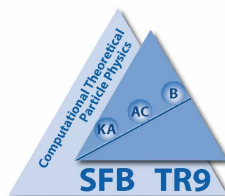


# From Moments to Functions in Higher Order QCD

J. Blümlein, DESY

in collaboration with M. Kauers (RISC), S. Klein (DESY), and C. Schneider (RISC)



- Introduction
- Single Scale Feynman Integrals as Recurrent Quantities
- Establishing and Solving Recurrences
- Application to 3-Loop Anomalous Dimensions and Wilson coefficients
- Conclusions

# 1. Introduction

- Higher order calculations in Quantum Field Theories easily become tedious due to the larger number of terms and the sophistication of the Feynman parameter integrals.
- This even applies to **Zero Scale** and **Single Scale** Quantities.
- Even more this is the case for **higher scale** problems.
- While in the latter case the mathematical structure of the solution for the Feynman Integrals is widely unknown, it is explored to **a certain extent** for **Zero Scale** and **Single Scale** quantities.
- **Zero Scale** quantities are the expansion coefficients of the **running couplings** and **masses**, **fixed moments** of **splitting functions** etc.
- They can be expressed by **rational numbers** and certain **special numbers** as **multiple  $\zeta$ -values** and related quantities.

# Introduction

- **Single Scale** quantities depend on a scale  $z \in [0, 1]$ , with  $z$  a ratio of Lorentz invariants. One may perform a **Mellin Transform** over  $z$

$$\int_0^1 dz z^{N-1} f(z) = M[f](N)$$

- Here one assumes  $N \in \mathbf{N}, N > 0$ . Due to this the problem on hand becomes **discrete**.
- One may seek a description in terms of **difference equations**.
- **Zero Scale** problems are obtained from **Single Scale** problems treating  $N$  as a fixed integer or considering the limit  $N \rightarrow \infty$ .

## Some Remarks about MZV's

- General question on the bases of MZV's: length in the non-alternating and alternating cases.
- Do Zero Scale Feynman integrals always lead to MZV's ?
- No! e.g. Y. Andre, 2008.
- At lower orders in perturbation theory one has just MZV's even in single-mass problems.
- J.B., Broadhurst, Vermaseren, 2008: explicit calculation of bases and all relations of alternating MZV's to  $w=12$  and non-alternating MZB's to  $w=22$ . [World Record.]; Verification to  $w=26$ .
- Broadhurst 1996 conjecture is proven. shuffles, stuffles, doubling, gen. doubling relations However, we did not find further reductions - which still may exist.

# Introduction

- Can one reconstruct the general formula for **Single Scale** quantities out of a **finite number** of fixed moments ?
- This is possible for recurrent quantities.
- At least up to **3-loop order**, presumably to higher orders, single scale quantities belong to this class.
- Goal : design a general formalism to solve the problem.

## 2. Single Scale Feynman Integrals as Recurrent Quantities

- Can one reconstruct the general formula for **Single Scale** quantities out of a **finite number** of fixed moments ?
- **Polynomials and Nested Harmonic Sums** obey recurrence relations, so do their polynomials.
- Example: Harmonic Sums or linear combinations thereof:

$$F(N + 1) - F(N) = \frac{\text{sign}(a)^{N+1}}{(N + 1)^{|a|}}$$

is solved by  $S_a(N)$ ; and similarly for deeper nested sums

$$S_{a,\vec{b}}(N) = \sum_{k=1}^N \frac{(\text{sign}(a))^k}{k^{|a|}} S_{\vec{b}}(k)$$

.

# Single Scale Feynman Integrals as Recurrent Quantities

- Feynman integrals have often a form like

$$\int_0^1 dz \frac{z^{N-1} - 1}{1 - z} H_{\vec{a}}(z), \quad \int_0^1 dz \frac{(-z)^{N-1} - 1}{1 + z} H_{\vec{a}}(z)$$

- This structure leads to recurrences.
- It is very likely that single scale Feynman diagrams do always obey difference equations.

