hexagon.m, CSectors.m, AMBRE.m

– New Results for Loop Integrals –

Tord Riemann, DESY, Zeuthen

ACAT, 03–07 November 2008

Ettore Majorana Foundation and Centre for Scientific Culture, Erice, Sicily

Prepared together with J. Gluza and K. Kajda (U. Silesia, Katowice) and Valery Yundin (DESY), thanks also to T. Diakonidis, J. Fleischer, B. Tausk

- **Introduction** – Feynman integrals, M-point functions with L loops
- **hexagon.m** – Reduce 1-loop tensors $\longrightarrow$ scalar 2-,3-,4-point functions [new]
- **CSectors.m** – Use sector_decomposition $\longrightarrow$ numerical approach [an interface]
- **AMBRE.m** – Use Mellin-Barnes representations $\longrightarrow$ numerics or sums [an update]
- A Short **Summary**

Introduction

In recent years, we observe that

- higher energy
- higher luminosity
- higher precision
- more massive particles

at

- LHC
- ILC
- but also at low energy meson factories

lead to completely new demands on the efficient evaluation of Feynman diagrams, e.g.:

- need some massive two-loop diagrams, including boxes
- need many n-point one-loop diagrams, massive and massless

Few of the approaches to answer the requests will be shortly introduced, concentrating on our own activities and on our publicly available packages.

Loop momentum integrations

with Feynman parameters – or without

for L-loop n-point functions and tensor rank R

Consider an arbitrary L-loop integral $G(X)$ with loop momenta k_l, with E external legs with momenta p_e, and with N internal lines with masses m_i and propagators $1/D_i$,

$$I = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \ldots d^d k_L \quad k_1^{\alpha_1} \cdots k_L^{\alpha_L}}{D_1^{\nu_1} \ldots D_i^{\nu_i} \ldots D_N^{\nu_N}}.$$

$$D_i^{\nu_i} = \left(q_i^2 - m_i^2\right)^{\nu_i} = \left\{ [\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{E} d_i^e p_e]^2 - m_i^2 \right\}^{\nu_i}$$

where we call $d = 4 - 2\epsilon$ the generic dimension and ν_i the index of the propagator.
The numerator may contain a tensor structure

$$N = 1$$

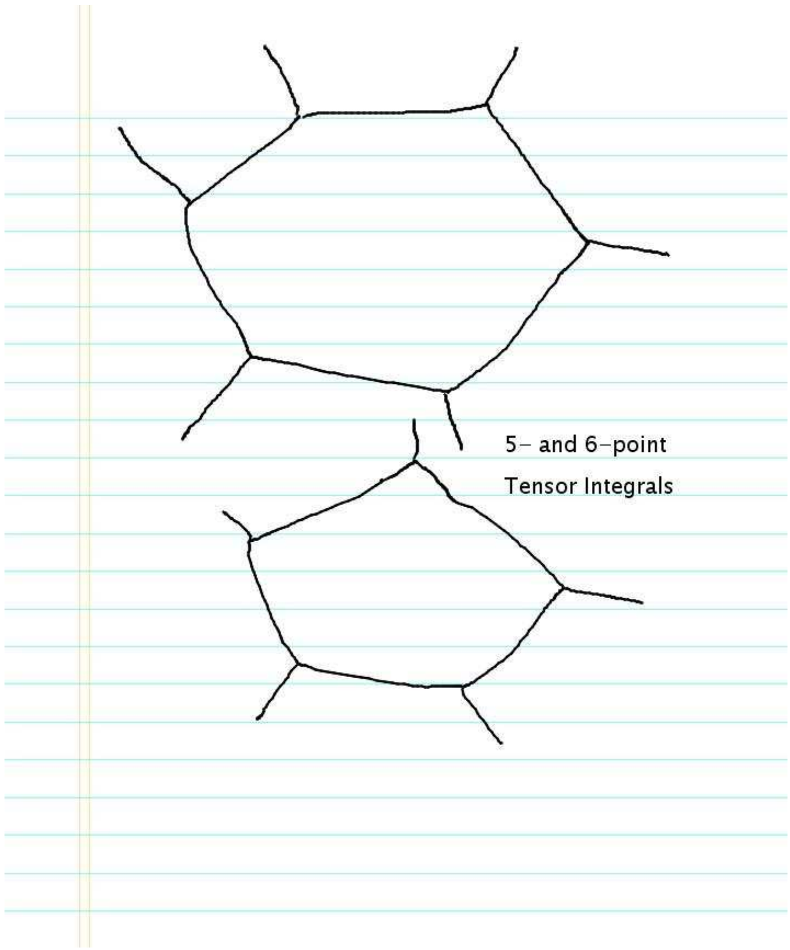
$$N = k_1^{\alpha_1} k_1^{\beta_1}$$

$$N = k_1^{\alpha_1} \cdots k_L^{\alpha_L}$$

etc.

