New implementation of the sector decomposition on FORM

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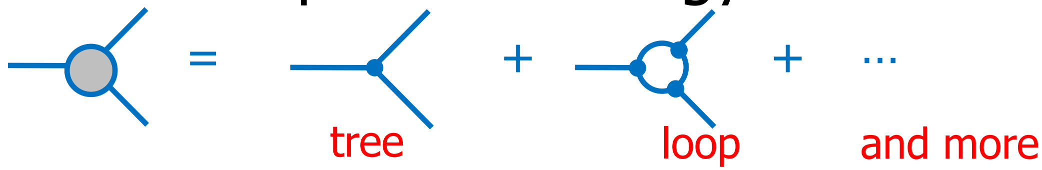
Outline

- Introduction / Motivation
  - Loop integration ⇨ numerical approach ⇨ sector decomposition
  - Why FORM?
- Quick review for sector decomposition
  - Formalism
- Some demonstrations
  - Examples of multi-loop integrals (space-like region)
- Advance beyond the thresholds…
  - Very preliminary results with extrapolation (1-loop)
    → Yuasa-san's talk
- Summary

Introduction - Loop Integrals

- High precision experiments in future require high precision theoretical calculations
- We must evaluate various loop diagrams to get higher order corrections to processes in phenomenology



e.g., LHC (pp-collider) ⇒ Large QCD background, EW corrections ⇒ needs of method to evaluate loops

How to evaluate loop integrals ?

- Analytic approach
  - Find formulae for loop integrals as functions of kinematic parameters
  - But difficult for many loops, many legs, many parameters
- Numerical approach ⇐ Today's topic
  - Evaluate loop integrals numerically
  - But usually costs a lot of CPU powers

Numerical Approach with Sector Decomposition

- We can executes the momentum integration by introducing parametric representation (e.g., Feynman parameters)
- In the context of the dimensional regularization, UV-divergences can be removed by usual way, but IR-divergences may appear as singularities on the edge of the integration domain
- Sector decomposition algorithm by T.Binoth & G.Heinrich can extract such IR-divergences as (finally) $1/\epsilon$ poles
- Then, we can numerically evaluate parametric integration and obtain coefficients of Laurent expansion
- Numerical approach with sector decomposition serves us a powerful method to perform (any) loop integrals

Why FORM

- FORM by J.Vermaseren, is rather fast, especially in the manipulation we need in HEP calculations, e.g., Dirac matrices
- De-facto in HEP calculation, now GRACE group also use it, therefore it is easy to combine them into the whole system
- More important thing is memory problem
  - In an intermediate stage of the sector decomposition, one needs to handle a lot of terms and very large expression

	B	DB	TB	QB
# sectors	12	293	10915	661484

(Generated sectors for massless on-shell boxes)

  - Most of other CAS try to keep them in the memory, and cannot handle it

    They assume there are infinite physical memories

  - Tough recently other solution to avoid memory problem appears, Qlink on Mathematica (by A.V.Smirnov)
    FORM is originally designed to treat such a huge expressions and has strong advantage for it

Sector Decomposition - Formalism

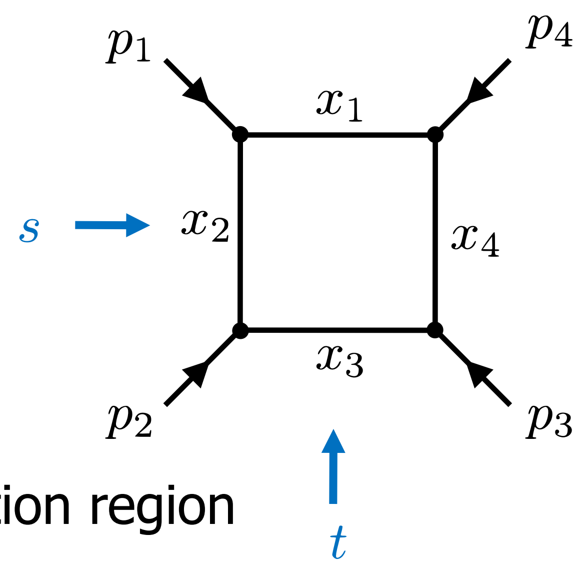
- Example: 1-loop massless on-shell scalar box

$(D = 4 - 2\epsilon)$

$$I_4^{0m} = \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2 + i0)[(k+p_1)^2 + i0][(k+p_{12})^2 + i0][(k+p_{123})^2 + i0]}$$

after introducing Feynman parameters and momentum integration

$$= \Gamma(2+\epsilon) \int_0^1 d^4x \frac{\delta(1 - x_{1234})}{(-s x_1 x_3 - t x_2 x_4 - i0)^{2+\epsilon}}$$