### 3. Establishing and Solving Recurrences

- One seeks the relation

$$\sum_{k=0}^l \left[ \sum_{i=0}^d c_{i,k} N^i \right] F(N+k) = 0 .$$

- The corresponding linear system is dense.
- Rational number arithmetics is **not applicable** for the large systems to be determined;  $C_{2,q,C_F}^{(3)}$  would require 11 Tb of memory.
- Use arithmetic in **finite fields** together with **Chinese remaindering**  
 $\implies$  few Gb of memory
- The linear system approximately minimizes for  $l \approx d$ .
- Join different recurrences found to reduce  $d$  to a minimal value.



# Establishing and Solving Recurrences

- For the solution of the recurrence low degrees are clearly preferred.
- The linear difference equation of order  $d$  with polynomial coefficients is equivalent to a linear system in  $d$  variables.
- It is solved in  $\Pi - \Sigma$  fields.
- Apply advanced symbolic summation methods: telescoping, creative telescoping and its refinement. Code: `sigma`.
- The solutions are found as linear combinations of rational terms in  $N$  combined with functions, which cannot be further reduced in the  $\Pi - \Sigma$  fields. In the present application they turn out all to be harmonic sums  $S_{\vec{b}}(N)$ .
- Other or higher order applications may consist of other sums too, which are uniquely found by the algorithm.

## 4. Application to 3-Loop Anomalous Dimensions and Wilson coefficients

- We apply the method for the unfolding of the unpolarized **anomalous dimensions** and **Wilson coefficients** up to **3-loop** order.
- $\implies$  analyze for individual color factors; **141** contributions from **1 – 3 loops**
- Input: Moch, Vermaseren, Vogt, 2004/05. The expressions are given in terms of harmonic sums.
- Calculate the moments (**rational numbers**) recursively through recursions for the harmonic sums; **MAPLE** code.
- Establish the corresponding difference equation by a **recurrency finder**; build a difference equation of **minimal** order possible; test the recurrency.
- Solve the difference equation order by order with the summation package **sigma** C. Schneider.; most complicated cases: 4 weeks @  $\leq 10\text{Gb}$ , 2 GHz Proc.

**Input**

C2qq3CF<sup>3</sup>

N=3:

#11 digits / #10 digits

-98268084191 / 1166400000

N=500:

#1262 digits / #1256 digits

1641840770424196780953020619176376506284303544481262083057197600746507008493793994  
4224110323441591630311482222058287688942209570859151121677307585313995100978363179  
2518952817622034037186132846974627021672678012913675099511203807811938593043910803  
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72017877003947396562261659860366839154407853462338171648227013134266795320251847  
/

3057444614247225372882570514367358697278130741348282122206492932820352440850471902  
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496586232475091126597629976797375278827111116774593003520000000000000000

N=5114:

#13388 digits / #13381 digits

Table 1: Run parameters for the unfolding of the non-singlet anomalous dimensions

