Hypergeometric Functions, Differential Reduction and Feynman Diagrams

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Introduction

Forty-five years ago, Regge [1967] proposed that any Feynman diagram can be understood in terms of a special class of hypergeometric functions satisfying some system of differential equations so that the singularity surface of the relevant hypergeometric function coincides with the surface of the Landau singularities of the original Feynman diagram. Based on Regge's conjecture, explicit systems of differential equations for particular types of diagrams have been constructed:
the hypergeometric representation for N-point one-loop diagrams via a series representation (Appell functions and Lauricella functions appear here)
(Kershaw, 1973; Wu, 1974; Mano, 1975);
the system of differential equations and its solution in terms of Lappo-Danilevsky functions (Lappo-Danilevsky, 1934) has been constructed by (Barucchi,Ponzano, 1973) and the monodromy structure of some Feynman diagrams has been studied by (Ponzano, Regge, Speer, Westwater, 1969)
It was known at mid-1970's that each Feynman diagram is a function of the "Nilsson class." This means that the Feynman diagram is a multivalued analytical function in complex projective space $\mathbb{CP}^n$. The singularities of this function are described by Landau's equation. Later, Kashiwara, Kawai, 1977, showed that any regularized Feynman integral satisfies some holonomic system of linear differential equations whose characteristic variety is confined to the extended Landau variety.

Hypergeometric Functions: I

Let us recall that there are several different ways to describe special functions:

- as an integral of the Euler or Mellin-Barnes type;
- by a series whose coefficients satisfy certain recurrence relations;
- as a solution of a system of differential and/or difference equations (holonomic approach).

These approaches and interrelations between them have been discussed in series of a papers by
I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky,
Adv. Math. **84** (1990) 255;
I.M. Gel'fand, M.I. Graev, V.S. Retakh,
Russian Math. Surveys **47** (1992) 1;
I.M. Gelfand, M.I. Graev,
Russian Math. Surveys **52** (1997) 639;
Russian Math. Surveys **56** (2001) 615.

Integral & Mellin-Barnes representations

An Euler integral has the form

$$\Phi(\vec{\alpha},\vec{\beta},P) = \int_\Sigma \Pi_i P_i(x_1,\cdots,x_k)^{\beta_i} x_1^{\alpha_1}\cdots x_k^{\alpha_k} dx_1\cdots dx_k \ ,$$

where P_i is some Laurent polynomial with respect to variables $x_1,\cdots,x_k$:
$P_i(x_1,\cdots,x_k) = \sum c_{\omega_1\cdots\omega_k} x_1^{\omega_1}\ldots x_k^{\omega_k}$,
with $\omega_j \in \mathbb{Z}$, and $\alpha_i,\beta_j \in \mathbb{C}$. We assume that the region Σ is chosen such that the integral exists.

$$\Phi\left(a_{js},b_{kr},c_i,d_j,\gamma,\vec{x}\right) = \int_{\gamma+i\mathbb{R}} dz_1\ldots dz_m \frac{\Pi_{j=1}^p \Gamma\left(\sum_{s=1}^m a_{js}z_s + c_j\right)}{\Pi_{k=1}^q \Gamma\left(\sum_{r=1}^m b_{kr}z_r + d_k\right)} x_1^{-z_1}\ldots x_m^{-z_m} \ ,$$

where $a_{js},b_{kr},c_i,d_j \in \mathbb{R}$, $\alpha_\neg \in \mathbb{C}$, and γ is chosen such that the integral exists. This integral can be expressed in terms of a sum of the residues of the integrated expression.

Series representation

We will take the Horn definition of the series representation. In accordance with this definition, a formal (Laurent) power series in r variables,

$$\Phi(\vec{x}) = \sum C(\vec{m})\vec{x}^{m} \equiv \sum_{m_1,m_2,\cdots,m_r} C(m_1,m_2,\cdots,m_r) x_1^{m_1} \cdots x_r^{m_r},$$

is called hypergeometric if for each $i = 1, \cdots, r$ the ratio

$$C(\vec{m}+\vec{e}_i)/C(\vec{m})$$

is a rational function in the index of summation: $\vec{m} = (m_1, \cdots, m_r)$, where $\vec{e}_j = (0, \cdots, 0, 1, 0, \cdots, 0)$, is unit vector with unity in the $j^{\rm th}$ place. Ore[1930], Sato[1990] found that the coefficients of such a series have the general form

$$C(\vec{m}) = \Pi_{i=1}^{r} \lambda_i^{m_i} R(\vec{m}) \left(\Pi_{j=1}^{N} \Gamma(\mu_j(\vec{m}) + \gamma_j + 1) \right)^{-1},$$

where $N \geq 0$, $\lambda_j, \gamma_j \in \mathbb{C}$ are arbitrary complex numbers, $\mu_j : \mathbb{Z}^{\grave{}} \to \mathbb{Z}$ are arbitrary linear maps, and R is an arbitrary rational function. The fact that all the Γ factors are in the denominator is inessential: using the relation $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, they can be converted to factors in the numerator.

