Numerical Calculations of Multiple Polylog functions

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Outline

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- Multiple PolyLogarithms
- Integration representation
- Numerical Contoure Integral
- Numerical Results
- Summary

Motivation

- Multiple Polylog functions are often appered in the higher loop calculations.

- Fast and numerically stable evaluation for MPLs is highly desireble.

ex. S. Moch arXiv:math-ph/0509057v1

Multiple-Sum Representation of MPL

$$Li(w_1,\ldots,w_n;x_1,\ldots,x_n)=\sum_{0<i_1<i_2<\ldots<i_n}^{\infty}\frac{x_1^{i_1}}{i_1^{w_1}}\frac{x_2^{i_2}}{i_2^{w_2}}\cdots\frac{x_n^{i_n}}{i_n^{w_n}}$$

Weight:W=w1+...wn depth: n

A.B. Goncharov Math. Res. Lett. 5, 497 (1998)

Multiple Polylogarithm

Z-sums

We define the Z-sums by

$$Z(n) = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases}$$

$$Z(n; m_1, \ldots, m_k; x_1, \ldots, x_k) = \sum_{i=1}^{n} \frac{x_1^i}{i^{m_1}} Z(i-1; m_2, \ldots, m_k; x_2, \ldots, x_k),$$

k is called the depth, $w = m_1 + \ldots + m_k$ the weight. An equivalent definition is given by

$$Z(n; m_1, \ldots, m_k; x_1, \ldots, x_k) = \sum_{n \geq i_1 > i_2 > \ldots > i_k > 0} \frac{x_1^{i_1}}{{i_1}^{m_1}} \cdots \frac{x_k^{i_k}}{{i_k}^{m_k}}.$$

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Z-sum, MPL and oter functions

For $n = \infty$ the Z-sums are the multiple polylogarithms of Goncharov [5]:

$$Z(\infty; m_1, \ldots, m_k; x_1, \ldots, x_k) = \mathrm{Li}_{m_k,\ldots,m_1}(x_k, \ldots, x_1).$$

For $x_1 = \ldots = x_k = 1$ the definition reduces to the Euler-Zagier sums [9, 10]:

$$Z(n; m_1, \ldots, m_k; 1, \ldots, 1) = Z_{m_1,\ldots,m_k}(n).$$

For $n = \infty$ and $x_1 = \ldots = x_k = 1$ the sum is a multiple ζ-value [6]:

$$Z(\infty; m_1, \ldots, m_k; 1, \ldots, 1) = \zeta(m_k, \ldots, m_1).$$

The S-sums reduce for $x_1 = \ldots = x_k = 1$ (and positive m_i) to harmonic sums [12]:

$$S(n; m_1, \ldots, m_k; 1, \ldots, 1) = S_{m_1,\ldots,m_k}(n).$$

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Relation to sipler functions

$$\mathrm{Li}_0(x) = \frac{x}{1-x}, \qquad \mathrm{Li}_1(x) = -\ln(1-x)$$

$$\mathrm{Li}_n(x) = \int_0^x dt \frac{\mathrm{Li}_{n-1}(t)}{t}.$$

Nielsen's generarized polylog

$$S_{n,p}(x) \quad = \quad \mathrm{Li}_{1,\ldots,1,n+1}(\underbrace{1,\ldots,1}_{p-1},x),$$

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Hermonic polylog (Remidi & Vermaseren)

$$H_{m_1,\ldots,m_k}(x) \;=\; \mathrm{Li}_{m_k,\ldots,m_1}(\underbrace{1,\ldots,1}_{k-1},x).$$

$$H_0(x) = \ln(x)$$

$$H_1(x) = -\ln(1-x)$$

$$H_{-1}(x) = \ln(1+x)$$

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Integration Representation of MPL

We first define the notation for iterated integrals

$$\int_0^{\Lambda} \frac{dt}{a_n - t} \circ \ldots \circ \frac{dt}{a_1 - t} = \int_0^{\Lambda} \frac{dt_n}{a_n - t_n} \int_0^{t_n} \frac{dt_{n-1}}{a_{n-1} - t_{n-1}} \times \ldots \times \int_0^{t_2} \frac{dt_1}{a_1 - t_1}.$$

