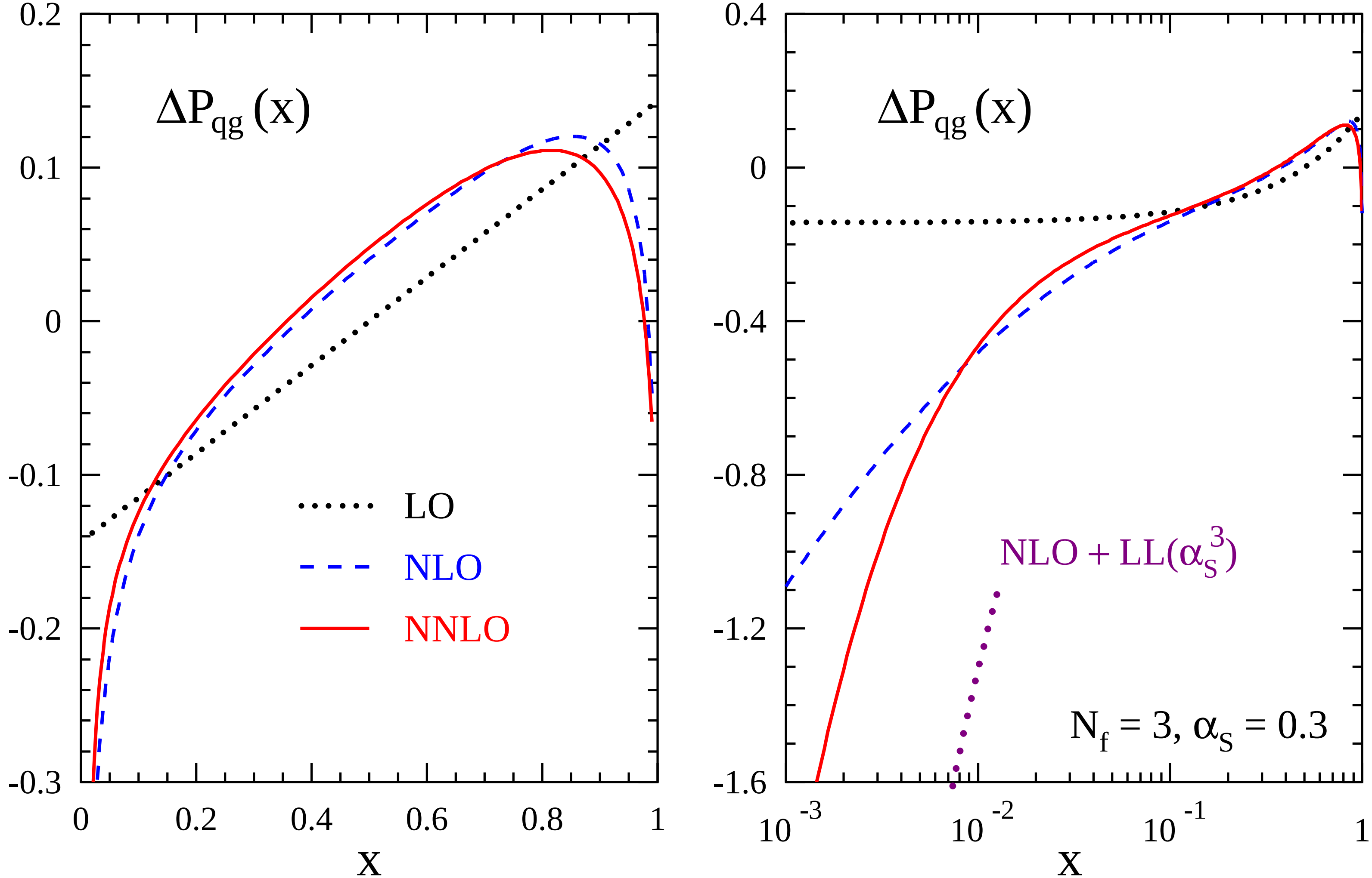

NNLO corr. $\leq 15\%$ for $0.005 \leq x < 0.9$

Summary

- The *recipe* for higher order DIS calculations has been described
- Application examples:
  - Charged Current DIS
    S. Moch, M. Rogal, Nucl.Phys.B782 ’07
    S. Moch, M. Rogal, A. Vogt, Nucl.Phys.B790:317-335, ’07
    M. Rogal, arXiv:0711.0521 [hep-ph]
  - 3-loop splitting functions for polarized DIS
    A. Vogt, S. Moch, M. Rogal, J.A.M. Vermaseren, arXiv:0807.1238 [hep-ph] ’08