A series of this type is called a "Horn-type" hypergeometric series. In this case, the system of differential equations has the form

$$Q_j \left(\sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k} \right) \frac{1}{x_j} \Phi(\vec{x}) = P_j \left(\sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k} \right) \Phi(\vec{x}) \,,$$

where P_j and Q_r are polynomials satisfying

$$\frac{C(\vec{m} + e_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})}.$$

Holonomic representations

A combination of differential and difference equations can be found to describe functions of the form

$$\Phi(\vec{z},\vec{x},W) = \sum_{k_1,\cdots,k_r=0}^{\infty} \left(\Pi_{a=1}^{m} \frac{1}{z_a + \sum_{b=1}^{r} W_{ab} k_j} \right) \Pi_{j=1}^{r} \frac{x_j^{k_j}}{k_j!} \,,$$

where W is an $r \times m$ matrix. In particular, this function satisfies the equations

$$\frac{\partial \Phi(\vec{z},\vec{x},W)}{\partial x_j} = \Phi(\vec{z}+\omega_j, x, W) \;, \quad j = 1,\cdots,r,$$

$$\frac{\partial}{\partial z_i} \left(z_i \Phi + \sum_{j=1}^{r} W_i x_j \frac{\partial \Phi}{\partial x_j} \right) = 0 \;, \quad i = 1,\cdots,m,$$

where ω_j is the $j^{\rm th}$ column of the matrix W.

Example

To illustrate difference between series representation and combination of differential/difference equation, let us consider the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ which we introduce via series representation

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \; , $$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol. The differential equation is

$$ \frac{d}{dz}\left(z\frac{d}{dz} + c - 1 \right) {}_2F_1(a,b;c;z) = $$
$$ \left(z\frac{d}{dz} + a \right) \left(z\frac{d}{dz} + b \right) {}_2F_1(a,b;c;z) \; . $$

Holonomic definition of Gauss hypergeometric function:

$$ \frac{d}{dz} {}_2F_1(a,b;c;z) = {}_2F_1(a+1,b+1;c+1;z) \; , $$
$$ {}_2F_1(a+1,b;c;z) - z {}_2F_1(a+1,b+1;c+1;z) = $$
$$ a {}_2F_1(a,b;c;z) \; , $$
$$ {}_2F_1(a,b+1;c;z) - z {}_2F_1(a+1,b+1;c+1;z) = $$
$$ b {}_2F_1(a,b;c;z) \; , $$
$$ {}_2F_1(a,b;c-1;z) - z {}_2F_1(a+1,b+1;c+1;z) = $$
$$ (c-1) {}_2F_1(a,b;c;z) \; . $$

Differential Reduction of Hypergeometric Functions

Let us introduce the differential operator

$$\theta = z\frac{d}{dz} \ ,$$

so that

$$\theta z^n = n z^n \ ,$$

and

$$\theta \ {}_pF_q(\vec{a};\vec{b};z) = z\frac{\Pi_{i=1}^p a_i}{\Pi_{j=1}^q b_j} {}_pF_q(\vec{a}+1;\vec{b}+1;z) \ .$$

In terms of this operator, the differential equation for the hypergeometric function ${}_pF_q$ can be written as

$$\left[z\Pi_{i=1}^p(\theta+a_i) - \theta\Pi_{i=1}^q(\theta+b_i-1)\right] {}_pF_q(\vec{a};\vec{b};z) = 0.$$

The universal differential identities are

$$\begin{aligned} a_1 F(a_1+1,\vec{a};\vec{b};z) &= B^+_{a_1} F(a_1,\vec{a};\vec{b};z) \\ &= (\theta + a_1)\, F(a_1,\vec{a};\vec{b};z) \ , \\ (b_1-1)\, F(\vec{a};b_1-1,\vec{b};z) &= H^-_{b_1} F(\vec{a};b_1,\vec{b};z) \\ &= \, = (\theta+b_1-1)\, F(\vec{a};b_1,\vec{b};z) \ , \end{aligned}$$

which are directly follows from series representation.

Consider the ring

$$\mathbb{R} = \mathbb{C}(x_1, \cdots x_n)[\partial/\partial x_1, \cdots, \partial/\partial x_n]$$

of linear partial differential operators and it maximal left-ideal I_λ parametrized by complex number λ. Denote by S_λ the collection of functions annihilated by I_λ.