Kinematic invariants: $s = (p_1 + p_2)^2 \quad t = (p_2 + p_3)^2$

abbrev. $p_{12} = p_1 + p_2 \quad p_{123} = p_1 + p_2 + p_3$

$x_{1234} = x_1 + x_2 + x_3 + x_4$

- IR-singularity comes from edges of the integration region

$$J_4^{0m} = \int_0^1 d^4x \frac{\delta(1 - x_{1234})}{(-s x_1 x_3 - t x_2 x_4 - i0)^{2+\epsilon}}$$

when two variables simultaneously go to zero

$x_1, x_2 \to 0$ or $x_1, x_4 \to 0$ or $x_3, x_2 \to 0$ or $x_3, x_4 \to 0$

How Sector Decomposition Works

- Sector decomposition algorithm disentangle such singularities

(for example)

T.Binoth & G.Heinrich, NPB585 (2000) 741
T.Binoth & G.Heinrich, NPB680 (2004) 375

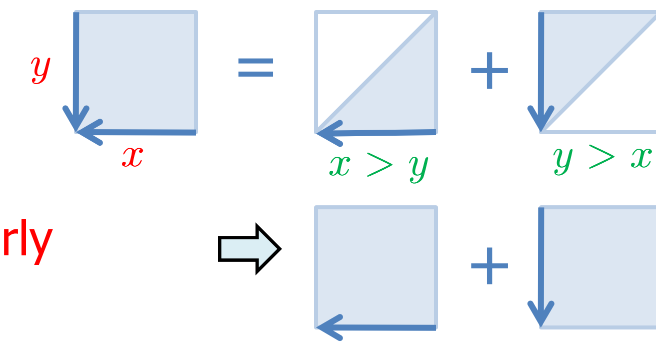
$$\int_0^1 dx \int_0^1 dy \frac{f(x,y)}{(x+y)^{2-\epsilon}}$$

singular when $x, y \to 0$

split integration domain

$$= \int_0^1 dx \int_0^x dy \frac{f(x,y)}{(x+y)^{2-\epsilon}} + \int_0^1 dy \int_0^y dx \frac{f(x,y)}{(x+y)^{2-\epsilon}}$$

$x > y$ $\quad$ $y > x$



map to [0,1] $\quad y \to xy$

similarly

$$\int_0^1 dx \int_0^1 x dy \frac{f(x,xy)}{(x+xy)^{2-\epsilon}}$$

factorize x

$$= \int_0^1 dx \int_0^1 dy\, x^{-1+\epsilon} \frac{f(x,xy)}{(1+y)^{2-\epsilon}} + \int_0^1 dx \int_0^1 dy\, y^{-1+\epsilon} \frac{f(xy,y)}{(1+x)^{2-\epsilon}}$$

singular when $x \to 0$ $\quad$ singular when $y \to 0$

- Then one can easily extract poles as

$$\int_0^1 dx\, x^{-1+\epsilon} f(x) = \frac{f(0)}{\epsilon} + \int_0^1 dx\, x^{\epsilon} \frac{f(x) - f(0)}{x}$$

singular when $x \to 0$ $\quad$ pole $\quad$ non-singular

How Sector Decomposition Works

- The cases, in which there is δ-function or three or more variables, can be handled in a similar way
- Iterative use of sector decomposition allows us to treat more complex singularities in a systematic way

(box example becomes)

$$J_4^{0m} = 2\int_0^1 d^3x\, x_1^{-1-\epsilon} \frac{(1+x_{12}+x_1x_3)^{2\epsilon}}{(-s-x_2x_3t)^{2+\epsilon}}$$
$$+2\int_0^1 d^3x\, x_1^{-1-\epsilon} x_2^{-1-\epsilon} \frac{(1+x_{12}+x_1x_2x_3)^{2\epsilon}+(1+x_1+x_2x_{13})^{2\epsilon}}{(-s-x_3t)^{2+\epsilon}} \quad +(s\leftrightarrow t)$$

expand with respect to ϵ

$$= \frac{C_2}{\epsilon^2} + \frac{C_1}{\epsilon} + C_0$$

- Here C_0, C_1, C_2 are written by sums of multi-dimensional integrals

Then all we have to do is to perform these integral numerically

Strategy for Choosing Sub-sectors

- If there are many integration variables, the choice of sub-sectors is not unique (# generated sectors can change)

