

Unitarity Methods For 1-loop Amplitudes

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Outline

- Motivation
- Technical preamble
- Overview of the unitarity technique
- Implementations
- Examples

Motivation

- NLO contributions to QCD processes important for backgrounds at LHC
 - e.g. $2g \longrightarrow (n-2)g$
- Feynman diagrams
 - an ‘in principle’ solution
 - too many diagrams
 - give vital information about the form of the amplitude
- Need a practical method

Colour Ordering

Amplitudes with n external gluons can be decomposed into colour-ordered partial amplitudes

$$\mathcal{A}_n^{\text{tree}}(k_i, \lambda_i, a_i) = g^{n-2} \sum_{\sigma \in S_n / Z_n} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) \times A_n^{\text{tree}}(k_{\sigma(1)}^{\lambda_{\sigma(1)}}, \dots, k_{\sigma(n)}^{\lambda_{\sigma(n)}})$$

where:

k_i is the momentum of the i -th external particle

λ_i is its helicity (+/-)

a_i is its colour index

The sum is over non-cyclic permutations of $\{1, 2, \dots, n\}$

(Paton & Chan; Berends & Giele; Mangano; Mangano & Parke; Mangano, Parke & Xu)

Colour Ordering

Similarly at 1-loop for adjoint particles of spin J in the loop:

$$\mathcal{A}_n(k_i, \lambda_i, a_i) = g^n \sum_J \sum_{c=1}^{L.I.(n/2)+1} \sum_{\sigma \in S_n / S_{n;c}} Gr_{n;c}(\sigma) A_{n;c}^{[J]}(\sigma)$$

where:

L.I.(x) is the largest integer less than or equal to x

$$Gr_{n;1}(1) = N_c \text{Tr}(T^{a_1} \dots T^{a_n})$$

$$Gr_{n;c}(1) = \text{Tr}(T^{a_1} \dots T^{a_{c-1}}) \text{Tr}(T^{a_c} \dots T^{a_n})$$

$S_{n;c}$ is the set of permutations of the two sets of legs in $Gr_{n;c}$

(Bern & Kosower)

Spinor – Helicity 1

$|i^\pm\rangle$ is a massless Weyl spinor with momentum k_i and chirality \pm

We can define spinor products via:

$$\langle ij \rangle = \langle i^- | j^+ \rangle \quad [ij] = \langle i^+ | j^- \rangle$$

Alternatively, define : $\lambda_m^i \bar{\lambda}_{\dot{n}}^i = k_\nu^i \sigma_{m\dot{n}}^\nu$

leading to: $\langle ij \rangle = \epsilon^{mn} \lambda_m^i \lambda_n^j \quad [ij] = -\epsilon^{\dot{m}\dot{n}} \bar{\lambda}_{\dot{m}}^i \bar{\lambda}_{\dot{n}}^j$

(Mangano & Parke; Xu, Zhang & Chang)

Spinor – Helicity 2

We can use these spinor products to express momentum invariants:

$$[ij]\langle ji\rangle = 2k_i \cdot k_j \equiv s_{ij}$$

and polarisation tensors:

$$\epsilon_{im\dot{n}}^- = \frac{\lambda_{im}\bar{\eta}_{\dot{n}}}{[\bar{\lambda}_i\bar{\eta}]} \quad \epsilon_{im\dot{n}}^+ = \frac{\eta_m\bar{\lambda}_{i\dot{n}}}{\langle\eta\lambda_i\rangle}$$

leading to very compact expressions:-

(Mangano & Parke; Xu, Zhang & Chang)

Spinor – Helicity 3

$$A_n^{\text{tree}}(1^+, 2^+, \dots, n^+) = 0$$

$$A_n^{\text{tree}}(1^+, \dots, j^-, \dots, n^+) = 0$$

$$A_n^{\text{tree}}(1^+, \dots, j^-, \dots, k^-, \dots, n^+) = i \frac{\langle jk \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}$$