Theorem (Takayama):

If we have step-up operators,

$$H_\lambda^+ : S_\lambda \to S_{\lambda+1} \ ,$$

then the step-down operators

$$B_\lambda^- : S_{\lambda+1} \to S_\lambda \ ,$$

could be constructed by solving the equation

$$B_{\lambda+1}^- H_\lambda^+ \equiv 1 \quad (mod \quad I_\lambda)$$

Similarly, if step-down operators B_λ^- are known, the step-up operators H_λ^+ are defined from equation:

$$H_\lambda^+ B_{\lambda+1}^- \equiv 1 \quad (mod \quad I_\lambda)$$

Example: Gauss hypergeometric function

Let us consider Gauss hypergeometric function

$$ {}_2F_1(a_1, a_2; b; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b)_k} \frac{z^k}{k!} \, , $$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol.
The universal differential operators:

$$ \begin{aligned} a_1 F(a_1+1, a_2; b; z) &= (\theta + a_1) \, F(a_1, a_2; b; z) \, , \\ (b-1) \, F(a_1, a_2; b-1; z) &= (\theta + b - 1) \, F(a_1, a_2; b; z) \, , \end{aligned} $$

Inverse operators are:

$$ {}_2F_1 \left(\begin{array}{c} a_1 - 1, a_2 \\ b \end{array} \middle| z \right) = $$
$$ \frac{1}{b - a_1} \left[(1-z)\theta + b - a_1 - a_2 z \right] {}_2F_1 \left(\begin{array}{c} a_1, a_2 \\ b \end{array} \middle| z \right) \, , $$
$$ {}_2F_1 \left(\begin{array}{c} a_1, a_2 \\ b+1 \end{array} \middle| z \right) = $$
$$ \frac{b}{(b-a_1)(b-a_2)} \left[(1-z)\theta + b - a_1 - a_2 \right] {}_2F_1 \left(\begin{array}{c} a_1, a_2 \\ b \end{array} \middle| z \right) \, . $$

Ideal:

$$ \left[z(\theta + a_1)(\theta + a_2) - \theta(\theta + b - 1) \right] {}_2F_1 \left(\begin{array}{c} a_1, a_2 \\ b \end{array} \middle| z \right) = 0 \, . $$

Example: Appell Function F_1

Let us consider Appell Function F_1

$$F_1(a,b_1,b_2;c;z_1,z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}} \frac{z_1^m}{m!}\frac{z_2^n}{n!} ,$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol. The system of equations are:

$$[\partial_{z_1}(\theta_{z_1}+\theta_{z_2}+c-1) - (\theta_{z_1}+\theta_{z_2}+a)(\theta_{z_1}+b_1)]w(z_1,z_2) = 0$$
$$[\partial_{z_2}(\theta_{z_1}+\theta_{z_2}+c-1) - (\theta_{z_1}+\theta_{z_2}+a)(\theta_{z_2}+b_2)]w(z_1,z_2) = 0$$
$$(z_1-z_2)\frac{\partial^2 w(z_1,z_2)}{\partial z_1 \partial z_2} - \alpha_1 \frac{\partial w(z_1,z_2)}{\partial z_1} - \alpha_2 \frac{\partial w(z_1,z_2)}{\partial z_2} = 0$$

Universal relations:

$$aF_1(a+1) = (a+\theta_1+\theta_2)F_1$$
$$(c-1)F_1(c-1) = (\theta_1+\theta_2+c-1)F_1$$

Inverse operators:

$$(c-a)F_1(a-1) = [\theta_1+\theta_2-z_1(b_1+\theta_1)-z_2(b_2+\theta_2)+c-a]F_1 ,$$

$$(c-a)(c-b_1-b_2)F_1(c+1) = c[c-a-b_1-b_2-(1-z_1)\theta_1-(1-z_2)\theta_2]F_1 .$$

HYPERDIRE

HYP ergeometric DI fferential RE duction

by Vladimir Bytev, M.K. & Bernd Kniehl

MATHEMATICA based program for differential reduction of hypergeometric functions to basis. Current version allow to work with hypergeometric functions of one variable ${}_PF_{P-1}$ and some functions of two-variables: F_1, F_2, F_3, F_4

Construction of ε-expansion: Formulation of problem

The elaboration of the algorithm for reduction and analytical evaluation of the higher order terms of the ε-expansion of any hypergeometric functions of several variables with arbitrary set of parameters.

There is not universal agreement on what it means to express a solution in terms of known special functions. One reasonable answer has been presented by Kitaev, when he quotes R. Askye's Forward to the book *Symmetries and Separation of Variables* by W. Miller, Jr., which says "One term which has not been defined so far is 'special function'. My definition is simple, but not time invariant. A function is a special function if it occurs often enough so that it gets a name".

Kitaev adds, "... most of the people who apply them . . . understand, under the notion of special functions, a set of functions which can be found in one of the well-known reference books. . . ." To this, we may add "functions which can be found in one of the well-known computer algebra systems."