	number of terms needed	order of recurrence	degree of recurrence	total time [sec]	length of recurrence [kbyte]	number of harm. sums a [b]	solution time [sec]
$P_{NS,0}$	14	2	3	0.05	0.087	1 [1]	0.55
$P_{NS,1,C_F^2}^-$	142	5	31	3.32	4.666	6 [10]	7.45
$P_{NS,1,C_A C_F}^-$	109	4	24	1.91	2.834	6 [7]	6.28
$P_{NS,1,C_F N_F}^-$	24	2	7	0.13	0.271	2 [2]	0.92
$P_{NS,1,C_F^2}^+$	142	5	31	3.35	4.707	6 [10]	7.45
$P_{NS,1,C_A C_F}^+$	109	4	23	1.88	2.703	6 [7]	6.23
$P_{NS,1,C_F N_F}^+$	24	2	7	0.09	0.271	2 [2]	0.89
$P_{NS,2,C_F^3}^-$	1079	16	192	3152.19	529.802	25 [68]	1194.41
$P_{NS,2,C_F^3 \zeta_3}^-$	48	3	11	0.49	0.643	1 [1]	1.56
$P_{NS,2,C_A C_F^2}^-$	974	15	181	1736.08	450.919	25 [62]	1194.41
$P_{NS,2,C_A C_F^2 \zeta_3}^-$	48	3	11	0.53	0.643	1 [1]	1.53
$P_{NS,2,C_A^2 C_F}^-$	749	12	147	1004.12	242.892	25 [62]	1100.88
$P_{NS,2,C_A^2 C_F \zeta_3}^-$	48	3	11	0.56	0.643	1 [1]	1.56
$P_{NS,2,C_F N_F^2}^-$	39	2	11	0.31	0.369	3 [3]	1.20
$P_{NS,2,C_F^2 N_F}^-$	377	8	68	76.34	33.946	12 [24]	72.22
$P_{NS,2,C_F^2 N_F \zeta_3}^-$	14	2	3	0.12	0.101	1 [1]	0.53
$P_{NS,2,C_A C_F N_F}^-$	356	7	62	65.25	23.830	12 [20]	52.67
$P_{NS,2,C_A C_F N_F \zeta_3}^-$	14	2	3	0.12	0.101	1 [1]	0.55
$P_{NS,2,C_F^3}^+$	1079	16	192	4713.27	527.094	25[68]	1165.22
$P_{NS,2,C_F^3 \zeta_3}^+$	48	3	11	0.55	0.643	1[1]	1.562
$P_{NS,2,C_A C_F^2}^+$	974	15	178	1715.03	442.031	25[62]	889.047
$P_{NS,2,C_A C_F^2 \zeta_3}^+$	48	3	11	0.61	0.643	1[1]	1.531
$P_{NS,2,C_A^2 C_F}^+$	749	12	146	991.22	240.325	25[50]	516.812
$P_{NS,2,C_A^2 C_F \zeta_3}^+$	48	3	11	0.61	0.643	1[1]	1.593
$P_{NS,2,C_F^2 N_F}^+$	377	8	69	111.38	33.872	12[24]	71.235
$P_{NS,2,C_F^2 N_F \zeta_3}^+$	14	2	3	0.15	0.101	1[1]	0.531
$P_{NS,2,C_A C_F N_F}^+$	307	7	61	48.62	23.808	12[24]	71.235
$P_{NS,2,C_A C_F N_F \zeta_3}^+$	14	2	3	0.15	0.101	1[1]	0.547
$P_{NS,2,C_F N_F^2}^+$	39	2	11	0.40	0.369	3[3]	1.172
$P_{NS,2,N_F^3}^-$	39	2	11	0.55	0.369	3 [3]	1.19

Table 2: Run parameters for the unfolding of the unpolarized quarkonic Wilson Coefficients for the structure function  $F_2(x, Q^2)$ .

	number of terms needed	order of recurrence	degree of recurrence	total time [sec]	length of recurrence [kbyte]	number of harm. sums a [b]	solution time [sec]
$C_{2,q,C_F}^{(1)}$	35	3	7	0.26	0.429	2[3]	1.13
$C_{2,q,C_F^2}^{(2)}$	689	11	137	1134.10	177.806	13[39]	258.24
$C_{2,q,C_A C_F}^{(2)}$	545	10	121	413.33	127.893	12[35]	178.73
$C_{2,q,C_F^2\zeta_3}^{(2)}$	15	2	3	0.27	0.100	1[1]	0.54
$C_{2,q,C_A C_F \zeta_3}^{(2)}$	15	2	3	0.27	0.112	1[1]	0.55
$C_{2,q,N_F C_F}^{(2)}$	71	4	16	2.68	1.655	4[10]	3.95
$C_{2,q,C_F^3}^{(3)}$	5114	35	938	$1.78886 \times 10^6$	30394.173	58[289]	$0.50924 \times 10^6$
$C_{2,q,C_F^3\zeta_3}^{(3)}$	284	8	64	31.02	32.363	7 [18]	27.60
$C_{2,q,C_F^3\zeta_4}^{(3)}$	19	2	5	0.08	0.163	1 [1]	0.47
$C_{2,q,C_F^3\zeta_5}^{(3)}$	19	2	5	0.08	0.163	1 [1]	0.47
$C_{2,q,C_F^2 C_A}^{(3)}$	5059	35	930	$1.69267 \times 10^6$	30122.380	60 [290]	$0.47780 \times 10^6$
$C_{2,q,C_F^2 C_A \zeta_3}^{(3)}$	284	8	64	34.00	33.400	7 [18]	28.53
$C_{2,q,C_F^2 C_A \zeta_4}^{(3)}$	48	3	11	0.32	0.643	1[1]	1.01
$C_{2,q,C_F^2 C_A \zeta_5}^{(3)}$	19	2	5	0.08	0.167	1 [1]	0.42
$C_{2,q,C_F C_A^2}^{(3)}$	4564	33	863	$1.38918 \times 10^6$	24567.518	60 [258]	$0.34941 \times 10^6$
$C_{2,q,C_F C_A^2 \zeta_3}^{(3)}$	284	8	63	26.83	29.918	7 [17]	30.46
$C_{2,q,C_F C_A^2 \zeta_4}^{(3)}$	48	3	11	0.32	0.643	1 [1]	1.01
$C_{2,q,C_F C_A^2 \zeta_5}^{(3)}$	19	2	5	0.08	0.175	1 [1]	0.42
$C_{2,q,C_F^2 N_F}^{(3)}$	1762	20	348	40237.45	2339.516	29 [107]	7548.56
$C_{2,q,C_F^2 N_F \zeta_3}^{(3)}$	87	4	21	1.94	2.354	3 [5]	2.83
$C_{2,q,C_F^2 N_F \zeta_4}^{(3)}$	15	2	3	0.07	0.101	1 [1]	0.34
$C_{2,q,C_F C_A N_F}^{(3)}$	1847	20	360	47661.64	2507.362	29 [111]	7525.89
$C_{2,q,C_F C_A N_F \zeta_3}^{(3)}$	89	4	24	2.47	2.935	3 [8]	3.19
$C_{2,q,C_F C_A N_F \zeta_4}^{(3)}$	15	2	3	0.06	0.101	1 [1]	0.34
$C_{2,q,C_F N_F^2}^{(3)}$	131	5	30	58.00	5.347	7 [22]	8.97
$C_{2,q,C_F N_F^2 \zeta_3}^{(3)}$	15	2	3	0.06	0.101	1 [1]	0.38
$C_{2,q,dabc}^{(3)}$	1199	14	242	6583.27	738.498	14 [62]	841.24
$C_{2,q,dabc\zeta_3}^{(3)}$	109	4	25	2.33	3.164	2[7]	2.40
$C_{2,q,dabc\zeta_5}^{(3)}$	8	1	2	0.03	0.041	0[0]	0.10

## A complicated example

$$\underline{C_{2,q} \propto C_F^3} :$$

- 5114 moments needed. Use a clever way to calculate the input.
  - Largest moment: fraction: numerator 13388 digits; denominator 13381 digits.
  - CPU time to determine the recurrence: 20.7 days.
- modular prediction of the dimension: 4 h; modular LEQ's: 5.8 days; modular operator GCDs: 11 days; Chinese Remainder + Rat. Reconstruction: 3.8 days. 140 large primes needed.
- 31 MB recurrence is established; largest integer: 1227 digits; order: 35; degree: 938
- Solved by sigma within about one week.
  - 3 loop anomalous dimensions: much smaller recurrences & shorter computation times.
- ⇒ In practice no method does yet exists to calculate such a high number of moments.
- ⇒ Existence proof of a quite general and powerful automatic difference-equation solver, standing rather demanding tests.

## Structure of the Results

- We carry out all algebraic reductions, J.B. 2003.
- Different color factor contributions lead to the same or nearly the same amount of sums at a given quantity.
- This points to the fact that the amount of harmonic sums is governed by topology rather than the fields and color.
- The linear harmonic sum representations by Vermaseren et al. 2004/05 require many more sums than our representation.
- There are reductions in the number of sums as  $264 \longrightarrow 29$ .
- Further use of structural relations will lead to maximally 35 sums for the 3-loop Wilson coefficients; J.B. 2008.