Currently, we have implemented one strategy for choosing sub-sectors, which is based on the original idea of Binoth & Heinrich

- Take smallest set of variables S such that the polynomial $P(x)$ vanishes when all parameters in S are set to zero
- Give priority to the variable which has large exponent

(for example)

$$x_1 x_2^2 + x_3^2$$

$$S = \{x_2, x_3\} \leftarrow \{x_2, x_3, x_1\}$$

$x_2 \to x_2 x_3$: $x_3^2(x_1 x_2^2 + 1)$

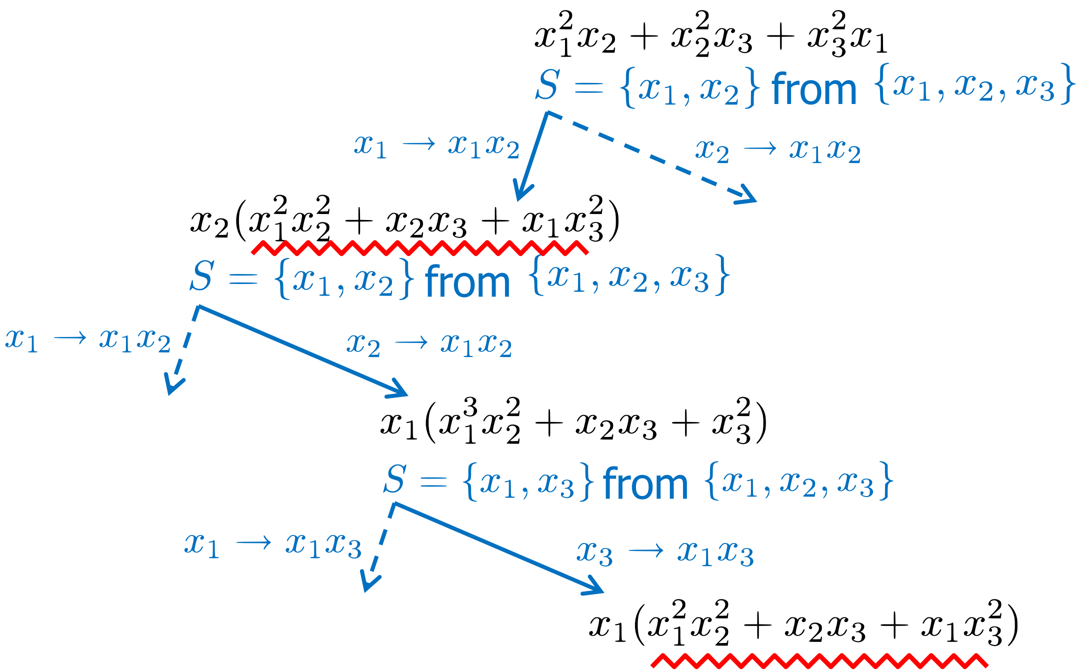
$x_3 \to x_2 x_3$: $x_2^2(x_1 + x_3^2)$

possibly $\{x_2, x_3\}$, $\{x_2, x_1\}$, $\{x_3, x_1\}$, $\{x_2, x_3, x_1\}$ are checked in this order, but first one $\{x_2, x_3\}$ is adopted

decomposed two steps at most

Strategy for Choosing Sub-sectors

- With this heuristic strategy one may enter the infinite loop for more complex case (e.g., $x_1 x_2^2 + x_2 x_3^2 + x_3 x_1^2$)