Package hexagon

for 1-loop five- and six-point functions, rank R tensors in the loop momentum



Schematic view

e.g.: $gg \longrightarrow \bar{t}tg, \bar{t}tgg$

or: $e^+e^- \longrightarrow e^+e^-\gamma$

Package hexagon.m

To our knowledge, there was, until quite recently, no publicly available numerical package for the evaluation of

- massive 5- and 6-point tensor integrals

Now we have 'on the market':

- hexagon.m – a Mathematica program (K.Kajda) for tensor reduction
  0807.2984 (Diakonidis, Fleischer, Gluza, Kajda, Riemann, Tausk)
  Needs a library for 2-,3-,4-point scalar functions
  A Fortran version (T. Diakonidis, B. Tausk) is available, but not yet released

- Golem95 – a Fortran program for one-loop amplitudes with massless internal legs
  0810.0992 (T. Binoth, J. Guillet, G. Heinrich, E. Pilon, T. Reiter)

There are also the packages for tensor integrals LoopTools&FF (Th. Hahn, v.Oldenborgh) and OPP by Ossola,Papadopoulos,Pittau.

Perform Algebraic Tensor Reduction (in one loop)

This is an old idea (t'Hooft,Veltman 1979), and there are several more or less sophisticated realizations.

- **See e.g. Denner, Dittmaier, hep-ph/0212259, hep-ph/0509141 and ref's. therein, with unpublished code**

We realize the following approach:

- **Eliminate the numerators of the Feynman integrals (i.e. the tensor structure) by shifting the dimension $d = 4 - 2\epsilon$ in units of two, and shifting the indices of the propagators $\longrightarrow$ Davydychev, PLB 1991**
- **Return to the generic dimension d and the original indices (usually one) by use of integration-by-parts and generalized relations with shift of $d \longrightarrow$ Tarasov, PRD 1996**
- **The reduction introduces inverse Gram determinants; this is extremely unwanted because of potential instabilities in practical applications and it might be avoided; see Campbell,Glover,Miller 1996, but also Denner,Dittmaier and Binoth et.al.**
  We explicitely derived in the Davydychev-Tarasov approach algebraic relations for tensor integrals which are free of the inverse Gram determinants, see Fleischer,Gluza,Kajda,Riemann, 0710.5100, but also DFGKRT@Loops&Legs08, use of work by Melrose 1965

Davydychev's shifts

Following Davydychev

[Davydychev:1991va]

we decompose the tensor integrals $J^{(N)}_{\mu_1\ldots\mu_R}$ **into a basis of scalar integrals** $J^{(N)}$ **constructed from** $g^{\mu\nu}$ **and the momenta** q_j

$$\begin{aligned} J^{(N)}_{\mu_1\ldots\mu_R}(d;\nu_1,\ldots,\nu_N) &= (-1)^R \sum_{\lambda,\kappa_1,\ldots,\kappa_N} (-\frac{1}{2})^{\lambda} \\ & \left\{[g]^{\lambda}[q_1]^{\kappa_1}\ldots[q_N]^{\kappa_N}\right\}_{\mu_1\ldots\mu_R} \frac{\Gamma(\nu_1+\kappa_1)}{\Gamma(\nu_1)}\ldots\frac{\Gamma(\nu_N+\kappa_N)}{\Gamma(\nu_N)} \\ & J^{(N)}(d+2(R-\lambda);\nu_1+\kappa_1,\ldots,\nu_N+\kappa_N) \end{aligned}$$

where the sum runs over non-negative integers such that $2\lambda+\kappa_1+\ldots+\kappa_N=R$**.**

The next step is to use recurrence relations to reduce the scalar coefficients $J^{(N)}$ **appearing in the decomposition to a set of master integrals.**

Tarasov's Reduction to generic dimension d and original indices

The tensor integral of e.g. degree 2 can be written:

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} q_i^{\mu}\, q_j^{\nu} I_{5,ij}$$

with (Fleischer,Jegerlehner,Tarasov 2000):

$$I_{5,ij} \quad = \quad \frac{1}{()_5}\left\{ -\frac{\binom{0}{j}_5}{\binom{0}{0}_5}\sum_{s=1}^{5}\binom{0i}{0s}_5 I_4^s - \sum_{s=1,s\neq i}^{5} \frac{\binom{s}{j}_5\binom{0s}{is}_5}{\binom{s}{s}_5} I_4^s + \sum_{s,t=1,s\neq i,t}^{5} \frac{\binom{s}{j}_5\binom{ts}{is}_5}{\binom{s}{s}_5} I_3^{st}\right\} \tag{1}$$

Crash Course on Gram Determinants, Signed Minors etc.