$$\gamma_{qq,(0)}(N) = S_1(N) - \frac{3N^2 + 3N + 2}{N(N+1)} \quad (1)$$

$$\begin{aligned} \gamma_{qq,1}^{(+)}(N) &= C_F N_F \left[ S_1(N) - \frac{3N^2 + 3N + 2}{N(N+1)} \frac{23N^4 + 6N^3 + 47N^2 + 20N - 12}{9N^2(N+1)^2} - \frac{40}{9}S_1(N) + \frac{8}{3}S_2(N) \right] \\ &+ C_A C_F \left[ \frac{-51N^5 - 153N^4 - 757N^3 - 144(-1)^N N^2 - 851N^2 - 208N + 132}{18N^2(N+1)^3} \right. \\ &\quad \left. + 8S_{-3}(N) + \frac{268}{9}S_1(N) + S_{-2}(N) \left( 16S_1(N) - \frac{8}{N(N+1)} \right) - \frac{44}{3}S_2(N) + 8S_3(N) \right. \\ &\quad \left. - 16S_{-2,1}(N) \right] \\ &+ C_F^2 \left[ \frac{-3N^6 - 9N^5 - 9N^4 + 32(-1)^N N^3 - 59N^3 - 40N^2 - 32N - 8}{2N^3(N+1)^3} - 16S_{-3}(N) \right. \\ &\quad \left. + S_{-2}(N) \left( \frac{16}{N(N+1)} - 32S_1(N) \right) + S_1(N) \left( \frac{8(2N+1)}{N^2(N+1)^2} - 16S_2(N) \right) \right. \\ &\quad \left. + \frac{4(3N^2 + 3N + 2)S_2(N)}{N(N+1)} - 16S_3(N) + 32S_{-2,1}(N) \right] \quad (2) \end{aligned}$$

$$\begin{aligned} \gamma_{qq,1}^{(-)}(N) &= C_F N_F \left[ \frac{3N^4 + 6N^3 + 47N^2 + 20N - 12}{9N^2(N+1)^2} - \frac{40}{9}S_1(N) + \frac{8}{3}S_2(N) \right] \\ &+ C_A N_F \left[ \frac{-144(-1)^N N^3 - (N+1)(51N^5 + 102N^4 + 655N^3 + 484N^2 + 12N + 144)}{18N^3(N+1)^3} \right. \\ &\quad \left. + 8S_{-3}(N) + \frac{268}{9}S_1(N) + S_{-2}(N) \left( 16S_1(N) - \frac{8}{N(N+1)} \right) - \frac{44}{3}S_2(N) + 8S_3(N) \right. \\ &\quad \left. - 16S_{-2,1}(N) \right] \\ &+ C_F^2 \left[ \frac{-3N^6 - 9N^5 - 9N^4 + 32(-1)^N N^3 + 5N^3 + 24N^2 + 32N + 24}{2N^3(N+1)^3} - 16S_{-3}(N) \right. \\ &\quad \left. + S_{-2}(N) \left( \frac{16}{N(N+1)} - 32S_1(N) \right) + S_1(N) \left( \frac{8(2N+1)}{N^2(N+1)^2} - 16S_2(N) \right) \right. \\ &\quad \left. + \frac{4(3N^2 + 3N + 2)S_2(N)}{N(N+1)} - 16S_3(N) + 32S_{-2,1}(N) \right] \quad (3) \end{aligned}$$