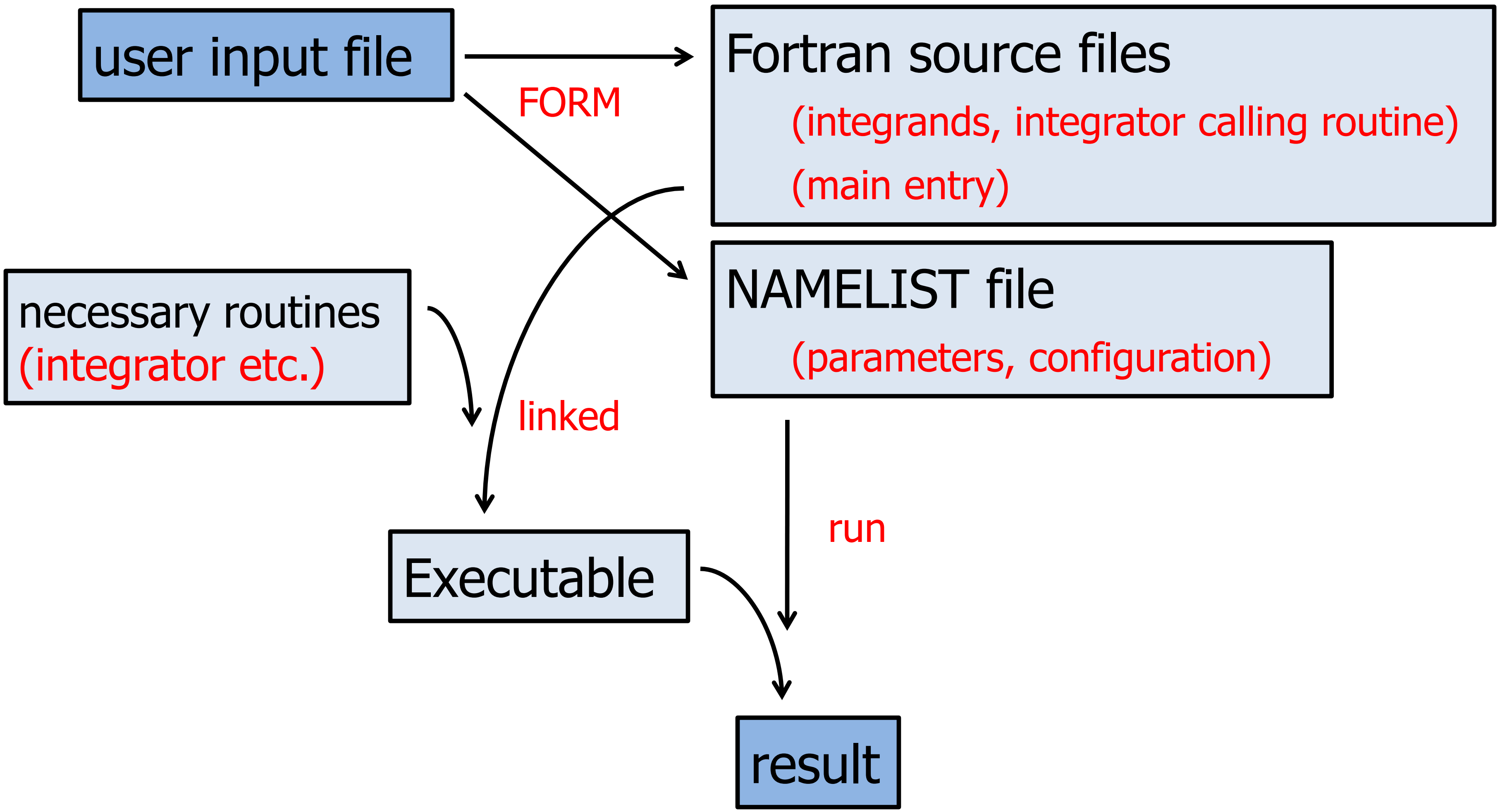


- But it is enough for usual Feynman loop evaluation, and it gives smallest # sectors

  Discussion about the other strategies, in which the iteration is guaranteed to stop, can be found in C.Bogner & S.Weinzierl, CPC178 (2008) 596
  A.V.Smirnov & M.N.Tentyukov, arXiv:0807.4129 [he-ph]

Example Calculations - System

- System



Example Calculations (I)

- On-shell massless double box (input file)

```
Vector p1,p2,p3;
Symbol s,t;
```

$$\int \frac{d^D k_1}{i\pi^{D/2}} \int \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{[-(k_1+p_1)^2](-k_1^2)[-(k_1+p_{12})^2][-(k_1-k_2)^2](-k_2^2)[-(k_2+p_{12})^2][-(k_2+p_{123})^2]}$$

```
Local DB = FD(k1+p1) * FD(k1) * FD(k1+p1+p2) * FD(k1-k2)
          * FD(k2) * FD(k2+p1+p2) * FD(k2+p1+p2+p3);
```