Maximally Helicity Violating (MHV) amplitudes

Passarino-Veltman Reduction

$I_n(f^p(l))$:

- 1-loop integral
- loop momentum l
- n propagators
- numerator of order p in the loop momentum

For $n > 5$, collapsing propagators gives

$$I_n(f^p(l)) = \sum_i B_i I_{n-1}^i(f^{p-1}(l))$$

Passarino-Veltman Reduction

For lower n the process leaves residual scalar (p=0) integrals:

4 to 3 leaves scalar boxes

3 to 2 leaves scalar triangles

Process ends at bubbles

Quadratic bubbles also give rational terms (no logarithms)

For massless particles:

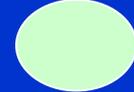
$$A_n^{1\text{-loop}} = \sum_i c_i I_4^I + \sum_j d_j I_3^j + \sum_k e_k I_2^k + R$$

Supersymmetric Decomposition

$$A_n^{\mathcal{N}=4} \equiv A_n^{[1]} + 4A_n^{[\frac{1}{2}]} + 3A_n^{[0]}$$



$$A_n^{\mathcal{N}=1\text{chiral}} \equiv A_n^{[\frac{1}{2}]} + A_n^{[0]}$$



Inverting we have:

$$A_n^{[1]} = A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1\text{chiral}} + A_n^{[0]}$$

$$A_n^{[\frac{1}{2}]} = A_n^{\mathcal{N}=1\text{chiral}} - A_n^{[0]}$$

Supersymmetric Decomposition

Divide and conquer

Physically motivated split

- Susy is only a tool

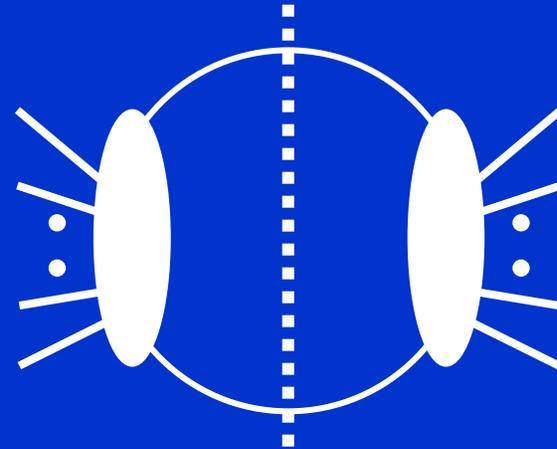
- the hard part is now the scalar loop

Unitarity Cuts

Take a 1-loop amplitude
'cut' two propagators

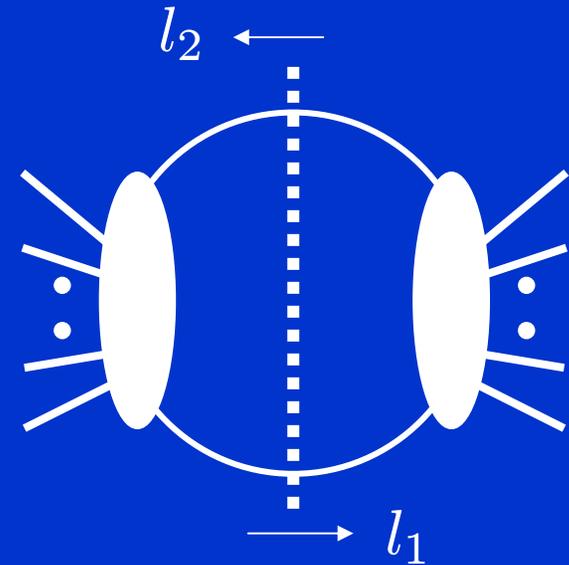
$$\frac{1}{(l-K)^2 + i\epsilon} \rightarrow \delta((l-K)^2)$$

Propagators now on shell
Blobs are tree amplitudes



(Bern, Dixon, Dunbar & Kosower)

Unitarity Cuts



The result is

$$C_2 = \int dLIPS(l_1, l_2) A_{\text{left}}^{\text{tree}}(l_1, \dots, -l_2) \times A_{\text{right}}^{\text{tree}}(l_2, \dots, -l_1)$$

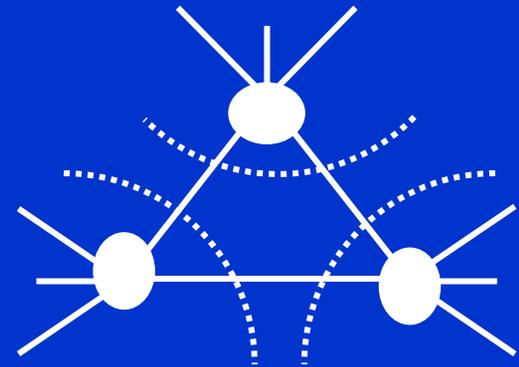
In terms of the basis of boxes etc.

$$C_2 = \sum_i c_i I_4^i|_{\text{disc}} + \sum_j d_j I_3^j|_{\text{disc}} + \sum_k e_k I_2^k|_{\text{disc}}$$

We can find the coefficient of any basis function that has this cut

Generalised Unitarity

We can cut more propagators
Triple cut



$$C_3 = \int dLIPS A_1^{\text{tree}}(l_1, \dots, -l_2) \times A_2^{\text{tree}}(l_2, \dots, -l_3) \\ \times A_3^{\text{tree}}(l_3, \dots, -l_1)$$

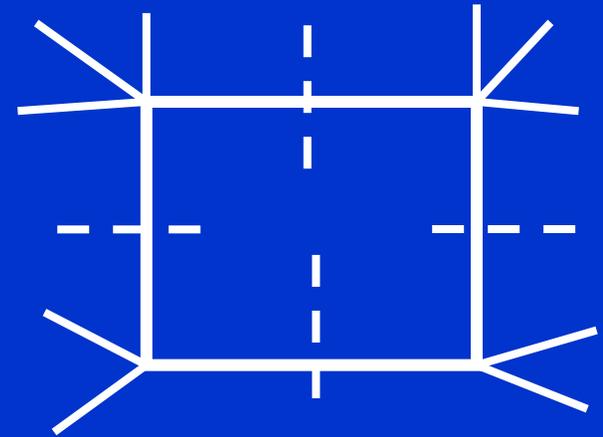
$$C_3 = \sum_i c_i I_4^i|_{\text{disc}} + \sum_j d_j I_3^j|_{\text{disc}}$$

Generalised Unitarity

Quad cuts

No integral

Algebraically isolates
a single box coefficient



$$c_i = \frac{1}{2} \sum A_1^{\text{tree}}(l_1, \dots, -l_2) \times A_2^{\text{tree}}(l_2, \dots, -l_3) \\ \times A_3^{\text{tree}}(l_3, \dots, -l_4) \times A_4^{\text{tree}}(l_4, \dots, -l_1)$$

Massless corners non-vanishing for
complex cut momenta

(Britto, Cachazo & Feng)

The Aim of the Game

Extract coefficients from the product of tree amplitudes algebraically/numerically

Our strategy

$$C_4 \longrightarrow c_i$$

$$C_3 \longrightarrow \{c_i, d_j\}$$

$$C_2 \longrightarrow \{c_i, d_j, e_k\}$$

Recent Implementations

Boxes: quad cut, algebra – job done

Triangles: triple cut leaves a 1 parameter
integral – contour methods

(Forde; Bjerrum-Bohr, Dunbar & WBP)

Bubbles: double cut leaves 2 parameter integral
Fermionic integration

(Britto & Feng; Britto, Feng & Mastrolia, Britto, Feng & Yang)

Direct parameterisation

(Forde)

Recent Implementations

Rational pieces

$d = 4 - 2\epsilon$ dimensional cuts

(Britto & Feng; Britto, Feng & Mastrolia, Britto, Feng & Yang)

On-shell recursion

(Bern, Dixon & Kosower)

Numerical Implementations

CutTools

(Ossola, Papadopoulos & Pittau)

BlackHat

(Berger, Bern, Dixon, Febres Cordero, Forde, Ita, Kosower, Maitre)

(compare traditional 'semi-numerical': Ellis, Giele & Zanderighi)

Canonical Basis

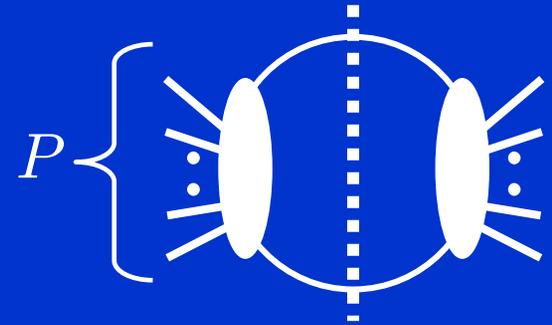
A limited number of structures occur in either the triple or double cuts

Determine the contribution of each structure once → canonical basis

Expressions can be cleaned up:

- explicit rationality
- remove (some) spurious singularities

A Simple Example



If the product of tree amplitudes in a double cut contains a term,

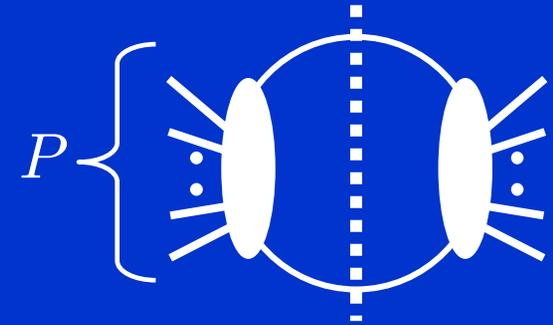
$$H_1 = \frac{\langle al \rangle}{\langle bl \rangle} = \frac{[b|l|a\rangle}{(l + k_b)^2}$$

This is a linear triangle

Standard reduction/direct parameterisation/...

$$H_1 \rightarrow \triangle + \frac{[b|P|a\rangle}{[b|P|b\rangle} I_2$$

Another Example



$$G_1 = \frac{[B|l]\langle l|c\rangle\langle l|d\rangle}{\langle l|a\rangle(l+Q)^2}$$

$$\rightarrow \blacksquare + \blacktriangle + \left(\frac{[B|P[Q,P]|a\rangle\langle c|[Q,P]|d\rangle}{8\Delta\langle a|QP|a\rangle} - \frac{[B|P|a\rangle(\langle c|a\rangle[a|P|d\rangle + \langle d|a\rangle[a|P|c\rangle])}{2\langle a|QP|a\rangle[a|P|a\rangle} \right) I_2$$

where

$$\Delta = P.Q^2 - P^2.Q^2$$

Using partial fractions this is all that is needed for N=1, n<8.
N=0 just has more, longer expressions

A real calculation

The (12) cut of $A_7^{1\text{-loop}}(1^-, 2^+, 3^-, 4^+, 5^+, 6^+, 7^-)$
in N=1

The product of tree amplitudes is,

$$A_4^{\text{tree}}(-l_1, 1^-, 2^+, l_2) \times A_7^{\text{tree}}(-l_2, 3^-, 4^+, 5^+, 6^+, 7^-, l_1)$$

The 7 point tree amplitude has 6 terms
Illustrate with one of these:

This term in the tree amplitude is,

$$\begin{aligned}
 & A_{\text{piece}}^{\text{tree}}(l_1^{-H}, -l_2^{+H}, 3^-, 4^+, 5^+, 6^+, 7^-) \\
 &= i \frac{[4|P_{56}|7\rangle^3 \langle 7l_2\rangle^{2-H} \langle 7l_1\rangle^{1+H}}{\langle 56\rangle \langle 67\rangle [34] \langle l_1 l_2\rangle [3|(l_1 - l_2)|7\rangle \langle 7|P_{56}P_{34}|l_2\rangle \langle 7|(l_1 - l_2)P_{34}|5\rangle}
 \end{aligned}$$

The corresponding cut integrand is,

$$\begin{aligned}
 \mathcal{C} &= \frac{[4|P_{56}|7\rangle^3}{s_{12}[12]\langle 56\rangle \langle 67\rangle [34][3|P_{12}|7\rangle \langle 7|P_{12}P_{34}|5\rangle} \\
 &\quad \times \sum_H \frac{[2l_1]^{2+H} [2l_2]^{1-H} \langle 7l_2\rangle^{2-H} \langle 7l_1\rangle^{1+H}}{[l_1 1] \langle 7|P_{56}P_{34}|l_2\rangle}
 \end{aligned}$$

Summing over the multiplet gives,

$$\mathcal{C} \rightarrow \frac{[4|P_{56}|7\rangle^3 \langle 17\rangle^2}{\langle 21\rangle \langle 56\rangle \langle 67\rangle [34] [3|P_{12}|7\rangle \langle 7|P_{12}P_{34}|5\rangle} \times \frac{\langle 1l_2\rangle \langle 7l_2\rangle}{\langle 2l_2\rangle \langle 7|P_{56}P_{34}|l_2\rangle}$$

Partial fractioning gives two H_1 type pieces,

$$\mathcal{C}|_{\text{bubble}} = \frac{[4|P_{56}|7\rangle^3 \langle 17\rangle^2}{\langle 21\rangle \langle 56\rangle \langle 67\rangle [34] [3|P_{12}|7\rangle \langle 7|P_{12}P_{34}|5\rangle} \times \frac{1}{\langle 7|P_{56}P_{34}|2\rangle} \left(\langle 17\rangle - \frac{\langle 1|P_{34}P_{56}|7\rangle [7|P_{56}P_{34}P_{12}|7\rangle}{[7|P_{56}P_{34}P_{12}P_{34}P_{56}|7\rangle} \right)$$

The other terms from this 7-pt tree follow similarly.
All bubbles in the 7-pt N=1 amplitudes involve the H_1 and G_1 forms only.

A Full Amplitude

The (71) cut of $A_7^{1\text{-loop}}(1^-, 2^-, 3^+, 4^-, 5^+, 6^+, 7^+)$
in $N=1$

The product of tree amplitudes is,

$$A_4^{\text{tree}}(-l_1, 7^+, 1^-, l_2) \times A_7^{\text{tree}}(-l_2, 2^-, 3^+, 4^-, 5^+, 6^+, l_1)$$

The 7 point tree amplitude is:

$$A_7^{\text{tree}}(1^{-h}, 2^h, 3^-, 4^+, 5^-, 6^+, 7^+)$$

$$\begin{aligned}
T_{1a}^B &= \frac{[4|P_{234}|5\rangle^2 [24]^2 \langle 51\rangle^2}{\langle 56\rangle \langle 67\rangle \langle 71\rangle [23] [34] t_{234} [2|P_{234}|5\rangle [4|P_{234}|1\rangle]} \times \left(\frac{[24] \langle 51\rangle}{[4|P_{234}|5\rangle} \right)^{2h} \\
T_{1b}^B &= \frac{-\langle 23\rangle \langle 13\rangle^2 [4|P_{67}|1\rangle^4}{\langle 67\rangle \langle 71\rangle \langle 12\rangle [45] [5|P_{67}|1\rangle [4|P_{23}|1\rangle \langle 1|P_{67}P_{45}|3\rangle \langle 6|P_{45}P_{23}|1\rangle]} \times \left(\frac{\langle 13\rangle}{\langle 23\rangle} \right)^{2f} \\
T_{1c}^B &= \frac{t_{671} [2|P_{671}|1\rangle^2 \langle 35\rangle^4}{\langle 67\rangle \langle 71\rangle \langle 34\rangle \langle 45\rangle [2|P_{71}|6\rangle [2|P_{345}|5\rangle \langle 1|P_{67}P_{45}|3\rangle t_{345}} \left(\frac{-[2|P_{671}|1\rangle}{t_{671}} \right)^{2h} \\
T_2^B &= \frac{-[27]^2 [17] \langle 35\rangle^4}{[12] \langle 34\rangle \langle 45\rangle \langle 56\rangle [2|P_{712}|6\rangle [7|P_{712}|3\rangle t_{712}} \times \left(-\frac{[27]}{[17]} \right)^{2h} \\
T_3^B &= \frac{\langle 13\rangle^2 \langle 23\rangle^2 [7|P_{123}|5\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 45\rangle \langle 56\rangle [7|P_{123}|3\rangle [7|P_{123}|4\rangle \langle 1|P_{23}P_{45}|6\rangle t_{123} t_{456}} \times \left(\frac{\langle 13\rangle}{\langle 23\rangle} \right)^{2h} \\
T_4^B &= \frac{-\langle 13\rangle^2 \langle 23\rangle^2 \langle 56\rangle^2 [67]^3}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 56\rangle [7|P_{56}|4\rangle [5|P_{567}|1\rangle t_{567} s_{56}} \times \left(\frac{\langle 13\rangle}{\langle 23\rangle} \right)^{2h}
\end{aligned}$$

The full bubble coefficient is

$$\begin{aligned}
 &= \frac{1}{\langle 56 \rangle \langle 67 \rangle \langle 12 \rangle [34]} G_5(1, 7, P_{12}P_{34}|5), P_{12}|3, P_{567}|4; 4; 2, 5, 5, X_{1a}, X_{1a}, P_{12}|4; Q_{567}) \\
 &+ \frac{\langle 23 \rangle^2}{\langle 12 \rangle \langle 67 \rangle [45]} \times H_6(1, 7, P_{67}|5, P_{567}|4, P_{67}P_{45}|3, P_{4567}P_{45}|6; 2, 3, P_{67}|4, P_{67}|4, P_{67}|4, P_{67}|4,) \\
 &+ \frac{\langle 35 \rangle^4}{\langle 12 \rangle \langle 34 \rangle \langle 45 \rangle \langle 67 \rangle t_{345}} \sum_{\substack{a \in \{1,2\} \\ b \in \{6,7\}}} [ab] H_5(2, a, b, P_{67}P_{345}|2, P_{67}P_{345}|2; 1, 7, P_{12}P_{345}|6, P_{12}P_{34}|5, P_{67}P_{45}|3) \\
 &+ \frac{\langle 35 \rangle^4 [71]^2 \langle 12 \rangle^2}{\langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 12 \rangle s_{12}[7|P_{712}|3]t_{712}} \times H_2(2, P_{12}P_{712}|6; 1, P_{12}|7) \\
 &+ \frac{[7|P_{123}|5]^4 \langle 23 \rangle^2}{\langle 12 \rangle \langle 45 \rangle \langle 56 \rangle [7|P_{123}|3][7|P_{123}|4]t_{123}t_{456}} \times H_2(2, P_{123}P_{45}|6; 1, 3) \\
 &+ \frac{[67]^3 \langle 23 \rangle^2}{\langle 12 \rangle \langle 34 \rangle [56] t_{567}[7|P_{56}|4]} \times H_2(1, P_{67}|5; 2, 3)
 \end{aligned}$$

Where $H_n(G_n)$ contains $n H_1(G_1)$ forms by partial fractioning.

Amplitude Repository

The full canonical forms along with the full six and seven point amplitudes will be available at:

<http://pyweb.swan.ac.uk/~dunbar/sixgluon.html>

and

<http://pyweb.swan.ac.uk/~dunbar/sevengluon.html>

Conclusions

Unitarity techniques provide an efficient tool for calculating 1-loop amplitudes

Already used extensively for 6 gluon scattering

A range of analytic and numerical implementations

Conclusions

Canonical basis approach produces clean expressions

Currently implemented for massless particles - natural extension to massive particles