Multiple polylogarithms (Hyperlogarithms)

The starting point of our consideration is integral

$$
\begin{aligned}
& I(z; a_k, a_{k-1} \cdots, a_1) \\
&= \int_0^z \frac{dt_k}{t_k - a_k} \int_0^{t_k} \frac{dt_{k-1}}{t_{k-1} - a_{k-1}} \cdots \int_0^{t_2} \frac{dt_1}{t_1 - a_1} \\
&= \int_0^z \frac{dt}{t - a_k} I(t; a_{k-1} \cdots, a_1) \, ,
\end{aligned}
$$

where we put that all $a_k \neq 0$. In early consideration by Kummer, Poincare, Lappo-Danilevky this integral was called as hyperlogarithms It was treated as analytical functions of one variable z, the upper limit of integration. Goncharov has analysed it as multivalued analytical functions on $a_1, \cdots, a_k, z$. One of the property of hyperlogarithms is the scaling invariance:

$$
I(z; a_1, \cdots, a_k) = I\left(1; \frac{a_1}{z}, \cdots, \frac{a_k}{z}\right) \, .
$$

By definition, the multiple polylogarithm

$$
\mathrm{Li}_{k_1, k_2, \cdots, k_n}(x_1, x_2, \cdots, x_n) = \sum_{m_n > \cdots > m_1 > 0}^{\infty} \frac{x_1^{m_1}}{m_1^{k_1}} \frac{x_2^{m_2}}{m_2^{k_2}} \cdots \frac{x_n^{m_n}}{m_n^{k_n}} \, ,
$$

where **weight** $k = k_1 + k_2 + \cdots + k_n$ and **depth** is equal to n.

Multiple polylogarithms II

The multiple polylogarithm is a special case of iterated integral

$$
\begin{aligned}
& G_{m_n,m_{n-1},\cdots,m_1}(z;x_n,\cdots,x_1) \\
& \equiv I(z;\underbrace{0,\cdots,0}_{m_n-1 \text{ times}},x_n,\cdots,\underbrace{0,\cdots,0}_{m_1-1 \text{ times}},x_1) \\
& = (-1)^n \mathrm{Li}_{m_1,m_2,\cdots,m_n}\left(\frac{x_2}{x_1},\frac{x_3}{x_2},\cdots,\frac{z}{x_n}\right) \,.
\end{aligned}
$$

The inverse relation is

$$
\begin{aligned}
& \mathrm{Li}_{k_1,k_2,\cdots,k_n}(y_1,y_2,\cdots,y_n) \\
& = (-1)^n G_{1;k_n,k_{n-1},\cdots,k_2,k_1}\left(\frac{1}{y_n},\frac{1}{y_n y_{n-1}},\cdots,\frac{1}{y_1\cdots y_n}\right) \,.
\end{aligned}
$$

The multiple polylogarithms satisfy two Hopf algebras, so called shuffle and stuffle ones. The first is related with integral representation, the second one with series.

Multiple polylogarithms: Particular cases

A particular case of the multiple polylogarithm is the "generalized polylogarithm" defined by

$$\mathrm{Li}_{k_1,k_2,\ldots,k_n}(z) = \sum_{m_n > m_{n-1} > \cdots > m_1 > 0}^{\infty} \frac{z^{m_n}}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}$$

where $|z| < 1$ when all $k_i \geq 1$, or $|z| \leq 1$ when $k_n \leq 2$.

Another particular case is a "multiple polylogarithm of a square root of unity," defined as

$$\mathrm{Li}_{\left(\substack{\sigma_1,\sigma_2,\cdots,\sigma_n \\ s_1,s_2,\cdots,s_n}\right)}(z) = \sum_{m_n > m_{n-1} > \cdots m_1 > 0} z^{m_n} \frac{\sigma_n^{m_n} \cdots \sigma_1^{m_1}}{m_n^{s_n} \cdots m_1^{s_1}} \, .$$

where $\vec{s} = (s_1, \cdots s_n)$ and $\vec{\sigma} = (\sigma_1, \cdots, \sigma_n)$ are multi-indices and σ_k belongs to the set of the square roots of unity, $\sigma_k = \pm 1$. This particular case of multiple polylogarithms has been analyzed in detail by Remiddi and Vermaseren, 2000

For the numerical evaluation of multiple polylogarithms: Vollinga & Weinzierl, 2005; Maître, 2006,2008

Integral & Series representation

In the Euler integral representation, the most important results are related to the construction of the all-order ε expansion of Gauss hypergeometric function with special values of parameters in terms of Nielsen polylogarithms
A.I.Davydychev , Phys.Rev.(1999);
A.I.Davydychev & M.K., Nucl.Phys.Proc.Suppl.89 (2000);
A.I.Davydychev & M.K., Nucl.Phys.B605 (2001)

The series representation is an intensively studied approach. Particularly impressive results were derived in the framework of the nested-sum approach for hypergeometric functions with a balanced set of parameters by
Moch,Uwer,Weinzierl,2002; Weinzierl,2004;
Computer realizations of nested sums approach to expansion of hypergeometric functions are given in
Weinzierl, 2002; Moch & Uwer, 2006;
Huber & Maître, 2006, 2008

Generating-function approach have been applied to construction of ε-expansion for hypergeometric functions with one unbalanced set of parameters
M.K,Ward,Yost,2007; M.K, Kniehl,2008

Iterated solution

An approach using the iterated solution of differential equations has been explored by Shu Oi, 2004, M.K.,Ward,Yost,2007, M.K.,Kniehl,2008

One of the advantages of the iterated-solution approach over the series approach is that it provides a more efficient way to calculate each order of the ε expansion, since it relates each new term to the previously derived terms, rather than having to work with an increasingly large collection of independent sums at each order. This technique includes two steps: (i) the differential-reduction algorithm (to reduce a generalized hypergeometric function to basic functions); (ii) iterative solution of the proper differential equation for the basic functions (equivalent to iterative algorithms for calculating the analytical coefficients of the ε expansion).

Results

Here, we will mention some of the existing results.

- If I_1, I_2, I_3 are arbitrary integers, the Laurent expansions of the Gauss hypergeometric functions

$$
\begin{aligned}
&{}_2F_1(I_1 + a\varepsilon, I_2 + b\varepsilon; I_3 + \tfrac{p}{q} + c\varepsilon; z) \; , \\
&{}_2F_1(I_1 + \tfrac{p}{q} + a\varepsilon, I_2 + \tfrac{p}{q} + b\varepsilon; I_3 + \tfrac{p}{q} + c\varepsilon; z) \; , \\
&{}_2F_1(I_1 + \tfrac{p}{q} + a\varepsilon, I_2 + b\varepsilon; I_3 + c\varepsilon; z) \; , \\
&{}_2F_1(I_1 + \tfrac{p}{q} + a\varepsilon, I_2 + b\varepsilon; I_3 + \tfrac{p}{q} + c\varepsilon; z)
\end{aligned}
$$

  are expressible in terms of multiple polylogarithms of arguments being powers of q-roots of unity and a new variable, that is an algebraic function of z, with coefficients that are ratios of polynomials.

- If $\vec{A}, \vec{B}$ are lists of integers and I, p, q are integers, the Laurent expansions of the generalized hypergeometric functions

$$
\begin{aligned}
&{}_pF_{p-1}(\vec{A} + \vec{a}\varepsilon, \tfrac{p}{q} + I; \vec{B} + \vec{b}\varepsilon; z) \; , \\
&{}_pF_{p-1}(\vec{A} + \vec{a}\varepsilon; \vec{B} + \vec{b}\varepsilon, \tfrac{p}{q} + I; z)
\end{aligned}
$$

  are expressible in terms of multiple polylogarithms of arguments that are powers of q-roots of unity and a new variable that is an algebraic function of z, with coefficients that are ratios of polynomials.

Results: II

- If $\vec{A}, \vec{B}$ are lists of integers, the Laurent expansion of the generalized hypergeometric function

$$ {}_pF_{p-1}(\vec{A} + \vec{a}\varepsilon; \vec{B} + \vec{b}\varepsilon; z) $$

are expressible via generalized polylogarithms.

- If p, q, I_k are any integers and $\vec{A}, \vec{B}$ are lists of integers, the generalized hypergeometric function

$$ {}_pF_{p-1}(\{\tfrac{p}{q}+\vec{A}+\vec{a}\varepsilon\}_r, \vec{I_1}+\vec{c}\varepsilon; \{\tfrac{p}{q}+\vec{B}+\vec{b}\varepsilon\}_r, \vec{I_2}+\vec{d}\varepsilon; z) $$

is expressible in terms of multiple polylogarithms of arguments that are powers of q-roots of unity and the new variable $z^{1/q}$, with coefficients that are ratios of polynomials.

- the coefficients of the ε expansion of the hypergeometric functions

$$ {}_{p+1}F_p\left(\left.\begin{array}{c} \vec{A}+\frac{r}{q}+\vec{a}\varepsilon \\ \vec{B}+\frac{r}{q}+\vec{b}\varepsilon \end{array}\right| z\right), \quad {}_{p+1}F_p\left(\left.\begin{array}{c} \vec{A}+\vec{a}\varepsilon \\ \vec{B}+\vec{b}\varepsilon, I+\frac{r}{q}+c\varepsilon \end{array}\right| z\right) $$

$$ {}_{p+1}F_p\left(\left.\begin{array}{c} I+\frac{r}{q}+c\varepsilon, \vec{A}+\vec{a}\varepsilon \\ \vec{B}+\vec{b}\varepsilon \end{array}\right| z\right), $$

where $\vec{A}, \vec{B}, \vec{a}, \vec{b}, c$ and I are all integers, are related to each other.