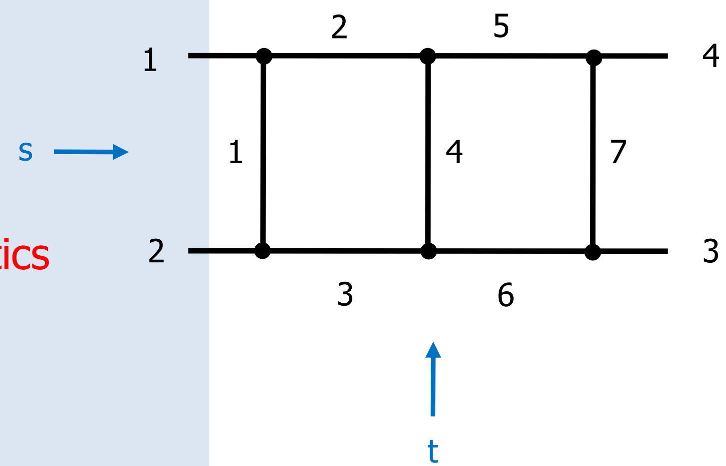
FD: denominator, `FD(k)=1/(-k.k)`

Procedure called after momentum integration

```
#procedure DefKine(l)

  multiply replace_(D,4-2*ep);

  id p1.p2 = (s - p1.p1 - p2.p2) / 2;
  id p2.p3 = (t - p2.p2 - p3.p3) / 2;
  id p1.p3 = (- s - t - p1.p1 - p3.p3) / 2;

  #do i=1,3
    id p`i'.p`i' = 0;
  #enddo

#endprocedure
```

$\leftarrow D = 4 - 2\epsilon$

kinematics




```
#call DefParam(s, -1)
#call DefParam(t, -1)
```

Register kinematic parameters and their default values
(One can modify these values at run-time)

(space-like)

Example Calculations (I)

- On-shell massless double box (result)

(analytic) V.A.Smirnov, PLB460 (1999) 297
(numerical) T.Binoth & G.Heinrich, NPB585 (2000) 741

```
*******************************************************************************
* Parameters:
*                  s =   -1.00000000000000
*                  t =   -1.00000000000000
*************************************************************************
* Integrator configurations:
*      VERBOSE_LEVEL =                          2
*         DCUHRE_KEY =                          9
*      DCUHRE_MINPTS =                          0
*      DCUHRE_MAXPTS =                  100000000
*     DCUHRE_ABS_TOL =   0.100000000000000E-15
*     DCUHRE_REL_TOL =   0.100000000000000E-05
*************************************************************************
* DB: Gamma(3+2*ep) * ep^-4                                      0.00s
*  0.200000000661904E+01 aerr= 0.12485517866572E-05 rerr= 0.6242759E-06
*************************************************************************
* DB: Gamma(3+2*ep) * ep^-3                                      0.00s
* -0.600000001069171E+01 aerr= 0.54096786712553E-05 rerr= 0.9016131E-06
*************************************************************************
* DB: Gamma(3+2*ep) * ep^-2                                      0.28s
* -0.491674205227 66E+01 aerr= 0.17982248269768E-04 rerr= 0.3657350E-05
*************************************************************************
* DB: Gamma(3+2*ep) * ep^-1                                      9.54s
*  0.114947381323 80E+02 aerr= 0.85267241358588E-04 rerr= 0.7417937E-05
*************************************************************************
* DB: Gamma(3+2*ep) * ep^0                                     134.61s
*  0.138011833924 83E+02 aerr= 0.29980115734943E-03 rerr= 0.2172286E-04
*************************************************************************
```

Here, we used DCUHRE, an adaptive multi-demensional integration routine with Genz-Malik cubature rules

(Xeon 3GHz, DCUHRE)

compared with analytical result

Example Calculations (II)

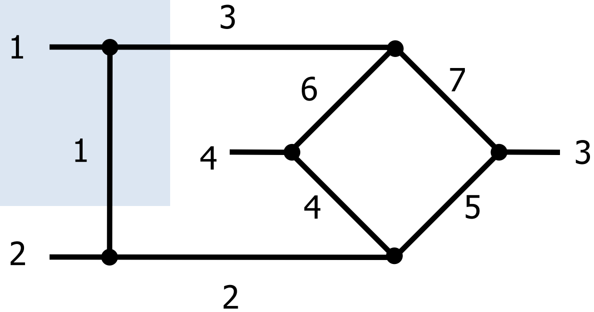
- On-shell massless non-planar double box (input file)

$$\Gamma(7-D)\int_0^1 dx_1\cdots\int_0^1 dx_7\,\delta(1-x_1-\cdots-x_7)\frac{\mathcal{U}^{7-3D/2}}{\mathcal{F}^{7-D}}$$

```
Symbol s,t,u;

Local NDB =
  Gamma(7-D) * int1(x1,...,x7)
    * upow( (x1+x2+x3)*(x4+x5+x6+x7) + (x4+x6)*(x5+x7), 7-3/2*D)
    * fpow( (-s)  * ( x2*x3*(x4+x5+x6+x7) + x2*x6*x7 + x3*x4*x5 )
          + (-t)  * x1*x4*x7 + (-u) * x1*x5*x6 , D-7)
;

multiply replace_(D,4-2*ep);

#call DefParam(s, -1)
#call DefParam(t, -1)
#call DefParam(u, -1)

#call DefParam(DCUHRE_REL_TOL, 1.d-2)
```

s+t+u≠0; unphysical

One can also use the expression after momentum integration as input



One can control integrator parameters in input file

(result)