We further use the following short hand notation:

$$\int_0^{\Lambda} \left(\frac{dt}{t} \circ \right)^m \frac{dt}{a - t} = \int_0^{\Lambda} \underbrace{\frac{dt}{t} \circ \ldots \frac{dt}{t}}_{m \text{ times}} \circ \frac{dt}{a - t}.$$

The integral representation for $\mathrm{Li}_{m_k,\ldots,m_1}(x_k,\ldots,x_1)$ reads:

$$\mathrm{Li}_{m_k,\ldots,m_1}(x_k,\ldots,x_1) = \int_0^{x_1 x_2 \ldots x_k} \left(\frac{dt}{t} \circ \right)^{m_1 - 1} \frac{dt}{x_2 x_3 \ldots x_k - t} \circ \left(\frac{dt}{t} \circ \right)^{m_2 - 1} \frac{dt}{x_3 \ldots x_k - t} \circ \ldots \circ \left(\frac{dt}{t} \circ \right)^{m_k - 1} \frac{dt}{1 - t}.$$

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Example of MPL Inegral Formura

$$Li(2,1,2,3\,;X_1,X_2,X_3,X_4)=$$

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \int_0^{t_3} dt_5 \int_0^{t_5} dt_6 \int_0^{t_6} dt_7 \int_0^{t_7} dt_8$$

$$\frac{1}{t_1 t_4 t_6 t_7 (t_2-\frac{1}{X_1})(t_3-\frac{1}{X_1 X_2})(t_5-\frac{1}{X_1 X_2 X_3})(t_8-\frac{1}{X_1 X_2 X_3 X_4})}$$