With

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2,$$

the "modified Cayley determinant" of a diagram with internal lines 1 $\ldots n$ **is**

$$()_n \equiv \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & Y_{11} & Y_{12} & \ldots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix},$$

labelling elements $0, \ldots, n$**. This object is called the Gram determinant; in fact it is (for** $q_n = 0$**):**

$$()_n = -G_{n-1} = -\det(2q_iq_k)$$

Cutting from $()_n$ rows $j_1, j_2, \ldots$ and columns $k_1, k_2, \ldots$, get the "Signed minors" (with a sign convention)(Melrose:1965kb). They are denoted by

$$\begin{pmatrix} j_1 & j_2 & \ldots \\ k_1 & k_2 & \ldots \end{pmatrix}_n ,$$

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n .$$

Simplest examples for a massive 1-loop QED pentagon diagram contributing in $e^+e^- \to e^+e^-\gamma$:

$$\binom{s}{j}_5 = D[s,j]$$

$$D[0,0] = -2stt'v_1v_2 + 2m_e^2(s^2(t-t')^2 + 2s(t+t')v_1v_2 + v_1^2v_2^2),$$

and

$$\begin{aligned}
D[0,1] &= s^2tt' - s^2t'^2 + st'^2v_1 - stt'v_2 - st'v_1v_2 \\
&\quad + m_e^2(-4st'v_1 + 2stv_2 + 2st'v_2 + 2v_1v_2^2), \\
D[0,2] &= D[0,1] \text{ (\textbf{with} } t, v_1 \leftrightarrow t', v_2) \\
D[0,3] &= -(tv_2(s(-t+t') + t'v_1 + (t-v_1)v_2)) \\
&\quad + m_e^2(2st^2 - 4stt' + 2st'^2 + 2tv_1v_2 + 2t'v_1v_2 + 4tv_2^2), \\
D[0,4] &= -(v_1v_2(s(t+t') - t'v_1 + (-t+v_1)v_2)) \\
&\quad + m_e^2(-2stv_1 + 2st'v_1 + 2stv_2 - 2st'v_2 - 2v_1^2v_2 - 2v_1v_2^2), \\
D[0,5] &= D[0,3] \text{ (\textbf{with} } t, v_1 \leftrightarrow t', v_2)..
\end{aligned}$$

$$\binom{ts}{ij}_5 = D[t,i,s,j],$$

$$\begin{aligned}
D[0,0,1,1] &= s^2t'^2 - 4m_e^2 s^2 t', \\
D[0,0,1,2] &= -(s^2tt') + m_e^2 s(2st + 2st' + 2v_1v_2), \\
D[0,0,1,3] &= stt'v_2 - m_e^2 v_2(2st + 2st' + 2v_1v_2), \\
D[0,0,1,4] &= st'v_1v_2 - m_e^2 s(2st - 2st' + 2v_1v_2), \\
D[0,0,1,5] &= -st'^2v_1 + 4m_e^2 st'v_1, \\
D[0,0,2,2] &= s^2t^2 - 4m_e^2 s^2 t, \\
D[0,0,2,3] &= -st^2v_2 + 4m_e^2 stv_2, \\
D[0,0,2,4] &= stv_1v_2 + m_e^2 s(2st - 2st' - 2v_1v_2), \\
D[0,0,2,5] &= stt'v_1 - m_e^2 v_1(2st + 2st' + 2v_1v_2), \\
D[0,0,3,3] &= +t^2v_2^2 - 4m_e^2 tv_2^2, \\
D[0,0,3,4] &= -(tv_1v_2^2) - m_e^2 v_2(2st - 2st' - 2v_1v_2), \\
D[0,0,3,5] &= tt'v_1v_2 + m_e^2(-2st^2 + 4stt' - 2st'^2 - 2tv_1v_2 - 2t'v_1v_2), \\
D[0,0,4,4] &= +v_1^2v_2^2 + 4m_e^2 sv_1v_2, \\
D[0,0,4,5] &= -(t'v_1^2v_2) + m_e^2 v_1(2st - 2st' + 2v_1v_2), \\
D[0,0,5,5] &= +t'^2v_1^2 - 4m_e^2 t'v_1^2.
\end{aligned}$$

Elimination of the inverse Gram determinants

We again show an example, $R = 2$ tensor:

The result contains 4-point functions I_4^s and 3-point functions I_3^{st}:

$$I_5^{\mu\nu} = \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5 \binom{s}{s}_5} \left[-\frac{1}{2} \binom{0}{s}_5 \binom{0s}{0s}_5 g^{\mu\nu} + \sum_{i,j=1}^{4} X_{ij}^{s0} q_i^\mu \, q_j^\nu \right] I_4^s$$

$$+ \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5 \binom{s}{s}_5} \sum_{t=1}^{5} \left[\frac{1}{2} \binom{0}{s}_5 \binom{0s}{ts}_5 g^{\mu\nu} - \sum_{i,j=1}^{4} X_{ij}^{st} q_i^\mu \, q_j^\nu \right] I_3^{st},$$

$$X_{ij}^{st} = -\binom{0s}{0j}_5 \binom{ts}{is}_5 + \binom{0i}{sj}_5 \binom{ts}{0s}_5 .$$

If there is no symmetry at all in the original Feynman integral:

- **five scalar four-point functions I_4^s with one off-shell leg**
- **ten scalar three-point functions $I_3 st$, five of them with one and five with two off-shell legs**

For the Bhabha QED case, we have e.g. three different four-point functions and six three-point functions (three plus three). **For tensors of degree 3, 4, 5 higher inverse powers of the Gram determinant have to be cancelled. This is more difficult, but has been done.**

Using hexagon.m

The package must be loaded in a `MATHEMATICA` environment by executing:

```
<<hexagon.m
```

The package is able to output:

- **the full result for a six or five point tensor integral**
- **a specific coefficient**
- **a list of all coefficients for a given rank**
- **both analytic and numerical results, depending on the user's input.**

Example files for use: See http://www-zeuthen.desy.de/theorie/research/CAS.html

Next slide:
Numerical examples, taken from arXive:0807.2984, presented also at Loops&Legs2008

A five-point tensor coefficient: $p_1^\mu p_1^\nu p_1^\lambda E_{\mu\nu\lambda}$

Point: $p_1^2 = p_2^2 = p_3^2 = p_5^2 = 1,\ p_4^2 = 0,\ m_1^2 = m_3^2 = 0,\ m_2^2 = m_4^2 = m_5^2 = 1,$

$s_{12} = -3,\ s_{23} = -6,\ s_{34} = -5,\ s_{45} = -7,\ s_{15} = -2$

In: `RedE3[` $p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$ `]/.{D4->D0,C3->C0,B2->B0}`

Out: `0.218741`

A six-point scalar function: F_0

Point: $p_1^2 = p_2^2 = p_3^2 = p_4^2 = p_5^2 = p_6^2 = -1,\ m_1^2 = m_2^2 = m_3^2 = m_4^2 = m_5^2 = m_6^2 = 1,$

$s_{12} = s_{23} = s_{34} = s_{45} = s_{56} = s_{16} = s_{123} = s_{234} = -1,\ s_{345} = -5/2$

In: `RedF0[` $p_1^2, \ldots, p_6^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}, m_1^2, \ldots, m_6^2$ `]/.{D4->D0}`

Out: `0.013526`

Several five-point vector and tensor coeff.: $E_0,\ E_1,\ E_2,\ E_3,\ E_4,\ E_{34},\ E_{123},\ E_{002}$

Point: $p_1^2 = p_2^2 = 0,\ p_3^2 = p_5^2 = 49/256,\ p_4^2 = 9/100,\ m_1^2 = m_2^2 = m_3^2 = 49/256,\ m_4^2 = m_5^2 = 81/1600,$

$s_{12} = 4,\ s_{23} = -1/5,\ s_{34} = 1/5,\ s_{45} = 3/10,\ s_{15} = -1/2$

In: `RedE0[` $p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$ `]/.D4->D0`

Out: `41.3403 - 45.9721*I`

In: `RedEget[rank1 ,` $p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$ `]/.D4->D0`

Out: `ee1 =-2.38605 + 5.27599*I, ee2 =-5.80875 + 0.597891*I,`

`ee3 =-14.4931 + 20.8149*I, ee4 =-11.3362 + 18.1593*I`

In: `RedEcoef[ee34 ,` $p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$ `]/.{D4->D0,C3->C0}`

Out: `7.1964 + 3.10115*I`

In: `RedEcoef[ee123 ,` $p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$ `]/.{D4->D0,C3->C0,B2->B0}`

Out: `-0.149527 - 0.31059*I`

In: `RedEcoef[ee002 ,` $p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$ `]/.{D4->D0,C3->C0,B2->B0}`

Out: `0.154517 - 0.387727*I`

Another Approach – Introduce Feynman Parameters

$$\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \ldots D_N^{\nu_N}} \quad = \quad \frac{\Gamma(\nu_1 + \ldots + \nu_N)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 dx_1 \ldots \int_0^1 dx_N \frac{x_1^{\nu_1-1} \ldots x_N^{\nu_N-1} \delta(1 - x_1 \ldots - x_N)}{(x_1 D_1 + \ldots + x_N D_N)^{N_\nu}},$$

with $N_\nu = \nu_1 + \ldots \nu_N$.

The denominator of G contains, after introduction of Feynman parameters x_i, the momentum dependent function m^2 with index-exponent N_ν:

$$(m^2)^{-(\nu_1+\ldots+\nu_N)} \quad = \quad (x_1 D_1 + \ldots + x_N D_N)^{-N_\nu} \quad = \quad (k_i M_{ij} k_j - 2Q_j k_j + J)^{-N_\nu}$$

Here M is an (LxL)-matrix, $Q = Q(x_i, p_e)$ an L-vector and $J = J(x_i x_j, m_i^2, p_{e_j} p_{e_l})$.
M, Q, J are linear in x_i. The momentum integration is now simple:
Shift the momenta k such that m^2 has no linear term in $\bar{k}$:

$$k \quad = \quad \bar{k} + (M^{-1})Q,$$

$$m^2 \quad = \quad \bar{k} M \bar{k} - QM^{-1}Q + J.$$

Finally, one gets for **Scalar integrals:**

$$G(1) \quad = \quad (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - D(L+1)/2}}{F(x)^{N_\nu - DL/2}}$$

with

$$U(x) \quad = \quad (\text{det } M) \; (\rightarrow 1 \text{ for } L = 1)$$

$$F(x) \quad = \quad (\text{det } M) \; \mu^2 \; = \; -(\text{det } M) \; J + Q \; \tilde{M} \; Q \; (\rightarrow -J + Q^2 \text{ for } L = 1)$$

Trick for one-loop functions:
$U = \text{det } M = 1 = \sum x_i$ and so U ‘disaapears’ and the construct $F_1(x)$ is bilinear in $x_i x_j$:

$$F_1(x) = -J(\sum x_i) + Q^2 = \sum A_{ij} x_i x_j .$$

The vector integral differs by some numerator $k_i p_e$ and thus there is a single shift in the integrand

$$k \to \bar{k} + U(x)^{-1}\tilde{M}Q$$

the $\int d^d\bar{k}\ \bar{k}/(\bar{k}^2+\mu^2) \to 0$, and no further changes:

$$G(k_{1\alpha}) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j\ x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu - D(L+1)/2-1}}{F(x)^{N_\nu - DL/2}} \left[\sum_l \tilde{M}_{1l}Q_l\right]_\alpha ,$$

Here also a tensor integral:

$$\begin{aligned} G(k_{1\alpha}k_{2\beta}) &= (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j\ x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu - 2 - D(L+1)/2}}{F(x)^{N_\nu - DL/2}} \\ &\times \sum_l \left[[\tilde{M}_{1l}Q_l]_\alpha [\tilde{M}_{2l}Q_l]_\beta - \frac{\Gamma\left(N_\nu - \frac{D}{2}L - 1\right)}{\Gamma\left(N_\nu - \frac{D}{2}L\right)} \frac{g_{\alpha\beta}}{2} U(x)F(x) \frac{(V_{1l}^{-1})^+(V_{2l}^{-1})}{\alpha_l} \right]. \end{aligned}$$

The 1-loop case may be used L times for a sequential treatment of an L-loop integral (remember $\sum x_j D_j = k^2 - 2Qk + J$ and $F(x) = Q^2 - J$):

$$G([1, kp_e]) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}\right)}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j\ x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^N x_i\right) \frac{[1, Q(x)p_e]}{F(x)^{N_\nu - D/2}}$$

Examples for one-loop F-polynomials

V3l2m: One-loop massive QED vertex

$$F(t,m^2) = m^2(x_1+x_2)^2 + [-t]x_1x_2$$

B4l2m: One-loop massive QED box

$$F(s,t,m^2) = m^2(x_1+x_2)^2 + [-t]x_1x_2 + [-s]x_3x_4$$

P5l3m: One-loop massive QED pentagon for $e^+e^- \to e^+e^-\gamma$:

$$F = m^2(x_1+x_3+x_4)^2 + [-t]x_1x_3 + [-t']x_1x_4 + [-s]x_2x_5 + [-v_1]x_3x_5 + [-v_2]x_2x_4$$

P5l2m: One-loop pentagon with one massive particle for $gg \to t\bar{t}g$:

$$F = m^2(x_3+x_4)^2 + [-(t-m^2)]x_2x_3 + [-v_2]x_1x_4 + [-(t'-m^2)]x_2x_4 + [-s]X_1x_5 + [-v_4]x_3x_5$$

B7l4m2: Massive QED 2-loop box, the sub-loop with 2 off-shell legs:

$$F^{-(a_{4567}-d/2)} = \Big\{[-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5+x_6)^2 + (m^2-Q_1^2)x_7(x_4+2x_5+x_6) + (m^2-Q_2^2)x_7x_5\Big\}^{-(a_{4567}-d/2)}$$

What to be done now?

Perform the x-integrations

Find an as-general-as-possible general formula

Make it ready for algorithmic analytical and/or numerical evaluation

Two approaches are worked out with more or less sophisticated software packages:

- Perform a **Sector decomposition by analyzing the IR-divergent structure in the x integrals, perform the remaining finite numerical integrations, also in more or less sophisticated ways.**
- **Introduce Mellin-Barnes integrals for sums of monomials in x_i and then try to evaluate them in more or less sophisticated ways.**

Sector decomposition

The integrand for the multi-dimensional x-integrations is positive semi-definite [for Euclidean variables].

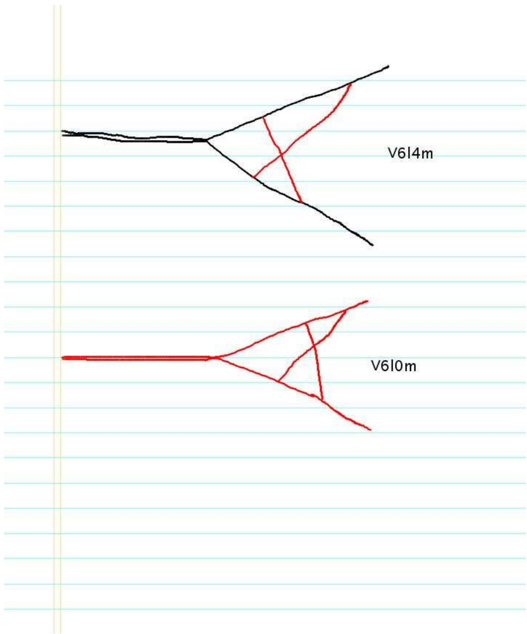
Avoid overlapping singularities
(see many papers, e.g. the review G. Heinrich, arXiv:0803.4177)

$$
\begin{aligned}
I &= \int_0^1 dx \int_0^1 dy \frac{1}{x^{1+a\epsilon} y^{b\epsilon}[x+(1-x)y]} \\
&= \int_0^1 \frac{dx}{x^{1+(a+b)\epsilon}} \int_0^1 \frac{dt}{t^{b\epsilon}[1+(1-x)t]} + \int_0^1 \frac{dy}{y^{1+(a+b)\epsilon}} \int_0^1 \frac{dt}{t^{1+a\epsilon}[1+(1-y)t]}
\end{aligned}
$$

The decomposition allows a unique (numerical) determination of the ϵ-expansion.
There are by now several strategies available.

Package CSectors

Numerics for ‘arbitrary’ rank-R–tensor loop integrals in the Euclidean



Examples:

V6l4m

non-planar massive two-loop vertex

V6l0m

non-planar massless two-loop vertex

`CSectors` **(K.Kajda) is an interface to the GINAC-package** `sector_decomposition` **of C. Bogner and S. Weinzierl, arXiv:0709.4092**

`CSectors` **prepares the Feynman diagram input a la** `Ambre.m` **for further use in** `sector_decomposition`**.**

It is being made public.

CSectors using for V6l4m

Program input text:

```
<< CSectors.m

Options[DoSectors]
SetOptions[DoSectors, TempFileDelete -> False, SetStrategy -> C]

n1 = n2 = n3 = n4 = n5 = n6 = n7 = 1;
m = 1; s = -11;
invariants = {p1^2 -> m^2, p2^2 -> m^2, p1 p2 -> (s - 2 m^2)/2};

DoSectors[{1},
   PR[k1, 0, n1]               PR[k2, 0, n2]               PR[k1 + p1, m, n3]
   PR[k1 + k2 + p1, m, n5] PR[k1 + k2 - p2, m, n6]  PR[k2 - p2, m, n7],
   {k2, k1}, invariants][-4, 2]
```

Output from CSectors.m for `V614m` * $(-s)^{2\epsilon}$:

```
Using strategy C
U = x3 x4+x3 x5+x4 x5+x3 x6+x5 x6+x2 (x3+x4+x6)+x1 (x2+x4+x5+x6)


F = x1 x4^2+13 x1 x4 x5+x4^2 x5+x1 x5^2+x4 x5^2+13 x1 x4 x6
+2 x1 x5 x6+13 x4 x5 x6+x5^2 x6+x1 x6^2+x5 x6^2+x3^2 (x4+x5+x6)
+x2(x3^2+x4^2+13 x4 x6+x6^2+x3 (2 x4+13 x6))+x3 (x4^2+(x5+x6)^2
+x4 (2x5+13 x6))
```

$$-0.0522099/\epsilon - 0.170037 + 0.246343\epsilon + 4.87728\epsilon^2$$

AMBRE.m & MB.m

(MB.m: M. Czakon)

Compare the above to MB-integration (not made optimal, only for illustrations here), resulting from up to 8-dimensional integrations:

$$-0.0522176/\epsilon - 0.165856 + 0.490976\epsilon + 4.31395\epsilon^2$$

AMBRE
for L-loop n-point functions

AMBRE - Automatic Mellin-Barnes REpresentation arXiv: 0704.2423

J. Gluza, K. Kajda (Silesia U.) , T. Riemann (DESY, Zeuthen)

AMBREv1.0.m

This version is described in arXiv:0704.2423 and Computer Physics Communications 177 (2007) 879,

the actual version is 1.2

AMBRE was designed for one-loop diagrams and for the sequential loop-by-loop-integration of n-loop diagrams.
This works well for planar diagrams, but has problems for non-planar diagrams

In final preparation: new version with option of non-sequential loop-by-loop integrations

Integrate over Feynman parameters with Mellin-Barnes–Integrals

Consider an off-shell one-loop box with indices a_4, a_5, a_6, a_7:

$$K_{\text{1-loop Box,off}} = \frac{(-1)^{a_{4567}}\Gamma(a_{4567}-d/2)}{\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)} \int\limits_0^\infty \Pi_{j=4}^7 dx_j x_j^{a_j-1} \frac{\delta(1-x_4-x_5-x_6-x_7)}{F^{a_{4567}-d/2}}$$

where $a_{4567} = a_4 + a_5 + a_6 + a_7$ and the function F is characteristic for the diagram:

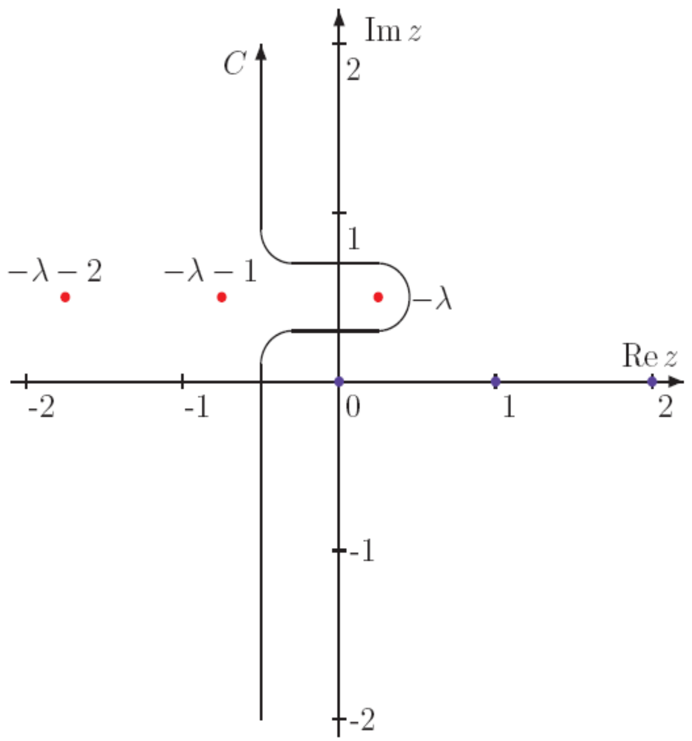
$$F^{-(a_{4567}-d/2)} = \Big\{[-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5+x_6)^2 + (m^2-Q_1^2)x_7(x_4+2x_5+x_6) + (m^2-Q_2^2)x_7x_5\Big\}^{-(a_{4567}-d/2)}$$

We want to inte3grate now over all x-variables with the following formula:

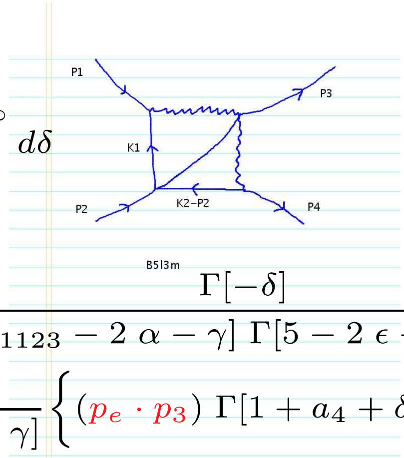
$$\int_0^1 \prod_{j=4}^7 dx_j \; x_j^{\alpha_j-1} \; \delta\left(1-x_4-x_5-x_6-x_7\right) = \frac{\Gamma(\alpha_4)\Gamma(\alpha_5)\Gamma(\alpha_6)\Gamma(\alpha_7)}{\Gamma\left(\alpha_4+\alpha_5+\alpha_6++\alpha_7\right)}$$

For this, we have to apply several MB-integrals here.

$$\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i\ \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma}\ \Gamma(a+\sigma)\Gamma(-\sigma)$$



. A nice box with numerator, B5l3m($p_e.k_1$)



$$\mathtt{B5l3m}(p_e \cdot k_1) = \frac{m^{4\epsilon}(-1)^{a12345}\ e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[5-2\epsilon-a_{123}](2\pi i)^4}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma\int_{-i\infty}^{+i\infty}d\delta$$

$$(-s)^{(4-2\ \epsilon)-a_{12345}-\alpha-\beta-\delta}\ \times\ (-t)^{\delta}$$

$$\frac{\Gamma[-4+2\ \epsilon+a_{12345}+\alpha+\beta+\delta]}{\Gamma[6-3\ \epsilon-a_{12345}-\alpha]}\frac{\Gamma[-\alpha]\ \Gamma[-\beta]}{\Gamma[7-3\ \epsilon-a_{12345}-\alpha]\ \Gamma[5-2\ \epsilon-a_{123}]}\frac{\Gamma[-\delta]}{\Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[5-2\ \epsilon-a_{1123}-2\ \alpha \ldots}$$

$$\frac{\Gamma[2-\epsilon-a_{13}-\alpha-\gamma]}{\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]}\frac{\Gamma[4-2\ \epsilon-a_{12345}-\alpha-\beta-\delta-\gamma]}{\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]}\Big\{(p_e\cdot p_3)\ \Gamma[1+a_4+\delta]\ \Gamma[6-3\ \epsilon- \ldots$$

$$\Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\ \Gamma[3-\epsilon-a_{12}-\alpha]\ \Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2 \ldots$$

$$\Gamma[5-2\ \epsilon-a_{1123}-\gamma]\ \Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[a_1+\gamma]\ \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[a_4+\delta]\Big[-(p_e\cdot p_1)\ \Gamma[7-3\ \epsilon- \ldots$$

$$\Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]$$

$$\Big[\Gamma[3-\epsilon-a_{12}-\alpha]\ \Gamma[5-2\ \epsilon-a_{1123}-\gamma]\Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[a_1+\gamma]+\Gamma[2-\epsilon-a_{12}-\alpha]\ \Gamma[4-2\ \epsilon-a_{1123}-\gamma]$$

$$\Gamma[5-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[1+a_1+\gamma]\Big]\ \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[6-3\ \epsilon-a_{12345}-\alpha]\ \Gamma[3-\epsilon-a_{12}-\alpha]$$

$$\Gamma[5-2\ \epsilon-a_{1123}-\gamma]\ \Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\Gamma[a_1+\gamma]\Big[((p_e\cdot(p_1+p_2))\ \Gamma[5-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[9-4\ \epsilon-a_{11223 \ldots}$$

$$\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+(p_e\cdot p_1)\ \Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]$$

$$\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]\Gamma[-1+\epsilon+a_{123}+\alpha+\delta+\gamma]\Big]\Big\}$$

General Tasks, first two steps automated by `MB.m`:

- **Find a region of definiteness of the n-fold MB-integral**

  $\Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10!$

- **Analytical continuation to the physical region where $\epsilon << 1$ by distorting the integration path step by step (adding each crossed residuum – per crossed residue, this means one integral less!!!)**

- **ϵ-expansion, get a sequence of multi-dimensional finite MB-integrals**

- **Perform numerical integration, – or –**

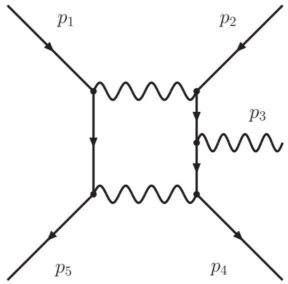
- **Take integrals by sums over residua, i.e. introduce infinite sums**

- **Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.**

Examples

The infrared divergent parts usually have less dimensional MB-representations and often may be summed up.

The massive QED pentagon one-loop example F5l3m:



The result for IR-divergent part in the limit $t' = t$:

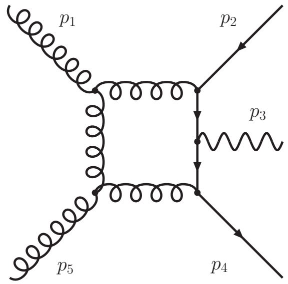
$$I^{\mathbf{5l3m(IR)}} = \frac{J_{-1}}{\epsilon} + J_0$$

$$J_{-1,i} = \frac{1}{2s}\frac{m^{2\epsilon}}{m^2}\frac{(-v_i)^{-2\epsilon}}{v_i}\sum_{n=0}^{\infty}\frac{(t/m^2)^n}{\binom{2n}{n}(2n+1)} = \frac{1}{2s}\frac{m^{2\epsilon}}{m^2}\frac{(-v_i)^{-2\epsilon}}{v_i}\frac{\arcsin\left(\sqrt{t/4m^2}\right)}{\sqrt{t/4m^2}\sqrt{1-(t/4m^2)}}$$

$$J_{0,i} = \frac{1}{2s}\frac{m^{2\epsilon}}{m^2}\frac{(-v_i)^{-2\epsilon}}{v_i}\sum_{n=0}^{\infty}\frac{(t/m^2)^n}{\binom{2n}{n}(2n+1)}\Big(-3S_1(n) + 2S_1(1+2n)\Big)$$

It is $v_1 = p_2 p_3, v_2 = p_4 p_3$.

The massive qcd pentagon one-loop example F5l2m:



The result for the IR-divergent part in the limit $t' = t = t_m + m^2$ is:

$$I^{\mathbf{5l2m(IR)}} = \frac{J_{-2}}{\epsilon^2} + \frac{J_{-1}}{\epsilon} + J_0$$

$$J_{-2} = m^{2\epsilon}(-t_m)^{-2\epsilon}\frac{1}{st_m^2} + \sum_{i=1,2} m^{2\epsilon}\frac{(-v_i)^{-2\epsilon}}{v_i}\frac{1}{2st_m}$$

$$J_{-1} = m^{2\epsilon}(-t_m)^{-2\epsilon}\frac{1}{st_m^2}\left(-\log(m^2) - \log(-s) + \log(-v_2) + \log(-v_4)\right)$$

$$J_0 = \frac{-8\pi^2}{12}\frac{1}{2st_m}\left(\frac{1}{v_2} + \frac{1}{v_4}\right) + \frac{-\pi^2}{12}m^{2\epsilon}(-t_m)^{-2\epsilon}\frac{1}{st_m^2} + \sum_{i=1,2}\frac{13\pi^2}{12}m^{2\epsilon}\frac{(-v_i)^{-2\epsilon}}{v_i}\frac{1}{2st_m}$$

Non-planar problems

Some time ago we thought that Ambre would go wrong when deriving MB-representations for non-planar diagrams. This is not the case, but there are subtleties.

- One-scale problems

Here the loop-by-loop method works well.
See e.g. our numerical example V6l4m, where we compared with a result from sectors. We have also a representation for it by HPL's.

- Multi-scale problems

Here the loop-by-loop method might give correct results, but there are cases where the final evaluation is problematic.
The massless non-planar two-loop box example is discussed by M. Czakon, arXiv:0707.4139.
The loop-by-loop approach gives a 6-dim. integral, the original representation was 4-fold (V. Smirnov, PLB 1999)
In short:

$$I^{4-dim}_{u\to -s-t} = \frac{1}{stu}\,\frac{2}{\epsilon^4} + \cdots$$

$$I^{6-dim}_{u\to -s-t} = \frac{1}{stu}\left[\frac{5}{2\epsilon^4} - \frac{1}{\epsilon^3}\ln(-s-t-u) + \cdots\right]$$

Summary

- **hexagon.m**
  A new code for algebraic reduction of one-loop tensor integrals to scalar 2-,3-,4-point functions
- **CSectors.m**
  A new interface for application of `sector_decomposition` **to tensor integrals**
- **AMBRE.m**
  An update in order to derive MB-presentations for multi-loop tensor integrals – without restriction to the loop-by-loop-method