NDB/Gamma(3+2*ep) =

+0.175001E+01 ϵ^{-4} -0.300027E+01 ϵ^{-3}

-0.228286E+02 ϵ^{-2} +0.113661E+03 ϵ^{-1}

-0.395206E+03 + $\mathcal{O}(\epsilon)$

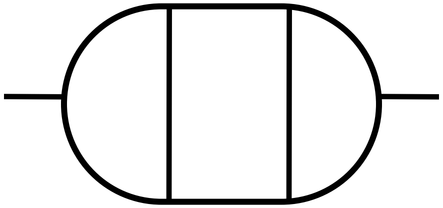
(analytic) J.B.Tausk, PLB469 (1999) 225

(numerical) T.Binoth & G.Heinrich, NPB585 (2000) 741

Example Calculations (III)

- Massless propagator-type diagrams

Analytic result can be found in
S.Bekavac, CPC175 (2006) 180



= Gamma(2+3*ep)*pow(-s,-2-3*ep) * (

+0.20738550050052E+02 — 17.21s

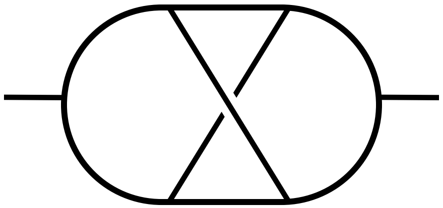
+0.93705981683747E+02 ϵ — 907.35s

+0.31742454422785E+03 ϵ^2 — 10127.17s

+ $\mathcal{O}(\epsilon^3)$

)

(Xeon 3GHz, DCUHRE)



= Gamma(2+3*ep)*pow(-s,-2-3*ep) * (

+0.20738548101272E+02 — 15.90s

+0.12838455747229E+03 ϵ — 447.77s

+0.51052943869464E+03 ϵ^2 — 11312.98s

+ $\mathcal{O}(\epsilon^3)$

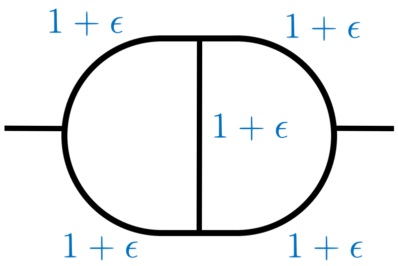
)

(Xeon 3GHz, DCUHRE)

Example Calculations (IV)

- Non-integer powers

Analytic result can be found in
I. Bierenbaum & S. Weinzierl Eur.Phys.J.C32 (2003) 67




```
= Gamma(1+7*ep)/Gamma^5(1+ep) * pow(-s,-2-3*ep) * (
```

+0.721234183388808E+01		< 0.01s
+0.241655928120011E+02	ϵ	0.14s
+0.161200329184407E+03	ϵ^2	2.19s
+0.572629703826290E+03	ϵ^3	20.70s
+0.348307134544090E+04	ϵ^4	141.41s
+0.141000697663510E+05	ϵ^5	616.67s
+0.813902170677060E+05	ϵ^6	1932.82s
+0.357480468775960E+06	ϵ^7	4870.39s
+ $\mathcal{O}(\epsilon^8)$		

)

(Xeon 3GHz, DCUHRE)

Advance beyond the Thresholds

- Consider box example again, but for $st < 0$

$$J_4^{0m} = \int_0^1 d^4x \frac{\delta(1 - x_{1234})}{(-sx_1x_3 - tx_2x_4 - i0)^{2+\epsilon}}$$

  - The denominator is not positive definite, and can become to zero at some points in the integration region
  - Infinitesimal $i0$ is not suitable for numerical calculation

We need some method to handle it

  - contour deformation Z.Nagy & D.E.Soper, PRD74, 093006 (2006)
    - Deform the integration path in the complex plane
    - Already used in some calculations

e.g., (1-loop pentagon) A.Lazopoulos, K.Melnikov & F.J.Petriello, PRD76 (2007) 014001
(2-loop vertex) C.Anastasiou, S.Beerli & A.Daleo, JHEP05 (2007) 071