$$\begin{aligned}
\gamma_{qq,2}^{(-)}(N) &= C_F^2 N_F \zeta_3 \left[ 32S_1(N) - \frac{8(3N^2 + 3N + 2)}{N(N+1)} \right] \\
&+ C_A C_F N_F \zeta_3 \left[ \frac{8(3N^2 + 3N + 2)}{N(N+1)} - 32S_1(N) \right] \\
&+ C_F N_F^2 \left[ \frac{51N^6 + 153N^5 + 57N^4 + 35N^3 + 96N^2 + 16N - 24}{27N^3(N+1)^3} - \frac{16}{27}S_1(N) - \frac{80}{27}S_2(N) \right. \\
&\quad \left. + \frac{16}{9}S_3(N) \right] \\
&+ C_F^3 \zeta_3 \left[ -\frac{24(5N^4 + 10N^3 + 9N^2 + 4N + 4)}{N^2(N+1)^2} - 192S_{-2}(N) \right] \\
&+ C_A C_F^2 \zeta_3 \left[ \frac{36(5N^4 + 10N^3 + 9N^2 + 4N + 4)}{N^2(N+1)^2} + 288S_{-2}(N) \right] \\
&+ C_A^2 C_F \zeta_3 \left[ -\frac{12(5N^4 + 10N^3 + 9N^2 + 4N + 4)}{N^2(N+1)^2} - 96S_{-2}(N) \right] \\
&+ C_A C_F N_F \\
&\quad \left[ -\frac{2(N+1)(270N^7 + 810N^6 - 463N^5 - 1392N^4 - 211N^3 - 206N^2 - 156N + 144)}{27N^4(N+1)^4} \right. \\
&\quad - 2\frac{96N^4(4N+1)}{27N^4(N+1)^4} + \frac{64}{3}S_4(N) + S_{-3}(N) \left( \frac{32}{3}S_1(N) - \frac{16(10N^2 + 10N + 3)}{9N(N+1)} \right) \\
&\quad + \frac{1336}{27}S_2(N) + S_{-2}(N) \left( \frac{16(16N^2 + 10N - 3)}{9N^2(N+1)^2} - \frac{320}{9}S_1(N) + \frac{64}{3}S_2(N) \right) \\
&\quad - \frac{8(14N^2 + 14N + 3)S_3(N)}{3N(N+1)} + \frac{80}{3}S_4(N) + \frac{32(10N^2 + 10N - 3)S_{-2,1}(N)}{9N(N+1)} \\
&\quad + S_1(N) \left( -\frac{4(209N^6 + 627N^5 + 627N^4 + 72N^3 + 281N^3 + 36N^2 + 36N + 18)}{27N^3(N+1)^3} \right. \\
&\quad \left. + 16S_3(N) + \frac{64}{3}S_{-2,1}(N) \right) - \frac{32}{3}S_{2,-2}(N) - \frac{64}{3}S_{3,1}(N) - \frac{128}{3}S_{-2,1,1}(N) \left. \right] \\
&+ \text{various pages more.} \tag{1}
\end{aligned}$$

$$\begin{aligned}
\gamma_{qq,1}^{(v)}(N) &= \frac{d_{abc}}{N_c} N_F \left[ \frac{51N^6 + 153N^5 + 57N^4 + 35N^3 + 96N^2 + 16N - 24}{27N^3(N+1)^3} - \frac{16}{27}S_1(N) - \frac{80}{27}S_2(N) \right. \\
&\quad \left. + \frac{16}{9}S_3(N) \right] \tag{2}
\end{aligned}$$

## Other Processes

- The present method can be applied irrespectively of the loop order to **all single scale** processes.
- As has been found before J.B. & Ravindran 2004/05, J.B. & Moch 2005, J.B. & S. Klein 2007 representing a large number of 2- and 3-loop processes in terms of harmonic sums, the **basis elements** emerging are always the same.  
 {anomalous dimensions, Wilson coefficients, space- and time-like, polarized/unpolarized, Drell-Yan process, hadronic Higgs Boson production in the heavy mass limit, HO QED corrections in  $e^+e^-$  annihilation, soft+virtual corrections to Bhabha scattering}.
- The formalism also applies to **Heavy Flavor Wilson Coefficients** at  $Q^2 \gg m^2$ , c.f. Bierenbaum, J.B., Klein 2007/08.
- **Basis** to  $w = 6$ , c.f. J.B. 2008.

## 5. Conclusions

- We established a general algorithm to calculate the **exact expression** for **single scale** quantities from a **finite** (suitably large) number of moments (zero scale quantities).
- The latter ones are much more easily calculable.
- We applied the method to the **anomalous dimensions** and **Wilson coefficients** up to **3-loop order**.
- To solve 3-loop problems this way is not possible at present, since the number of required moments is too large for the methods available.
- We attempted to solve the quantities for all **color projections** at once. This problem is too voluminous.
- Yet we showed that giant **difference equations** [order 35; degree  $\sim 1000$ ] can be reliably and fast **established** and **solved unconditionally** for advanced problems in Quantum Field Theory.