Generalized hypergeometric functions

The generalized hypergeometric function can be written as series

$$
{}_PF_Q\left(\begin{array}{c} \{A_1+a_1\varepsilon\}, \{A_2+a_2\varepsilon\}, \cdots \{A_P+a_P\varepsilon\} \\ \{B_1+b_1\varepsilon\}, \{B_2+b_2\varepsilon\}, \cdots \{B_Q+b_q\varepsilon\} \end{array} \middle| z \right)
= \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{\Pi_{s=1}^{P} (A_s+a_s\varepsilon)_j}{\Pi_{r=1}^{Q} (B_r+b_r\varepsilon)_j} \,,
$$

where $(\alpha)_j \equiv \Gamma(\alpha+j)/\Gamma(\alpha)$ is the Pochhammer symbol.

It is well known that any function

$$
{}_pF_{p-1}(\vec{a}+\vec{m}; \vec{b}+\vec{k}; z)
$$

is expressible in terms of p other functions of the same type:

$$
R_{p+1}(\vec{a},\vec{b},z) {}_pF_{p-1}(\vec{a}+\vec{m}; \vec{b}+\vec{k}; z) =
\sum_{k=1}^{p} R_k(\vec{a},\vec{b},z) {}_pF_{p-1}(\vec{a}+\vec{e_k}; \vec{b}+\vec{E}_k; z) \,,
$$

where $\vec{m}, \vec{k}, \vec{e}_k$, and $\vec{E}_k$ are lists of integers and R_k are polynomials in parameters $\vec{a}, \vec{b}$, and z.

Construction of all-order ε-expansion via Differential equation

M.K.,Ward,Yost, JHEP, 07.

$$\omega(z) = {}_pF_{p-1}(\vec{a}\varepsilon; \vec{1} + \vec{b}\varepsilon; z)$$

Defining the coefficients functions $w_k(z)$ at each order by

$$\omega(z) = \sum_{k=0}^{\infty} w_k(z)\varepsilon^k,$$

The differential equation is

$$\left[(1-z)\frac{d}{dz}\right]\left(z\frac{d}{dz}\right)^{p-1} w_k(z) =$$
$$\sum_{i=1}^{p-1}\left[P_i(\vec{a})-\frac{1}{z}Q_i(\vec{b})\right]\left(z\frac{d}{dz}\right)^{p-i} w_{k-i}(z)+P_p(\vec{a})w_{k-p}(z) \; ,$$

where $P_j(\vec{a})$ and $Q_j(\vec{b})$ are polynomials of order j depending on vectors $\vec{a}$ and $\vec{b}$, respectively.

$$z\frac{d}{dz}\rho_k^{(j)}(z) = \rho_k^{(j+1)}(z) \; , \quad j = 0, 1, \cdots, p-2$$
$$(1-z)\frac{d}{dz}\rho_k^{(p-1)}(z) = \sum_{i=1}^{p}\left[P_i(\vec{a})-\frac{1}{z}Q_i(\vec{b})\right]\rho_{k-i}^{(p-i)}(z) \; ,$$

The solution is iterated integral:

$$
\begin{aligned}
\rho_k^{(p-1)}(z) &= \sum_{i=1}^{p} \left[P_i(\vec{a}) - Q_i(\vec{b}) \right] \int_0^z \frac{dt}{1-t} \rho_{k-i}^{(p-i)}(t) \\
&\quad - \sum_{i=1}^{p-2} Q_i(\vec{b}) \rho_{k-i}^{(p-i-1)}(z) \\
&\quad - Q_{p-1}(\vec{b}) [w_{k-p+1}(z) - \delta_{0,k-p+1}] \;, \\
\rho_k^{(j-1)}(z) &= \int_0^z \frac{dt}{t} \rho_k^{(j)}(t) \;, \quad k \geq 1 \;, \quad j = 1, 2, \cdots, p-2 \;,
\end{aligned}
$$

Iterative solution of Gauss hypergeometric function:

$$\omega(z) = {}_2F_1\left(\begin{array}{c} a_1\varepsilon, a_2\varepsilon \\ \frac{1}{2}+f\varepsilon \end{array} \middle| z \right) .$$

Defining the coefficients functions $w_k(z)$ at each order by

$$\omega(z) = \sum_{k=0}^{\infty} w_k(z)\varepsilon^k,$$

Original equation is

$$\left[(1-z)\frac{d}{dz} - \frac{1}{2z}\right]\left(z\frac{d}{dz}\right) w_i(z)$$
$$= \left[(a_1+a_2)-\frac{f}{z}\right]\left(z\frac{d}{dz}\right) w_{i-1}(z)+a_1a_2w_{i-2}(z) .$$

Let us introduce the new variable y

$$y = \frac{1-\sqrt{\frac{z}{z-1}}}{1+\sqrt{\frac{z}{z-1}}}, \quad z = -\frac{(1-y)^2}{4y} ,$$

and define a set of a new functions $\rho_i(y)$

$$\begin{aligned} y\frac{d}{dy}\rho_i(y) &= (a_1+a_2)\frac{1-y}{1+y}\rho_{i-1}(y) \\ &\quad +2f\left(\frac{1}{1-y}-\frac{1}{1+y}\right)\rho_{i-1}(y)+a_1a_2w_{i-2}(y) , \\ y\frac{d}{dy}w_k(y) &= -\rho_k(y) . \end{aligned}$$

The solution of these differential equations has the form

$$
\begin{aligned}
\rho_i(y) &= \int_1^y dt \left[2f \frac{1}{1-t} - 2(a_1+a_2-f)\frac{1}{1+t} \right] \rho_{i-1}(t) \\
&\quad -(a_1+a_2)\left[w_{i-1}(y) - w_{i-1}(1) \right] \\
&\quad +a_1 a_2 \int_1^y \frac{dt}{t} w_{i-2}(t) \,, \quad i \geq 1 \,, \\
w_i(y) &= -\int_1^y \frac{dt}{t} \rho_i(t) \,, \quad i \geq 1 \,.
\end{aligned}
$$

Application to Feynman diagrams

The case of one-loop Feynman diagrams has been studied the most. The hypergeometric representations for N-point one-loop diagrams with arbitrary powers of propagators and an arbitrary space-time dimension have been derived for non-exceptional kinematics by
Davydychev & Boos, 1991; Davydychev 1991
His approach is based on the Mellin-Barnes technique. The results for N-point diagram are expressible in terms of Lauricella function $F_D^{(N)}$ depending on $1/2(N-1)(N+2)$ variables.

An alternative hypergeometric representation for one-loop diagrams has been derived recently by Fleischer, Jegerlehner, Tarasov, 2003 using a difference equation in the space-time dimension. In this approach, the one-loop N-point function was shown to be expressible in terms of hypergeometric functions of $N{-}1$ variables. One remarkable feature of the derived results is a one-to-one correspondence between arguments of the hypergeometric functions and Gram and Cayley determinants, which are two of the main characteristics of diagrams.

Application to Feynman diagrams: construction ε-expansion

The program of constructing the analytical coefficients of the ε-expansion is a more complicated matter. The finite parts of one-loop diagrams in $d = 4$ dimension are expressible in terms of the Spence dilogarithm function
Hooft, Veltman, 1979;
Denner, Nierste, Scharf, 1991
Only partial results for higher-order terms in the ε-expansion are known at one loop. The all-order ε-expansion of the one-loop propagator with an arbitrary values of masses and external momentum has been constructed in terms of Nielsen polylogarithms:
A.I.Davydychev, 1999; A.I.Davydychev & M.K., 2000, 2001;
The term linear in ε for the one-loop vertex diagram with non-exceptional kinematics has also been constructed in terms of Nielsen polylogarithms
Nierste, Müller, Böhm, 1993
The all-order ε expansion for the one-loop vertex with non-exceptional kinematics is expressible in terms of multiple polylogarithms of two variables:
Davydychev, 2006; Tarasov, 2008
Beyond these examples, the situation is less complete. The term linear in ε for the box diagram is still under construction. Some cases for particular masses have been analyzed
Fleischer, Riemann, Tarasov, 2003;
Körner, Merebashvili, Rogal, 2005,2006
Many physically interesting particular cases have been considered beyond one loop.

1-loop vertex: Particular case

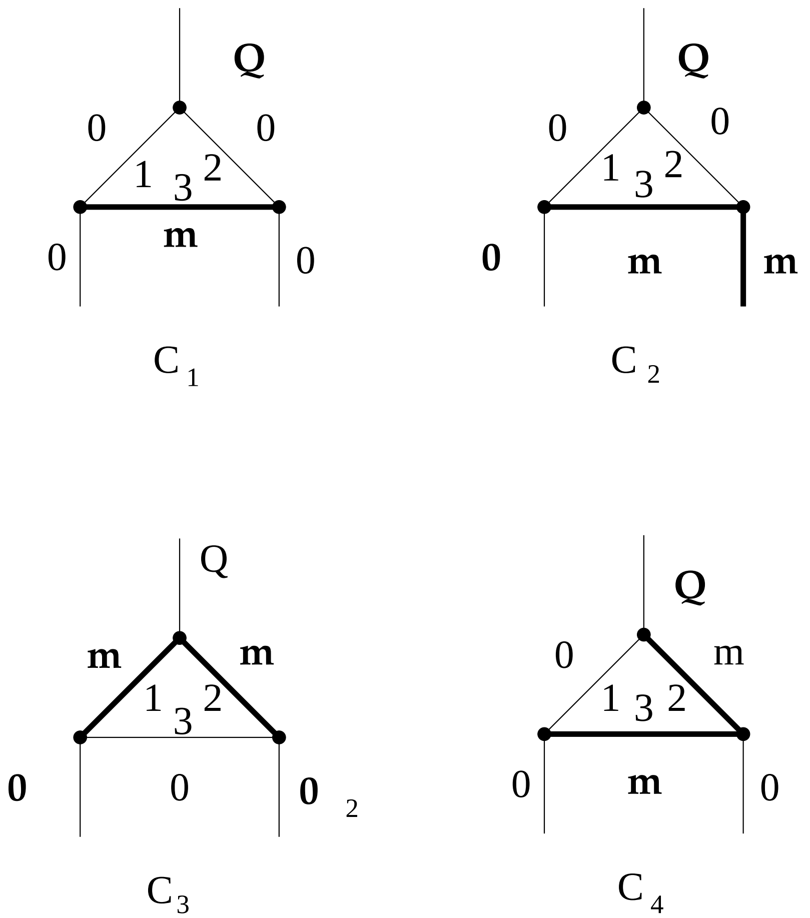

Figure 1: One-loop vertex-type diagrams expressible in terms of generalized hypergeometric functions. Bold and thin lines correspond to massive and massless propagators, respectively.

$$
\frac{C_1}{i^{1-n}\pi^{\frac{n}{2}}} = (-m^2)^{\frac{n}{2}-j_{123}}
\left\{ \frac{\Gamma\left(j_{123}-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-j_{12}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(j_3)} {}_3F_2\left(\begin{array}{c} j_{123}-\frac{n}{2}, j_1, j_2 \\ \frac{n}{2}, 1+j_{12}-\frac{n}{2} \end{array} \middle| -\frac{Q^2}{m^2} \right) \right.
+ \left(-\frac{Q^2}{m^2}\right)^{\frac{n}{2}-j_{12}} \frac{\Gamma\left(\frac{n}{2}-j_1\right)\Gamma\left(\frac{n}{2}-j_2\right)\Gamma\left(j_{12}-\frac{n}{2}\right)}{\Gamma\left(n-j_{12}\right)\Gamma(j_1)\Gamma(j_2)}
\left. \times {}_3F_2\left(\begin{array}{c} j_3, \frac{n}{2}-j_1, \frac{n}{2}-j_2 \\ n-j_{12}, \frac{n}{2}-j_{12}+1 \end{array} \middle| -\frac{Q^2}{m^2} \right) \right\} .
$$

$$
\frac{C_2}{i^{1-n}\pi^{\frac{n}{2}}} = (-m^2)^{\frac{n}{2}-j_{123}}
\left\{ \frac{\Gamma\left(j_{123}-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-j_{12}\right)\Gamma\left(n-j_{13}-2j_2\right)}{\Gamma\left(n-j_{123}\right)\Gamma\left(\frac{n}{2}-j_2\right)\Gamma(j_3)} \right.
{}_3F_2\left(\begin{array}{c} j_{123}-\frac{n}{2}, j_1, j_2 \\ 1+j_{12}-\frac{n}{2}, 1+j_{13}+2j_2-n \end{array} \middle| \frac{Q^2}{m^2} \right)
+ \left(-\frac{Q^2}{m^2}\right)^{\frac{n}{2}-j_{12}} \frac{\Gamma\left(\frac{n}{2}-j_1\right)\Gamma\left(j_{12}-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-j_{23}\right)}{\Gamma\left(n-j_{123}\right)\Gamma(j_1)\Gamma(j_2)}
\left. \times {}_3F_2\left(\begin{array}{c} j_3, \frac{n}{2}-j_2, \frac{n}{2}-j_1 \\ \frac{n}{2}-j_{12}+1, 1+j_{23}-\frac{n}{2} \end{array} \middle| \frac{Q^2}{m^2} \right) \right\} .
$$

In accordance with **HYPERDIRE**, the result of reduction are expressible in terms of two Gauss hypergeometric functions with upper integer parameter.

$$\frac{C_3}{i^{1-n}\pi^{\frac{n}{2}}} = (-m^2)^{\frac{n}{2}-j_{123}} \frac{\Gamma\left(j_{123}-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-j_3\right)}{\Gamma\left(j_{12}\right)\Gamma\left(\frac{n}{2}\right)}$$
$${}_4F_3\left(\begin{array}{c} j_{123}-\frac{n}{2}, j_1, j_2, \frac{n}{2}-j_3 \\ \frac{n}{2}, \frac{j_{12}}{2}, \frac{j_{12}+1}{2} \end{array} \middle| \frac{Q^2}{4m^2} \right) ,$$

In accordance with **HYPERDIRE**, this function could be written in terms of two ${}_3F_2$ functions.

$$\frac{C_4}{i^{1-n}\pi^{\frac{n}{2}}} = (-m^2)^{\frac{n}{2}-j_{123}} \frac{\Gamma\left(j_{123}-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-j_1\right)}{\Gamma\left(j_{23}\right)\Gamma\left(\frac{n}{2}\right)}$$
$${}_3F_2\left(\begin{array}{c} j_{123}-\frac{n}{2}, j_1, j_2 \\ \frac{n}{2}, j_{23} \end{array} \middle| \frac{Q^2}{m^2} \right) .$$

In accordance with **HYPERDIRE**, this function could be written in terms of two ${}_2F_1$ function with one integer parameter.