Here we try to another possibility

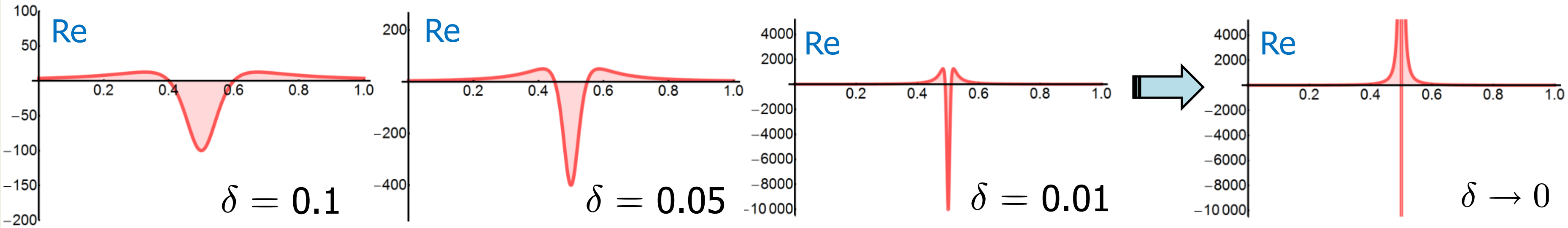
  - numerical extrapolation

E. de Doncker, Y.Shimizu, J.Fujimoto & F. Yuasa, CPC159 (2004) 145

Numerical Extrapolation

Simple example:

$$\int_0^1 dx \frac{1}{(x-a-i\delta)^2} = -\frac{1}{a(1-a)} \qquad \delta \to 0 \qquad 0 < a < 1$$



- If we take δ_k as geometric series

$$\delta_k = \delta_1 r^{k-1} \qquad k = 1, 2, 3, \ldots \qquad 0 < r < 1$$

Then the series $\{I(\delta_k)\}$ converges linearly

This convergence can be accelerated by adequate method,

e.g., Wynn's ϵ -algorithm

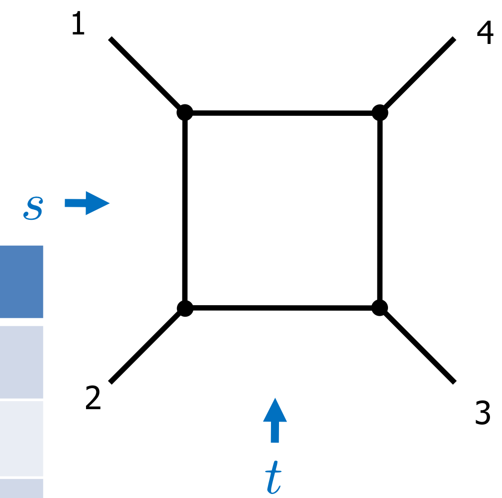
- Only we have to do for the integrand, is just to insert $i\delta$

More topics and applications about numerical integration with extrapolation method can be found in Yuasa-san's talk

Example Calculations (V)

- Massless on-shell scalar box

$I_4^{0m}/\Gamma(2+\epsilon)$ s=123, t=-200



		Numerical Result	Analytic Result(*)
ϵ^{-2}	Re	-0.162601625709E-03	-0.162601626016E-3
	Im	0.328048372972E-16	0
ϵ^{-1}	Re	0.984593629811E-03	0.984593635928E-3
	Im	0.255414040540E-03	0.255414036877E-3
ϵ^{0}	Re	-0.238880580424E-02	-0.238880579909E-2
	Im	0.160867868831E-02	0.160867866412E-2

1sec @ Core2 2.6GHz, DQAG
(double precision, requested $E_{rel}=10^{-8}$ for each integrals)

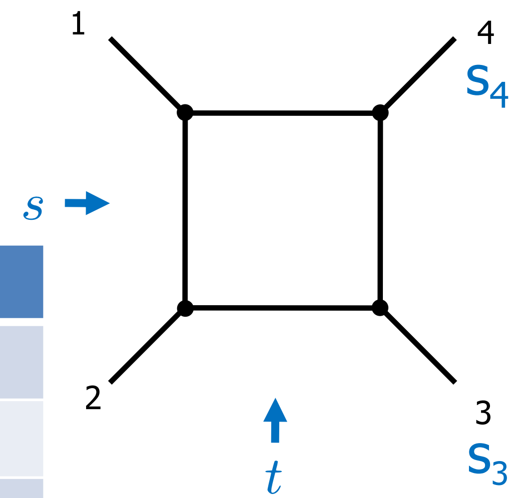
* Analytic result can be found in

G.Duplancic & B.Nizic, Eur.Phys.J.C20 (2001) 357

Y.Kurihara, Eur.Phys.J.C45 (2006) 427

Example Calculations (VI)