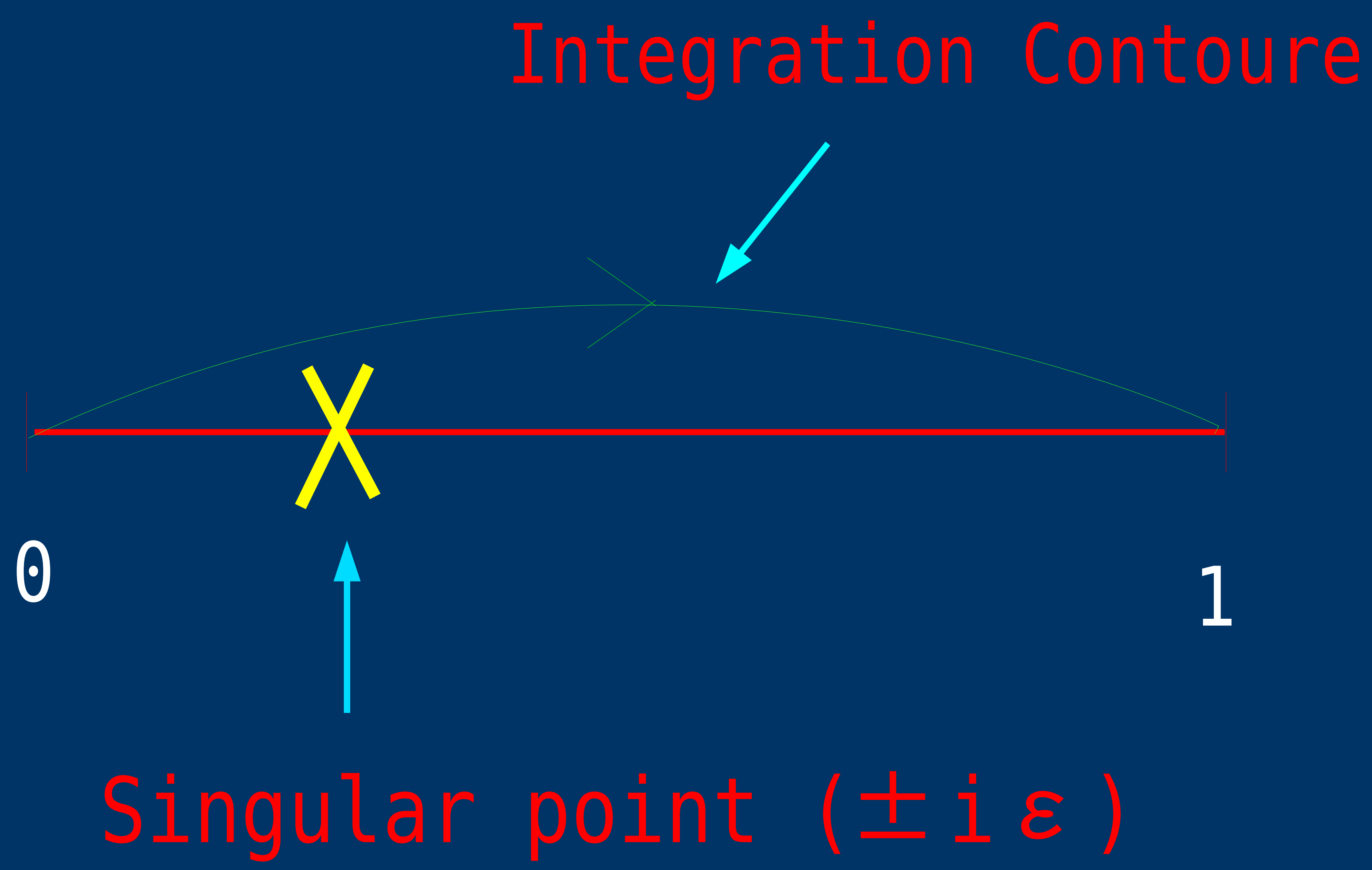
Pre-Modification

- Integrated by the last variable
- Separate singular variable using partial fractioning

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \int_0^{t_3} dt_5 \int_0^{t_5} dt_6 \int_0^{t_6} dt_7$$

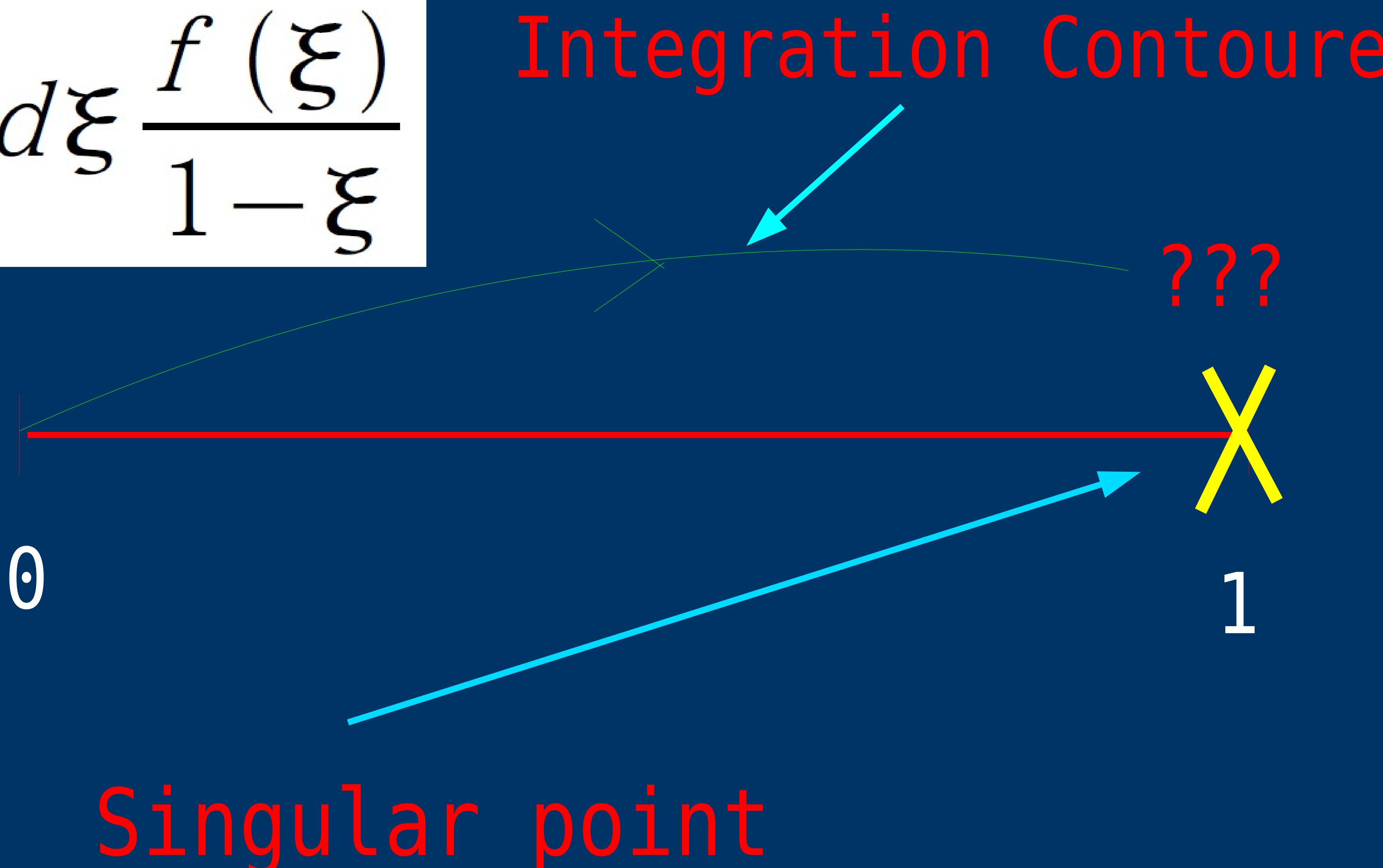
$$-\frac{x1\ x2\ x3\ \text{Log}[1 - t7\ x1\ x2\ x3\ x4]}{t1\ t4\ t6\ t7\ (t2 - t3\ x2)\ (-1 + t3\ x1\ x2)\ (t3 - t5\ x3)}$$

Numerical Contoure Integral



Numerical Contoure Integral

$$\int_0^1 d\xi \frac{f(\xi)}{1-\xi}$$

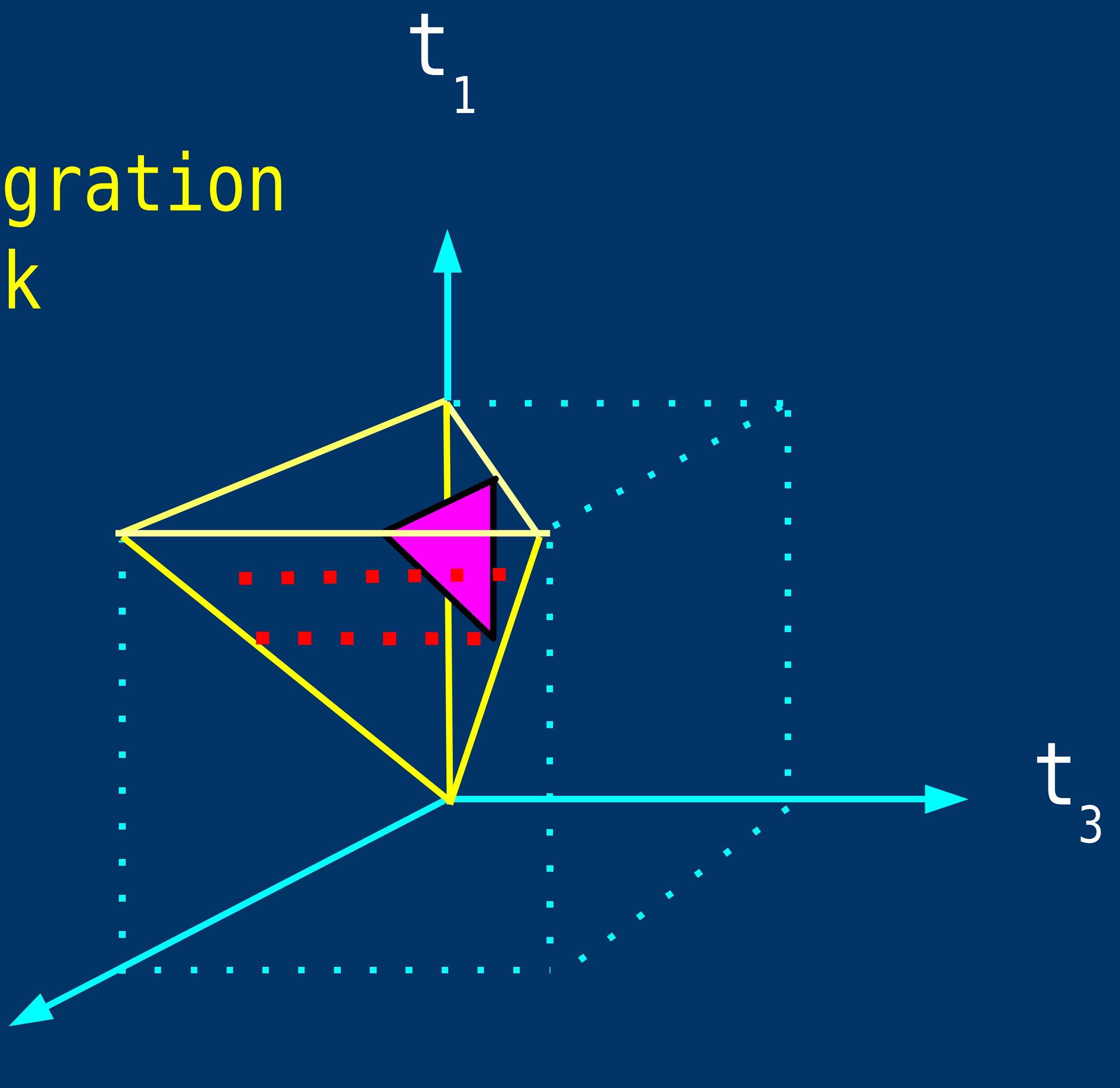


$$\int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \int_0^{t_3} dt_5 \int_0^{t_5} dt_6$$

$$-\frac{x1\ x2\ x3\ \mathrm{Log}[1 - t7\ x1\ x2\ x3\ x4]}{t1\ t4\ t6\ t7\ (t2 - t3\ x2)\ (-1 + t3\ x1\ x2)\ (t3 - t5\ x3)}$$

3-dimensional explanation

Simple integration doesn't work



Transformation

$$t_1 = \xi_1(1-\xi_k)+\xi_k$$
$$t_2 = \xi_2(1-\xi_k)+\xi_k$$
$$\cdots$$
$$t_{k-1} = \xi_{k-1}(1-\xi_k)+\xi_k$$
$$t_k = \xi_k$$
$$t_{k+1} = \xi_{k+1}\xi_k$$
$$\cdots$$
$$t_n = \xi_n\xi_k$$

$$\frac{d\vec{t}}{d\vec{\xi}} = \xi_k^{n-k}(1-\xi_k)^{(k-1)}$$

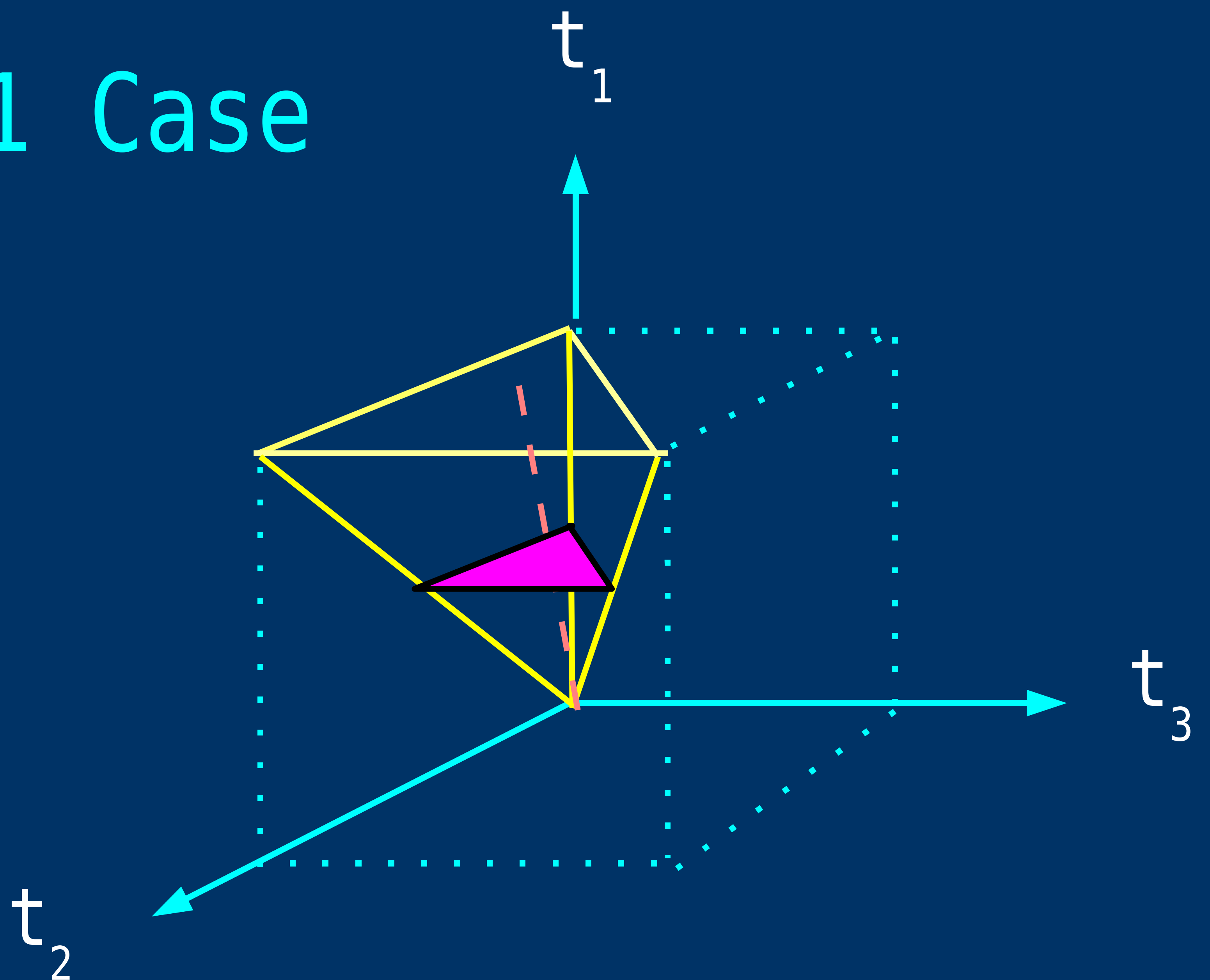
$$0 \le \xi_j \le \xi_{j-1} \; for \; j \ne 1, k, k+1$$
$$0 \le \xi_j \le 1 \; for \; j = 1, k, k+1$$

n:Integration dimension

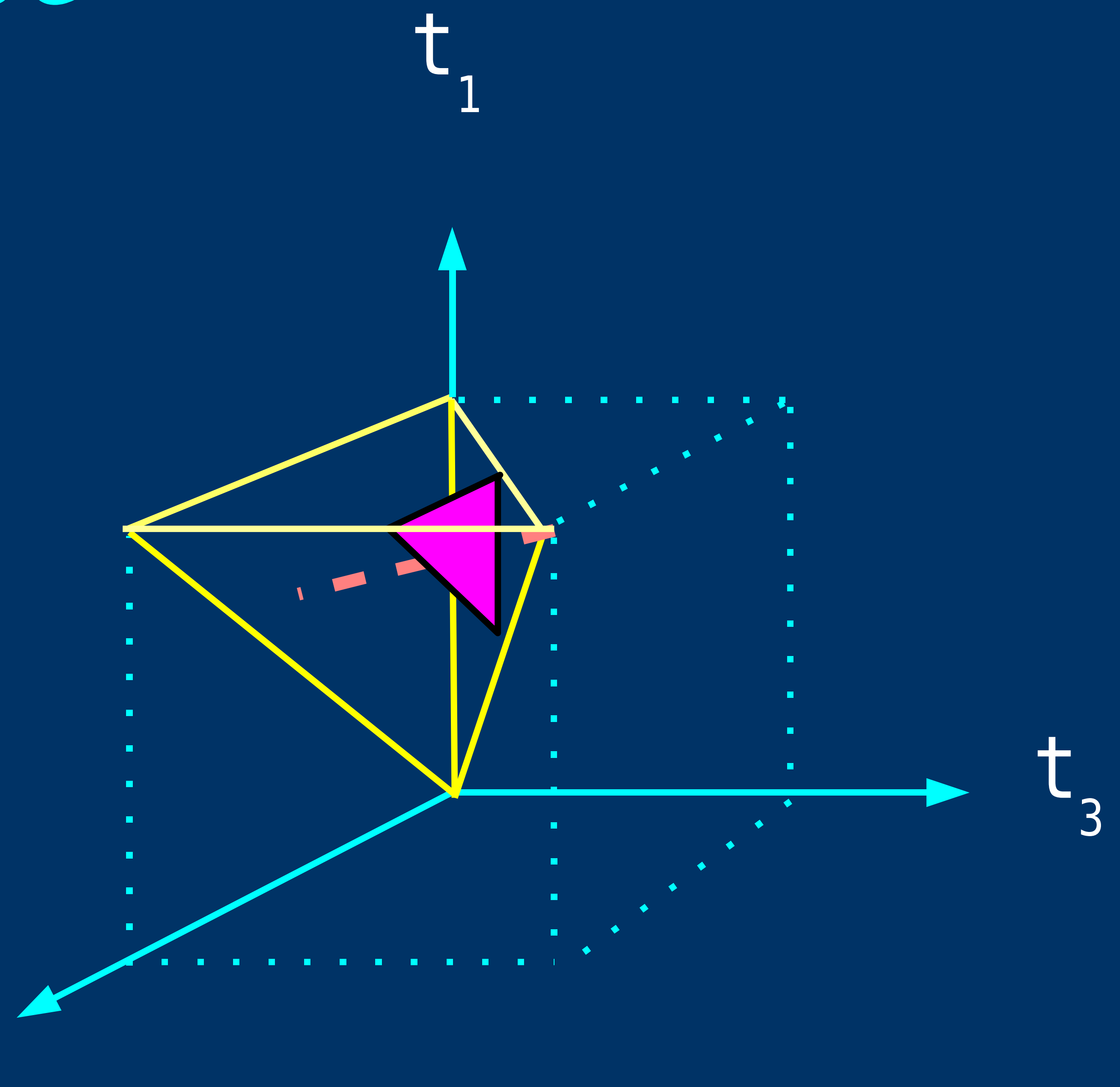
k:Target variable

3-dimensional explanation

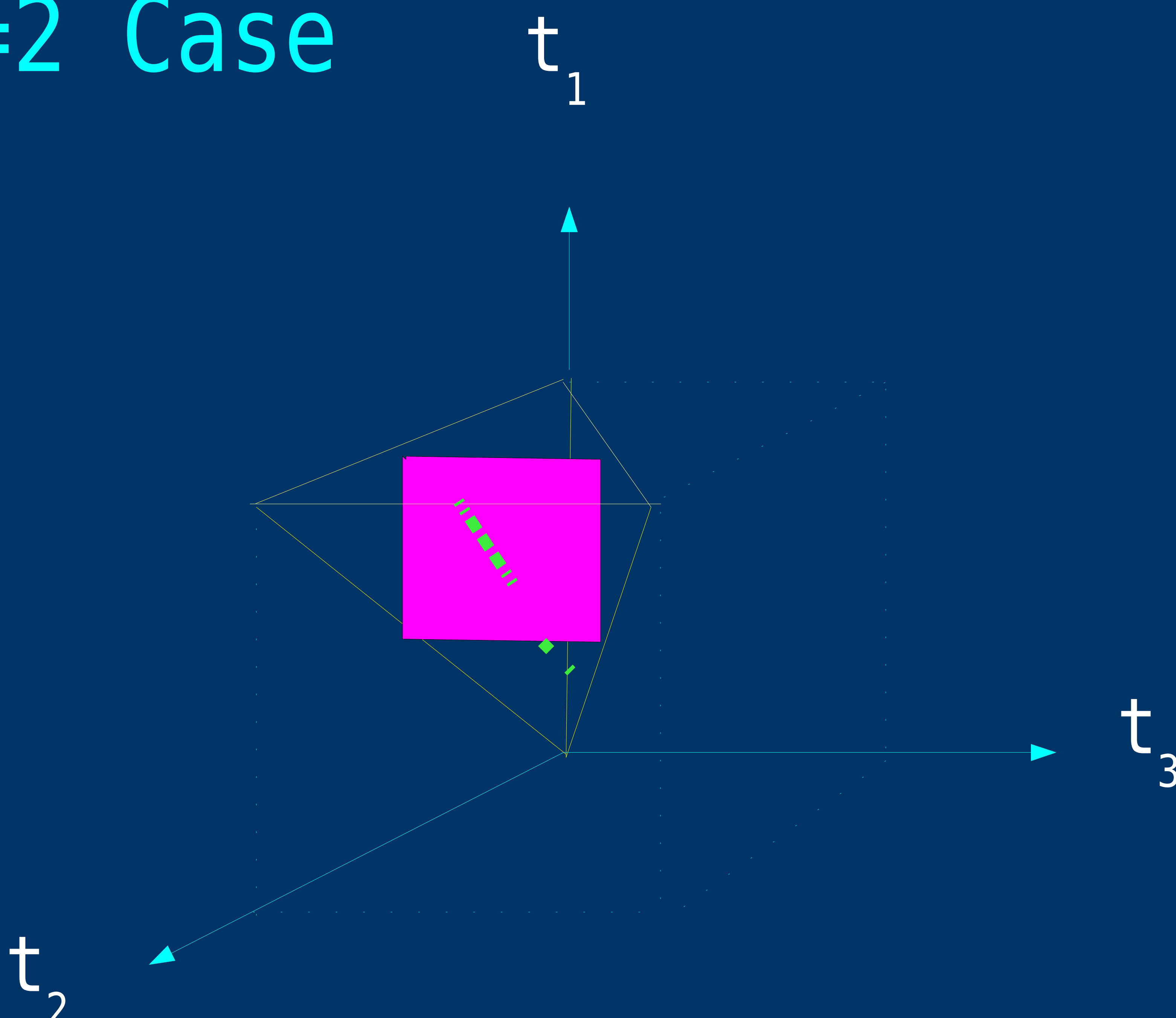
k=1 Case



k=3 Case



k=2 Case



Procedure

Mathematica

- Generation of MPL formura
- Integrate first variable
- Partial Fractioning
- Transformation

Fortran (BASES)

- Numeirical Integratin of NCI

Numerical Results

Preliminary

Li		NCI		GiNaC (*)	
w	x	Real	Imag	Real	Imag
1,1	8/3,1/5	-0.82059	-0.70103	-0.8205920	-0.7010261
2,2,1	0.1,0.2,0.3	3.5544E-07	0	3.55437E-07	0
2,2,1	3.0,2.0,0.2	-0.7889	0.5792	-0.78907	0.57917
2,3,2	0.1,0.2,0.3	1.7734E-07	0	1.77328E-07	0
2,3,2	2.0,3.0,4.0	-10.043	2.123	-	-
2,1,2,3	0.1,0.2,0.3,0.4	1.621E-10	0	1.62105E-10	0
2,1,2,3	2,3,4,5	75.38	-72.26	-	-

* J. Vollinga & S. Weinzierl hep-ph/0410259

Summary

- MPLs often appear in higher order calculations in high energy physics.
- Numerical evaluation of MPLs is necessary.
- The NCI method can be used to calculate MPLs for physical region (w/ higher wight, higher depth).