- Massless box with two adjacent off-shell legs



$I_4^{2mh}/\Gamma(2+\epsilon)$ s=123, t=-200, s_3=50, s_4=60

		Numerical Result	Analytic Result(*)
ϵ^{-2}	Re	-0.406504064273E-04	-0.406504065041E-4
	Im	0.763198172102E-16	0
ϵ^{-1}	Re	-0.341563066836E-03	-0.341563069952E-3
	Im	-0.127707015460E-03	-0.127707018439E-3
ϵ^{0}	Re	-0.149295025998E-02	-0.149295024564E-2
	Im	-0.287455792563E-03	-0.287455926817E-3

19min @ Core2 2.6GHz, DQAG
(double precision, requested $E_{rel}=10^{-8}$ for each integrals)

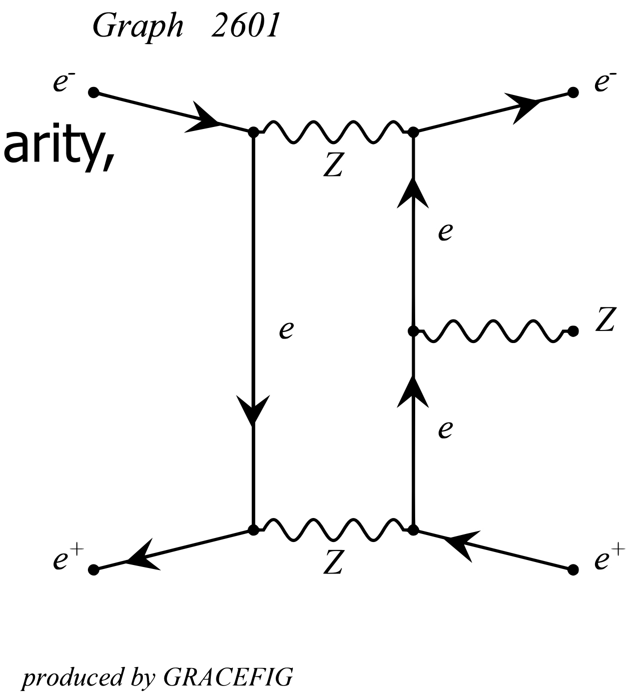
* Analytic result can be found in

G.Duplancic & B.Nizic, Eur.Phys.J.C20 (2001) 357

Y.Kurihara, Eur.Phys.J.C45 (2006) 427

Example Calculations (VII)

- 1-loop scalar pentagon
  - Actually this integral does not have IR-singularity, but sector decomposition does work



s_{12} = 100000.000000000

s_{23} = -30471.3126018059

s_{34} = 32384.1496580698

s_{45} = 37833.5682283554

s_{51} = -14146.0960752976

Re: -0.411918010140121E-13 AbsErr=0.269E-17 RelErr=0.654E-04

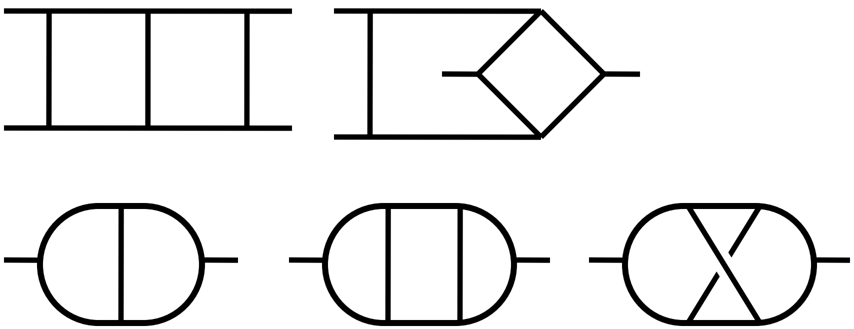
Im : 0.233630556003699E-12 AbsErr=0.139E-17 RelErr=0.596E-05

2.6hour @ Core2 2.6GHz, DQAG, w/ partial integration
(double precision, requested $E_{rel}=10^{-6}$ for each integrals)

Checked by other method in our group ⇨ Yuasa-san's talk
(in my case, electron mass is neglected, but it does not affect the result with this error)

Summary

- We have implemented the sector decomposition algorithm on symbolic manipulation program FORM
  - Thanks to its advantage for handling very large expression, we can overcome the memory problem
- We perform several loop integrals by using it



- As future work, we use sector decomposition + numerical extrapolation for physical region
  - Preliminary results: