# Impact Factors for Reggeon-Gluon Transitions

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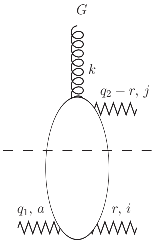
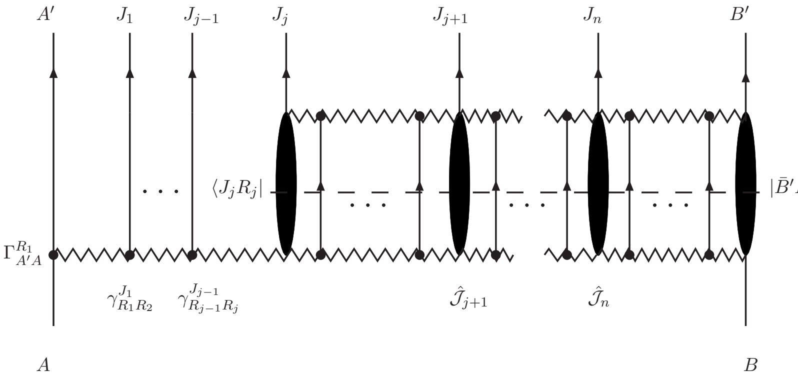
The impact factors for Reggeon-gluon transitions describe transitions of Reggeons(Reggeized gluons) into ordinary gluons due to interaction with Reggeized gluons.



# Reminder

These impact factors enter as an integral part in the $s_i$ discontinuities of multi-Regge amplitudes

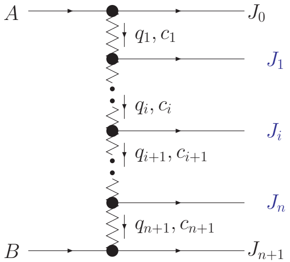


# Reminder

These discontinuities enter in the bootstrap relations

$$\sum_{l=j+1}^{n+1} \Delta_{jl} - \sum_{l=0}^{j-1} \Delta_{lj} = \frac{1}{2} \left( \omega(t_{j+1}) - \omega(t_j) \right) \Re \, \mathcal{A}_{AB}^{A'B'+n} ,$$

which follow from the requirement of compatibility of the $s_i$-channel unitarity with the Reggeized form of multi-particle amplitudes



# Reminder

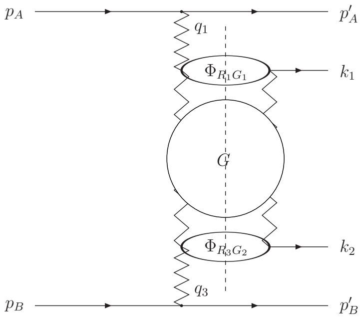
Fulfillment of these relations guarantees the multi–Regge form of scattering amplitudes.

All bootstrap relations are fulfilled if several bootstrap conditions imposed on the Reggeon vertices and the trajectory are satisfied.

All these conditions are proved to be satisfied in the NLO.

Another problem, where consideration of the $s_i$ channel discontinuities (which contain the impact factors for Reggeon-gluon transitions) is important is the high-energy behavior of the BDS ansatz for MHV amplitudes in N=4 SYM in the limit of large number of colours.

There is almost evident contradiction of the expressions for $s_i$ channel discontinuities with the ansatz for $n$ gluon amplitudes starting with $n = 6$.
Energy behaviour of the discontinuity of $A_{2\to4}$ in the $s_2$- channel ($s_2$ is the invariant mass square of produced gluons )

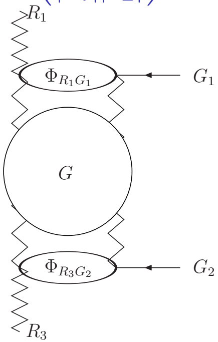



is not given by the Regge factor.

# Reminder

Instead, it is determined by the matrix element

$$\langle G_1 R_1 | e^{\hat{K} \ln\left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)} | G_2 R_3 \rangle = \left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)^{\omega(t_2)} \langle G_1 R_1 | e^{\hat{K}_m \ln\left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)} | G_2 R_3$$




where $\hat{K}_m = \hat{K} - \omega(t)$ – modified BFKL kernel.
Our aim now to calculate this factor.

# Reminder

Why it is interesting?

Violation of the BDS ansatz is known.
There factorization hypothesis states that correct amplitudes are given by the product of $M^{BDS}$ times remainder function $R$. Here $M^{BDS}$ contains infrared divergencies through the Regge factors, and the remainder function $R$ depends only on the anharmonic ratios of kinematic invariants.

This hypothesis is not proved.

Another hypothesis states that the remainder functions are given by expectation values of Wilson loops in N=4 SYM.

This hypothesis is also not proved.

All this makes important direct calculation of this matrix element

# Infrared safety of the modified kernl

Possible representations of the colour group in the $t$–channel

$\underline{1}, \underline{8_a}, \underline{8_s}, \underline{10}, \underline{\overline{10}}, \underline{27}.$

$\widehat{\mathcal{K}}_r$ — the "real" part, depending on the presentation.

In the leading order

$$\mathcal{K}_r^B(\vec{q}_1, \vec{q}_2; \vec{q}) = \frac{g^2 N_c c_{\mathcal{R}}}{(2\pi)^{D-1}} \left( \frac{\vec{q}_1^{\,2} \vec{q}_2^{\,\prime 2} + \vec{q}_2^{\,2} \vec{q}_1^{\,\prime 2}}{(\vec{q}_1 - \vec{q}_1^{\,\prime})^2} - \vec{q}^{\,2} \right)$$

Only one structure with coefficients depending from $t$–channel colour states.

In the NLO $\widehat{\mathcal{K}}_r$ can be written it terms of two independent structures (octet and singlet).

In the leading order at $D = 4 + 2\epsilon$:

$$\omega^{(1)}(\vec{q}) = -\frac{g^2 N_c \Gamma(1-\epsilon)}{(4\pi)^{2+\epsilon}} \frac{2}{\epsilon} (\vec{q})^{\epsilon}$$

# Infrared safety of the modified kernl

In the following we will consider only the representation $\underline{8_a}$.
Proper BFKL kernel for this represention is not infrared safe.
It is easily seen in the LO, because

$$\widehat{\mathcal{K}} = \widehat{\omega}_1 + \widehat{\omega}_2 + \widehat{\mathcal{K}}_r$$

and the real part of the kernel differs from corresponding part for the singlet kernel by the coefficient $1/2$.
Cancelation of infrared divergencies in the singlet channel must be.
But the kernels for odderon and remainder factor do not coinside neither with the proper BKKL kernel, nor with each other.
In the octet case trajectories must be taken with the coefficient $1/2$ (because there are three reggeized gluons, each with its trajectory, and three paired interactions between them), so that the odderon pair interaction kernel is

# Infrared safety of the modified kernl

$$\widehat{\mathcal{K}}_{12} = \frac{1}{2}\left(\widehat{\omega}_1 + \widehat{\omega}_2\right) + \widehat{\mathcal{K}}_r$$

In the LO this kernel differs from the colour singlet kernel only by the coefficient $1/2$. The kernel for the remainder factors is

$$\widehat{\mathcal{K}} = \left(\widehat{\omega}_1 + \widehat{\omega}_2 - \widehat{\omega}\right) + \widehat{\mathcal{K}}_r$$

where $\widehat{\omega}_{12}$ is the gluon trajectory for the total momenta, and is infrared safe also.

It is important that the singular part of the trajectory does not depend on momenta, and the singularities of the real parts of the singlet and octet kernels differ by the coefficient $1/2$, both in the leading and in the next-to leading orders.

# Eigenvalues of the modified kernel

In the limit

$$|q_1| \sim |q_1'| \ll |q| \approx |q_2| \approx |q_2'|\,,$$

with the denotation $\vec{q}_1$ and $\vec{q}_1{}'$ by $\vec{p}$ and $\vec{p}\,'$, respectively, the kernel for the reminder factor takes the form

$$K(\vec{p},\vec{p}\,') = -\delta^2(\vec{p}-\vec{p}\,')\,|p|^2\,\frac{\alpha N_c}{4\pi^2}\,\left(\left(1-\frac{\alpha N_c}{2\pi}\zeta(2)\right)\right.$$

$$\left.\times\int d^2p\,'\,\left(\frac{2}{|p\,'|^2}+\frac{2(\vec{p}\,',\vec{p}-\vec{p}\,')}{|p\,'|^2|p-p\,'|^2}\right)-3\alpha\,\zeta(3)\right)$$

$$+\frac{\alpha N_c}{4\pi^2}\,\left(1-\frac{\alpha N_c}{2\pi}\zeta(2)\right)\,\left(\frac{|p|^2+|p\,'|^2}{|p-p\,'|^2}-1\right)+\frac{\alpha^2 N_c^2}{32\,\pi^3}\,R(\vec{p},\vec{p}\,')\,,$$

where

# Eigenvalues of the modified kernel

$$R(\vec{p},\vec{p}\,') =$$

$$\left(\frac{1}{2} - \frac{|p|^2+|p\,'|^2}{|p-p\,'|^2}\right)\ln^2\frac{|p|^2}{|p\,'|^2} - \frac{|p|^2-|p\,'|^2}{2|p-p\,'|^2}\ln\frac{|p|^2}{|p\,'|^2}\ln\frac{|p|^2|p\,'|^2}{|p-p\,'|^4}$$

$$+4\frac{[\vec{p}\times\vec{p}\,']^2}{(\vec{p}-\vec{p}\,')^2}\int_0^1 dx\,\frac{1}{|(1-x)p+xp\,'|^2}\ln\frac{(1-x)|p|^2+x|p\,'|^2}{x(1-x)|p-p\,'|^2}\,.$$

Due to the rotational and dilatational invariance of the kernel its eigenfunctions have the simple form

$$\Phi_{\nu n}(\vec{p}) = |p|^{2i\nu}e^{i\phi n}\,, \qquad (1)$$

where $\phi$ is the angle of the transverse vector $\overrightarrow{p}$ with respect to the axis $x$. Note, that $\nu$ is real and $n$ is integer.
Corresponding eigenvalues are

# Eigenvalues of the modified kernel

$$\omega(\nu, n) = -a\left(E_{\nu n} + a\,\epsilon_{\nu n}\right)\ ,\ \ a = \frac{\alpha N_c}{2\pi}\ ,$$

where $E_{\nu n}$ is the "energy" in the leading approximation

$$E_{\nu n} = -\frac{1}{2}\frac{|n|}{\nu^2 + \frac{n^2}{4}} + \psi(1 + i\nu + \frac{|n|}{2}) + \psi(1 - i\nu + \frac{|n|}{2}) - 2\psi(1)$$

and the next-to-leading correction $\epsilon_{\nu n}$ can be written as follows

$$\epsilon_{\nu n} = -\frac{1}{4}\left(\psi''(1 + i\nu + \frac{|n|}{2}) + \psi''(1 - i\nu + \frac{|n|}{2}) + \frac{2i\nu\left(\psi'(1 - i\nu + \frac{|n|}{2}) - \psi'(1 + i\nu + \frac{|n|}{2})\right)}{\nu^2 + \frac{n^2}{4}}\right)$$

$$-\zeta(2)\,E_{\nu n} - 3\zeta(3) - \frac{1}{4}\frac{|n|\left(\nu^2 - \frac{n^2}{4}\right)}{\left(\nu^2 + \frac{n^2}{4}\right)^3}\ ,\quad \psi(x) = (\ln\Gamma(x))'.$$

# Eigenvalues of the modified kernel

Here the $\zeta$-functions are expressed in terms of polylogarithms

$$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}\,, \ \ \zeta(n) = Li_n(1)\,. \tag{2}$$

Note, that $\omega(\nu, n)$ has the important property

$$\omega(0,0) = 0\,. \tag{3}$$

It is in an agreement with the existence of the eigenfunction $\Phi = 1$ with a vanishing eigenvalue, which is a consequence of the bootstrap relation.

# Standard and Möbius invariant forms

The modified octet kernel can be written as follows:

$$K(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = K^B(\vec{q}_1, \vec{q}_1{}'; \vec{q}) \left(1 - \frac{\alpha_s N_c}{2\pi}\zeta(2)\right)$$

$$+\delta^{(2)}(\vec{q}_1 - \vec{q}_1{}')\frac{\vec{q}_1{}^2 \vec{q}_2{}^2}{\vec{q}^{\,2}}\frac{\alpha_s^2 N_c^2}{4\pi^2}3\zeta(3) + \frac{\alpha_s^2 N_c^2}{32\pi^3}R(\vec{q}_1, \vec{q}_1{}'; \vec{q}) \,,$$

$K^B$ is the leading order kernel, which can be written in the explicitly Möbius invariant form:

$$K^B(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = -\delta^{(2)}(\vec{q}_1 - \vec{q}_1{}')\frac{\vec{q}_1{}^2 \vec{q}_2{}^2}{\vec{q}^{\,2}}\frac{\alpha_s N_c}{4\pi^2}\int \frac{\vec{q}^{\,2} d^2 l}{(\vec{q}_1 - \vec{l})^2(\vec{q}_2 + \vec{l})^2}$$

$$\left(\frac{\vec{q}_1{}^2(\vec{q}_2 + \vec{l})^2 + \vec{q}_2{}^2(\vec{q}_1 - \vec{l})^2}{\vec{q}^{\,2}\vec{l}^{\,2}} - 1\right) + \frac{\alpha_s N_c}{4\pi^2}\left(\frac{\vec{q}_1{}^2 \vec{q}_2{}'^{\,2} + \vec{q}_1{}'^{\,2} \vec{q}_2{}^2}{\vec{q}^{\,2}\vec{k}^{\,2}} - 1\right) .$$

# Standard and Möbius invariant forms

$$R(\vec{q}_1,\vec{q}_1{}',\vec{q}) = \frac{1}{2}\left(\ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}}\right) + \ln\left(\frac{\vec{q}_1{}'^{\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{q}_2{}'^{2}}{\vec{q}^{\,2}}\right)\right.$$

$$\left. + \ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1{}'^{\,2}}\right)\right) - \frac{\vec{q}_1^{\,2}\vec{q}_2{}'^{\,2} + \vec{q}_2^{\,2}\vec{q}_1{}'^{\,2}}{\vec{q}^{\,2}\vec{k}^{\,2}}\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1{}'^{\,2}}\right) - \frac{\vec{q}_1^{\,2}\vec{q}_2{}'^{\,2} - \vec{q}_2^{\,2}\vec{q}_1{}'^{\,2}}{2\vec{q}^{\,2}\vec{k}^{\,2}}$$

$$\times \ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1{}'^{\,2}}\right)\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_1{}'^{\,2}}{\vec{k}^{\,4}}\right) + 4\frac{(\vec{k}\times\vec{q}_1)}{\vec{q}^{\,2}\vec{k}^{\,2}}\Big(\vec{k}^{\,2}(\vec{q}_1\times\vec{q}_2)$$

$$-\vec{q}_1^{\;2}(\vec{k}\times\vec{q}_2) - \vec{q}_2^{\;2}(\vec{k}\times\vec{q}_1)\Big) I_{\vec{q}_1,-\vec{k}} + (\vec{q}_1 \leftrightarrow -\vec{q}_2,\;\; \vec{q}_1{}' \leftrightarrow -\vec{q}_2{}') \;.$$

$$\vec{k} = \vec{q}_1 - \vec{q}_1{}' = \vec{q}_2{}' - \vec{q}_2,\;\; (\vec{a}\times\vec{b}) = a_x b_y - a_y b_x$$

$$I_{\vec{p},\vec{q}} = \int_0^1 \frac{dx}{(\vec{p}+x\vec{q})^2}\ln\left(\frac{\vec{p}^{\,2}}{x^2\vec{q}^{\,2}}\right) \;.$$

# Standard and Möbius invariant forms

The contribution $R(\vec{q}_1, \vec{q}_1{}', \vec{q})$ violates the Möbius invariance. In the paper

V.S. F., L.N. Lipatov, 2012

it was assumed that there is a conformal invariant representation of the kernel. Since its eigenvalues do not depend on the representation and on the total momentum transfer, they were found using the limit

$$|\vec{q}_1| \sim |\vec{q}_1{}'| \ll |\vec{q}| \approx |\vec{q}_2| \approx |\vec{q}_2{}'| \, .$$

In this limit

$$K(z) = K^B(z)\left(1 - \frac{\alpha_s N_c}{2\pi}\zeta(2)\right) + \delta^{(2)}(1-z)\frac{\alpha_s^2 N_c^2}{4\pi^2}3\zeta(3) + \frac{\alpha_s^2 N_c^2}{32\pi^3}R(z) \, ,$$

where $z = q_1/q_1'$,

$$K^B(z) = \frac{\alpha_s N_c}{8\pi^2}\left(\frac{z+z^*}{|1-z|^2} - \delta^{(2)}(1-z)\int \frac{d\vec{l}}{|l|^2}\frac{l+l^*}{|1-l|^2}\right) \, ,$$

# Standard and Möbius invariant forms

$$R(z) = \left(\frac{1}{2} - \frac{1+|z|^2}{|1-z|^2}\right)\ln^2|z|^2 - \frac{1-|z|^2}{2|1-z|^2}\ln|z|^2\ln\frac{|1-z|^4}{|z|^2|}$$

$$+\left(\frac{1}{1-z} - \frac{1}{1-z^*}\right)(z-z^*)\int_0^1 \frac{dx}{|x-z|^2}\ln\frac{|z|^2}{x^2}\,.$$

$p = p_x + ip_y$ and $p^* = p_x - ip_y$ for the two-dimensional vectors $\vec{p} = (p_x, ip_y)$.Vice versa, two complex numbers $z$ and $z^*$ are equivalent to the vector $\vec{z}$ with the components $(z+z^*)/2$ and $(z-z^*)/(2i)$.

# Standard and Möbius invariant forms

Due to the Möbius invariance, the kernel $K_c(\vec{q}_1, \vec{q}_1{}'; \vec{q})$ can be written as $K(z)$ with the argument $z = q_1 q_2'/(q_2 q_1')$. If we denote

$$K(\vec{q}_1, \vec{q}_1{}'; \vec{q}) - K_c(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = \frac{\alpha_s^2 N_c^2}{32\pi^3} \Delta(\vec{q}_1, \vec{q}_1{}'; \vec{q}),$$

then

$$\Delta(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = R(\vec{q}_1, \vec{q}_1{}'; \vec{q}) - R(z),$$

Since $R(\vec{q}_1, \vec{q}_1{}'; \vec{q})$ is not conformal invariant, $Delta(\vec{q}_1, \vec{q}_1{}'; \vec{q})$ cannot be written using the single variable $z$. Using relations between dilogarithms it can be written in the form

$$\Delta(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}) = \ln\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}} + \ln\frac{\vec{q}_1^{\,\prime\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,\prime\,2}}{\vec{q}^{\,2}} + \ln\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}_2^{\,\prime\,2}}$$

$$-2\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}+\vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\ln\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}_2^{\,\prime\,2}} + \frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}-\vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\left(\ln\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,\prime\,2}}{\vec{q}^{\,2}}\right.$$

$$\left.-\ln\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_1^{\,\prime\,2}}{\vec{q}^{\,2}}\right) + \frac{4}{\vec{q}^{\,2}\vec{k}^{\,2}}\left(\vec{k}^{\,2}[\vec{q}_1\times\vec{q}_2] - \vec{q}_1^{\,2}[\vec{k}\times\vec{q}_2] - \vec{q}_2^{\,2}[\vec{k}\times\vec{q}_1]\right)$$

$$\times\left([\vec{q}_1\times\vec{q}_2]I_{\vec{q}_1,\vec{q}_2} - [\vec{q}_1^{\,\prime}\times\vec{q}_2^{\,\prime}]I_{\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}}\right)\,.$$

Important properties of $\Delta$ are its symmetries with respect to the exchanges $\vec{q}_1 \leftrightarrow -\vec{q}_2$, $\vec{q}_1^{\,\prime} \leftrightarrow -\vec{q}_2^{\,\prime}$ and $\vec{q}_i \leftrightarrow -\vec{q}_i$, as well as the gauge invariance (vanishing at zero momentum of each reggeon), which are easily seen from this representation.

# Similarity transformation

If the kernels $\hat{\mathcal{K}}$ and $\hat{\mathcal{K}}_c$ coincide in the leading order and are related by a similarity transformation, there must exist an operator $\hat{\mathcal{O}}$ satisfying the commutation relation

$$[\hat{\mathcal{K}}^B, \hat{O}] = \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{8\pi}\hat{\Delta}\,.$$

One can find a formal expression for this operator allowing to construct the similarity transformation in perturbation theory. Indeed, it is enough to calculate the matrix element of the above commutation relation between the eigenfunctions of the Born kernel with the corresponding eigenvalues $\omega^B_{\nu n}$ in the form

$$\left(\omega^B_{\nu' n'} - \omega^B_{\nu n}\right)\langle \nu' n' |\hat{\mathcal{O}}|\nu n\rangle = \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{8\pi}\langle \nu' n' |\hat{\Delta}|\nu n\rangle .$$

It can be seen from this equation that the solution $\hat{\mathcal{O}}$ exists only if the operator $\hat{\Delta}$ has vanishing matrix elements between states with the same eigenvalues. In this case

# Similarity transformation

$$\hat{O} = \frac{\alpha_s^2}{32\pi^3} \sum_{n,n'} \int d\nu d\nu' \frac{|\nu' n'\rangle\langle\nu' n'|\hat{\Delta}|\nu n\rangle\langle\nu n|}{\omega^B_{\nu' n'} - \omega^B_{\nu n}}$$

$$\langle\vec{q}_1, q_2|\hat{\mathcal{O}}|\vec{q}_1{}', \vec{q}_2{}'\rangle = \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{8\pi} \sum_{n,n'} \int d\nu \int d\nu'$$

$$\frac{\langle\vec{q}_1, q_2|\nu' n'\rangle\langle\nu' n'|\hat{\Delta}|\nu n\rangle\langle\nu n|\vec{q}_1{}', \vec{q}_2{}'\rangle}{\omega^B_{\nu' n'} - \omega^B_{\nu n}} \, .$$

Since the kernel $\hat{\Delta}$ is known in the momentum space, we can transform it into the $(n, \nu)$ representation,

$$\langle\nu n|\hat{\Delta}|\nu' n'\rangle = \int \frac{\vec{q}\,^2 d\vec{q}_1}{\vec{q}_1^{\,2}(\vec{q} - \vec{q}_1)^2} \int \frac{\vec{q}\,^2 d\vec{q}_1{}'}{\vec{q}_1{}'^{\,2}(\vec{q} - \vec{q}_1{}')^2} \langle\nu' n'|\vec{q}_1{}', \vec{q}_2{}'\rangle$$

$$\times\langle\Delta(\vec{q}_1, \vec{q}_1{}'; \vec{q})\rangle\langle\vec{q}_1, \vec{q}_2|\nu n\rangle$$

using the known eigenfunctions in the momentum space, which allows to find the matrix element $\langle\vec{q}_1, \vec{q}_2|\hat{\mathcal{O}}|\vec{q}_1{}', \vec{q}_2{}'\rangle$.

# Similarity transformation

But the final expression for $\hat{\mathcal{O}}$ obtained by this method is rather complicated. It turned out more simple to guess the form of the operator $\hat{O}$. We supposed that the conformal invariant kernel can be obtained using the substraction procedure different from the standard one. If it is so, then the operator $\hat{\mathcal{O}}$ in the momentum representation must be proportional $K^B(\vec{q}_1, \vec{q}_1{}'; \vec{q})$. Then, from the symmetry arguments it follows that the most attractive candidate for $\hat{\mathcal{O}}$ is

$$\hat{\mathcal{O}}_t = C\left[\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right), \hat{\mathcal{K}}_r^B\right] ,$$

where $C$ is some coefficient. In the momentum space

$$O(\vec{q}_1, \vec{q}_1{}'; \vec{q}) \;=\; \frac{\alpha_s N_c}{16\pi^2}\left(\frac{\vec{q}_1^{\,2}\vec{q}_2{}'^{\,2} + \vec{q}_1{}'^{\,2}\vec{q}_2^{\,2}}{\vec{k}^{\,2}} - \vec{q}^{\,2}\right)\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{\vec{q}_1{}'^{\,2}\vec{q}_2{}'^{\,2}}\right)$$

Remind that in the NLO there is an ambiguity, analogous to the well known ambiguity of the NLO anomalous dimensions, because it is possible to redistribute radiative corrections between the kernel and the impact factors. It permits to make transformations

$$\hat{\mathcal{K}} \to \hat{\mathcal{K}} - \alpha_s[\hat{\mathcal{K}}^{(B)}, \hat{U}]$$

conserving the LO kernel $\hat{\mathcal{K}}^{(B)}$ (which is fixed in our case by the requirement of conformal invariance) and changing the NLO part of the kernel.

# Similarity transformation

Therefore the "standard" and conformal invariant kernels has to be connected by such transformation. In principle, one can write a formal expression for the operator $\hat{U}$, since the difference between these kernels is known. Indeed, let us denote the diffetence as $\hat{\Delta}$ and the Born kernel $\hat{\mathcal{K}}^B$ eigenstates $|\mu\rangle$, and corresponding eigenvalues $\omega_\mu^B$. Then, if $\hat{\Delta} = \alpha_s \left[\hat{\mathcal{K}}^B, \hat{U}\right]$,

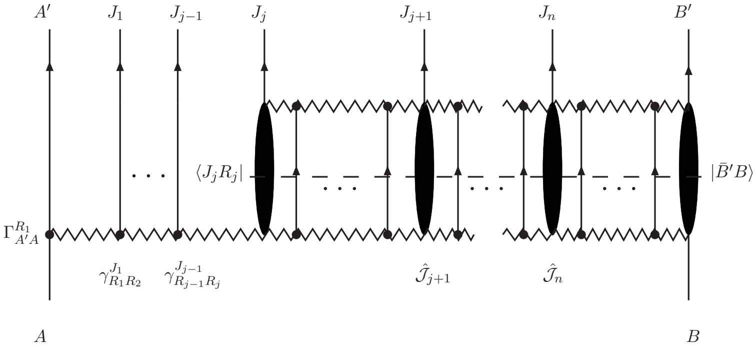
$$\left(\omega_{\mu'}^B - \omega_\mu^B\right) \langle\mu'|\alpha_s \hat{U}|\mu\rangle = \langle\mu'|\hat{\Delta}|\mu\rangle.$$

It is seen from here that the operator $\hat{U}$ exists only if the operator $\hat{\Delta}$ has zero matrix elements between states of equal energies. If so, supposing that the states $|\mu\rangle$ form a complete set,

$$\langle\mu'|\alpha_s \hat{U}|\mu\rangle = \sum_{\mu,\mu'} \frac{|\mu'\rangle\langle\mu'|\hat{\Delta}|\mu\rangle\langle\mu|}{\omega_{\mu'}^B - \omega_{\mu'}^B}.$$

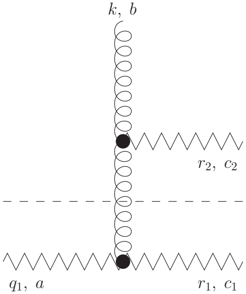
# Impact factor in the bootstrap scheme

The impact factor for Reggeon-gluon transition appears as one of components of the last bootstrap condition in partial discontinuites of MRK amplitudes



# Impact factor in the bootstrap scheme

In the Born approximation only gluon can be in the intermediate state




$$\langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle^{(B)} = 2g^2\delta(\vec{q}_1 - \vec{k} - \vec{r}_1 - \vec{r}_2)\left(T^aT^b\right)_{c_1c_2}\ \vec{e}^{\,*}\vec{C}_1,$$

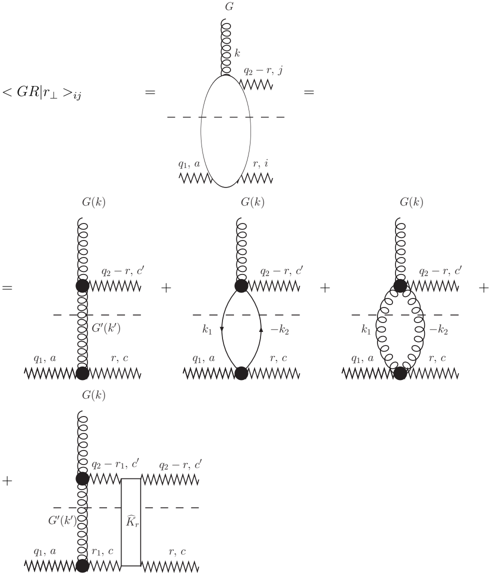
$\vec{e}^{\,*}$ is the conjugated transverse part of the polarization vector $e(k)$ in the gauge $e(k)p_2 = 0$ with the lightcone vector $p_2$ close to the vector $p_B$,

$$\vec{C}_1 \ = \ \vec{q}_1 - (\vec{q}_1 - \vec{r}_1)\frac{\vec{q}_1^{\,2}}{(\vec{q}_1 - \vec{r}_1)^2}.$$

# Impact factor in the bootstrap scheme

## In the NLO

$$< GR|r_\perp >_{ij} =$$



# Impact factor in the bootstrap scheme

In N=4 SYM the NLO impact factor contains gluon, fermion and scalar contributions. These contributions were found
M.G. Kozlov, A.V. Reznichenko, V.S. F. 2012,
M.G. Kozlov, A.V. Reznichenko, V.S. F. 2013,
for Yang-Mills theories with any number of fermions and scalars in arbitrary representations of the gauge group.

In general, the impact factors contain two parts with different colour structure. In the planar limit only parts with Born colour structure remain, so that the impact factor can be written as

$$\langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle = g^2\delta(\vec{q}_1-\vec{k}-\vec{r}_1-\vec{r}_2)\left(T^aT^b\right)_{c_1c_2}\ \vec{e}^{\,*}\left[2\vec{C}_1+\bar{g}^2\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1}\right].$$

The results of
M.G. Kozlov, A.V. Reznichenko, V.S. F. 2012,
M.G. Kozlov, A.V. Reznichenko, V.S. F. 2013,
are in dimensional regularization, which differs from the dimensional reduction used in supersymmetric theories.

# Impact factor in the bootstrap scheme

To take into account this difference one has to take the number of the scalar fields $n_S$ equal to $6-2\epsilon$ ($\epsilon=(D-4)/2$, $D$ is the space-time dimension).

With account of this, in the planar N=4 SYM

$$\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1*} = \vec{C}_1\left(\ln\left(\frac{(\vec{q}_1-\vec{r}_1)^2}{\vec{k}^{\,2}}\right)\ln\left(\frac{\vec{r}_2^{\,2}}{\vec{k}^{\,2}}\right)+\ln\left(\frac{(\vec{q}_1-\vec{r}_1)^2\vec{q}_1^{\,2}}{\vec{k}^{\,4}}\right)\ln\left(\frac{\vec{r}_1^{\,2}}{\vec{q}_1^{\,2}}\right)\right.$$

$$\left.-4\frac{(\vec{k}^{\,2})^{\epsilon}}{\epsilon^2}+6\zeta(2)\right)+\vec{C}_2\left(\ln\left(\frac{\vec{k}^{\,2}}{\vec{r}_2^{\,2}}\right)\ln\left(\frac{(\vec{q}_1-\vec{r}_1)^2}{\vec{r}_2^{\,2}}\right)\right.$$

$$\left.+\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,2}}{\vec{q}_2^{\,2}}\right)\right)-2\left[\vec{C}_1\times\left(\left[\vec{k}\times\vec{r}_2\right]I_{\vec{k},\vec{r}_2}-\left[\vec{q}_1\times\vec{r}_1\right]I_{\vec{q}_1,-\vec{r}_1}\right)\right]$$

$$\left.+2\left[\vec{C}_2\times\left(\left[\vec{k}\times\vec{r}_2\right]I_{\vec{k},\vec{r}_2}+\left[q_1\times\vec{k}\right]I_{\vec{q}_1,-\vec{k}}\right)\right]\right\}.$$

# Impact factor in the bootstrap scheme

Here $\bar{g}^2 = g^2\Gamma(1-\epsilon)/(4\pi)^{2+\epsilon}$,

$$\vec{C}_2 \;=\; \vec{q}_1 - \vec{k}\frac{\vec{q}_1^{\,2}}{\vec{k}^{\,2}},$$

$\Gamma(x)$ is the Euler gamma-function, $\zeta(n)$ is the Riemann zeta-function ($\zeta(2)=\pi^2/6$), $\left[\vec{a}\times c\left[\vec{b}\times\vec{c}\right]\right]$ is a double vector product, and

$$I_{\vec{p},\vec{q}} = \int_0^1 \frac{dx}{(\vec{p}+x\vec{q})^2}\ln\left(\frac{\vec{p}^{\,2}}{x^2\vec{q}^{\,2}}\right)\;,\quad I_{\vec{p},\vec{q}} = I_{-\vec{p},-\vec{q}} = I_{\vec{q},\vec{p}} = I_{\vec{p},-\vec{p}-\vec{q}}\,.$$

The NLO correction $\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1*}$ is obtained after huge cancellations between gluon, fermion and scalar contributions. In particular, solely due to these cancellations only two vector structures ($\vec{C}_1$ and $\vec{C}_2$) remain; each of the contributions separately contains three independent vector structures.

# Impact factor in the bootstrap scheme

As it is known, NLO corrections are scheme dependent. The scheme used in the derivation of $\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1*}$ was adjusted simplifying the verification of the bootstrap conditions (we call it bootstrap scheme). It is different from the standard scheme defined in
V.S. F., R. Fiore, M.G. Kozlov, A.V. Reznichenko, 2006.
The impact factors in these schemes are connected by the transformation

$$\langle GR_1|_* = \langle GR_1|_s - \langle GR_1|^{(B)}\widehat{\mathcal{U}}_k,$$

where subscript $s$ means the standard scheme and the the operator $\widehat{\mathcal{U}}_k$ is defined by the matrix elements

$$\langle \mathcal{G}_1'\mathcal{G}_2'|\widehat{\mathcal{U}}_k|\mathcal{G}_1\mathcal{G}_2\rangle = \frac{1}{2}\ln\left(\frac{\vec{k}^{\,2}}{(\vec{r}_1 - \vec{r}_1{}')^2}\right)\langle \mathcal{G}_1'\mathcal{G}_2'|\widehat{\mathcal{K}}_r^B|\mathcal{G}_1\mathcal{G}_2\rangle.$$

Here $\widehat{\mathcal{K}}_r^B$ is the part of the LO BFKL kernel related with the real gluon production:

# Transformation to the standard scheme

$$\langle \mathcal{G}_1' \mathcal{G}_2' | \widehat{\mathcal{K}}_r^B | \mathcal{G}_1 \mathcal{G}_2 \rangle = \delta(\vec{r}_1{}' + \vec{r}_2{}' - \vec{r}_1 - \vec{r}_2) \frac{g^2}{(2\pi)^{D-1}} T^i_{c_1 c_1'} T^i_{c_2' c_2}$$

$$\times \left( \frac{\vec{r}_1{}^2 \vec{r}_2{}'^{\,2} + \vec{r}_2{}^2 \vec{r}_1{}'^{\,2}}{\vec{l}^{\,2}} - \vec{q}_2{}^2 \right),$$

where $\vec{l} = \vec{r}_1 - \vec{r}_1{}' = \vec{r}_2{}' - \vec{r}_2, \quad \vec{q}_2 = \vec{r}_1 + \vec{r}_2 = \vec{r}_1{}' + \vec{r}_2{}'$.
At large $N_c$ we can write

$$\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1 s} \;=\; \vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1 *} + \vec{\mathcal{I}}_1,$$

$$\vec{\mathcal{I}}_1 = \int \frac{d\vec{l}}{\Gamma(1-\epsilon)\pi^{1+\epsilon}} \vec{C}_1{}' \frac{1}{\vec{r}_1{}'^{\,2} \vec{r}_2{}'^{\,2}} \left( \frac{\vec{r}_1{}^2 \vec{r}_2{}'^{\,2} + \vec{r}_2{}^2 \vec{r}_1{}'^{\,2}}{\vec{l}^{\,2}} - \vec{q}_2{}^2 \right) \ln\left( \frac{\vec{k}^{\,2}}{\vec{l}^{\,2}} \right),$$

where

$$\vec{C}_1{}' \;=\; \vec{q}_1 - \vec{q}_1{}^2 \frac{(\vec{q}_1 - \vec{r}_1{}')}{(\vec{q}_1 - \vec{r}_1{}')^2}.$$

The integration gives

$$\vec{\mathcal{I}}_1 = \frac{1}{2}\vec{C}_1\left[\ln\left(\frac{\vec{r}_2^{\,2}}{\vec{k}^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,4}}{(\vec{q}_1-\vec{r}_1)^2\vec{r}_2^{\,2}}\right)+\ln\left(\frac{\vec{r}_1^{\,2}}{\vec{k}^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,2}\vec{q}_1^{\,2}}{(\vec{q}_1-\vec{r}_1)^2\vec{r}_1^{\,2}}\right)\right.$$

$$\left.-\ln\left(\frac{\vec{k}^{\,2}}{(\vec{q}_1-\vec{r}_1)^2}\right)\ln\left(\frac{\vec{k}^{\,2}\vec{q}_1^{\,2}}{(\vec{q}_1-\vec{r}_1)^4}\right)+4\frac{(\vec{k}^{\,2})^{\epsilon}}{\epsilon^2}-4\zeta(2)\right]$$

$$-\frac{1}{2}\vec{C}_2\left[\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{r}_2^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,4}}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}\right)+\ln\left(\frac{(\vec{q}_1-\vec{r}_1)^2}{\vec{q}_1^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,4}}{\vec{r}_1^{\,2}(\vec{q}_1-\vec{r}_1)^2}\right)\right]$$

$$+\left[\vec{C}_1\times\left(\left[\vec{k}\times\vec{r}_2\right]I_{\vec{k},\vec{r}_2}-\left[\vec{q}_1\times\vec{r}_1\right]I_{\vec{q}_1,-\vec{r}_1}\right)\right]$$

$$+\left[\vec{C}_2\times\left(\left[\vec{r}_1\times\vec{r}_2\right]I_{\vec{r}_1,\vec{r}_2}+\left[\vec{q}_1\times\vec{r}_1\right]I_{\vec{q}_1,-\vec{r}_1}\right)\right],$$

# Transformation to the standard scheme

and the one-loop correction to the impact factor in the standard scheme

$$\vec{\Phi}^{\mathcal{G}_1\mathcal{G}_2}_{GR_1(s)} = \frac{1}{2}\vec{C}_1\left[\ln\left(\frac{\vec{q}_1^{\,2}}{\vec{r}_1^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,2}\vec{r}_1^{\,2}}{\vec{q}_1^{\,4}}\right)+\ln\left(\frac{\vec{r}_2^{\,2}}{\vec{k}^{\,2}}\right)\ln\left(\frac{(\vec{q}_1-\vec{r}_1)^2}{\vec{r}_2^{\,2}}\right)\right.$$

$$\left.+\ln\left(\frac{(\vec{q}_1-\vec{r}_1)^2}{\vec{k}^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,4}\vec{r}_1^{\,2}}{(\vec{q}_1-\vec{r}_1)^4\vec{q}_1^{\,2}}\right)-4\frac{(\vec{k}^{\,2})^{\epsilon}}{\epsilon^2}+8\zeta(2)\right]$$

$$+\left[\vec{C}_1\times\left(\left[\vec{q}_1\times\vec{r}_1\right]I_{\vec{q}_1,-\vec{r}_1}-\left[\vec{k}\times r_2\right]I_{\vec{k},\vec{r}_2}\right)\right]$$

$$+\frac{1}{2}\vec{C}_2\left[\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}}\right)\ln\left(\frac{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}{\vec{q}_2^{\,4}}\right)+\ln\left(\frac{\vec{r}_2^{\,2}}{(\vec{q}_1-\vec{r}_1)^2}\right)\ln\left(\frac{\vec{r}_2^{\,2}\vec{q}_1^{\,2}}{\vec{r}_1^{\,2}(\vec{q}_1-\vec{r}_1)^2}\right)\right]$$

$$+\left[\vec{C}_2\times\left(\left[\vec{r}_1\times\vec{r}_2\right]I_{\vec{r}_1,\vec{r}_2}+\left[\vec{q}_1,\vec{r}_1\right]I_{\vec{q}_1,-\vec{r}_1}+2\left[\vec{k},\vec{r}_2\right]I_{\vec{k},\vec{r}_2}+2\left[\vec{q}_1,\vec{k}\right]I_{\vec{q}_1,-\vec{k}}\right)\right].$$

# The impact factor in the Möbius scheme

This correction corresponds to the standard kernel and the energy scale $|\vec{k}_1||\vec{k}_2|$, where $\vec{k}_{1,2}$ are the transverse momenta of produced gluons in the two impact factors connected by the Green function of the two interacting Reggeons (BFKL ladder). Our goal is the impact factor in the Möbius scheme, that means the impact factor for Reggeon-gluon transition which can be used for the calculation of the remainder function with conformal invariant kernel and energy evolution parameter. Let us remind here that the kernel used for the calculation of the remainder function (which is called modified kernel $\hat{\mathcal{K}}_m$) is the BFKL kernel in $N = 4$ SYM for the adjoint representation of the gauge group with subtracted gluon trajectory depending on the total momentum transfer, $\hat{\mathcal{K}}_m = \hat{\mathcal{K}} - \omega(t)$ (the subtraction is made to avoid double counting of terms included in the BDS ansatz).

To obtain the impact factor in the Möbius scheme we have to perform two transformations, to reconcile the impact factor with

# The impact factor in the Möbius scheme

As it was shown, the conformal invariant $\hat{\mathcal{K}}_c$ and standard $\hat{\mathcal{K}}_m$ forms of the modified kernel are connected by the similarity transformation

$$\hat{\mathcal{K}}_c = \hat{\mathcal{K}}_m + \frac{1}{4}\left[\hat{\mathcal{K}}^B, \left[\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right), \hat{\mathcal{K}}^B\right]\right] ,$$

where $\hat{\mathcal{K}}^B$ is the usual LO kernel and $\hat{\vec{q}}_{1,2}$ are the operators of Reggeon momenta. Note that in the commutator there is no difference between usual and modifided kernels, so that $\hat{\mathcal{K}}^B$ is taken instead of $\hat{\mathcal{K}}_m^B$. The corresponding transformation for the impact factor is

$$\langle GR_1|_t = \langle GR_1|_s - \frac{1}{4}\langle GR_1|^{(B)}\left[\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right), \hat{\mathcal{K}}^{(B)}\right] ,$$

where the subscript $t$ means transformed to fit the conformal kernel.

# The impact factor in the Möbius scheme

For the NLO correction we obtain

$$\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1 t} = \vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1 s} + \vec{\mathcal{I}}_2,$$

$$\vec{\mathcal{I}}_2 = \int \frac{d\vec{l}}{\Gamma(1-\epsilon)\pi^{1+\epsilon}} \vec{C}_1{}' \frac{1}{\vec{r}_1{}'^{\,2}\vec{r}_2{}'^{\,2}} \left( \frac{\vec{r}_1^{\,2}\vec{r}_2{}'^{\,2} + \vec{r}_2^{\,2}\vec{r}_1{}'^{\,2}}{\vec{l}^{\,2}} - \vec{q}_2^{\,2} \right) \ln\left( \frac{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}{\vec{r}_1{}'^{\,2}\vec{r}_2{}'^{\,2}} \right).$$

This integral is infrared finite and can be calculated in two-dimensional space, with the help of the decomposition used before, the decomposition

$$\frac{1}{\vec{r}_1{}'^{\,2}\vec{r}_2{}'^{\,2}} \left( \frac{\vec{r}_1^{\,2}\vec{r}_2{}'^{\,2} + \vec{r}_2^{\,2}\vec{r}_1{}'^{\,2}}{\vec{l}^{\,2}} - \vec{q}_2^{\,2} \right) = -\left( \frac{1}{(r_1 - l)^+} + \frac{1}{l^+} \right)$$

$$\times \left( \frac{1}{(r_2 + l)^-} - \frac{1}{l^-} \right) - \left( \frac{1}{(r_1 - l)^-} + \frac{1}{l^-} \right) \left( \frac{1}{(r_2 + l)^+} - \frac{1}{l^+} \right).$$

The transformed impact factor takes the form:

$$\vec{\Phi}^{\mathcal{G}_1\mathcal{G}_2}_{GR_1t} = \vec{C}_1\left[\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}}\right)\ln\left(\frac{\vec{q}_2^{\,4}(\vec{q}_1-\vec{r}_1)^4}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}\vec{k}^{\,2}\vec{r}_1^{\,2}}\right)-\ln\left(\frac{\vec{q}_2^{\,2}(\vec{q}_1-\vec{r}_1)^2}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}}\right)\right.$$

$$\left.\times\ln\left(\frac{\vec{q}_2^{\,2}(\vec{q}_1-\vec{r}_1)^2}{\vec{k}^{\,2}\vec{r}_1^{\,2}}\right)\frac{3}{4}\ln^2\left(\frac{\vec{k}^{\,2}\vec{r}_1^{\,2}}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}}\right)-\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right)-2\frac{(\vec{k}^{\,2})^{\epsilon}}{\epsilon^2}+4\zeta(2)\right]$$

$$+\frac{1}{4}\vec{C}_2\ln\left(\frac{\vec{q}_2^{\,2}(\vec{q}_1-\vec{r}_1)^2}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}}\right)\ln\left(\frac{(\vec{q}_1-\vec{r}_1)^4\vec{q}_1^{\,2}\vec{k}^{\,2}\vec{r}_1^{\,2}}{\vec{r}_2^{\,6}\vec{q}_2^{\,4}}\right)$$

$$+\frac{3}{2}\left[\vec{C}_2\times\left(\left[\vec{r}_1\times\vec{r}_2\right]I_{\vec{r}_1,\vec{r}_2}+\left[\vec{q}_1\times\vec{r}_1\right]I_{\vec{q}_1,-\vec{r}_1}+\left[\vec{k}\times\vec{r}_2\right]I_{\vec{k},\vec{r}_2}+\left[\vec{q}_1\times\vec{k}\right]I_{\vec{q}_1,-\vec{k}}\right)\right].$$

# The impact factor in the Möbius scheme

The Möbius invariant kernel was used for the calculation of the NLO remainder function with the Möbius invariant convolution of the NLO BFKL impact factors (which was called for brevity simply impact factor) obtained from direct two-loop calculations and with the energy scale $s_0$ chosen in such a way that the ratio (energy evolution parameter) $s/s_0 = s\vec{q}_2^{\,2}/\sqrt{\vec{q}_1^{\,2}\vec{q}_3^{\,2}\vec{k}_1^{\,2}\vec{k}_2^{\,2}}$ is Möbius invariant. This energy scale differs from the energy scale used in the standard definition of the impact factor which is equal $|\vec{k}_1||\vec{k}_2|$. To adjust the impact factor to the energy scale $\sqrt{\vec{q}_1^{\,2}\vec{q}_3^{\,2}\vec{k}_1^{\,2}\vec{k}_2^{\,2}}/\vec{q}_2^{\,2}$, we need to perform an additional transformation:

$$\langle GR_1|_t \to \langle GR_1|_c = \langle GR_1|_t - \frac{1}{2}\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}}\right)\langle GR_1|^{(B)}\hat{\mathcal{K}}_m^{(B)}|\mathcal{G}_1\mathcal{G}_2\rangle\,,$$

# The impact factor in the Möbius scheme

where the subscript $c$ means transformed to fit the conformal energy scale and $\hat{\mathcal{K}}_m^{(B)}$ is the modified LO kernel. Let

$$\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1 c} = \vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1 t} + \vec{\mathcal{I}}_3,$$

then the integral for $\vec{\mathcal{I}}_3$ can be written as

$$\vec{\mathcal{I}}_3 = -\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}}\right)\int\frac{d\vec{l}}{\pi}\left(\vec{C}_1' - \vec{C}_1\right)\frac{1}{\vec{r}_1^{\,\prime\,2}\vec{r}_2^{\,\prime\,2}}\left(\frac{\vec{r}_1^{\,2}\vec{r}_2^{\,\prime\,2}+\vec{r}_2^{\,2}\vec{r}_1^{\,\prime\,2}}{\vec{l}^{\,2}} - \vec{q}_2^{\,2}\right),$$

where instead of $\vec{C}_1'$ the difference $\left(\vec{C}_1' - \vec{C}_1\right)$ is taken and instead of the full modified kernel only its part related to real gluon production is kept. Moreover, the integral is written in two-dimensional transverse space.

# The impact factor in the Möbius scheme

This result gives

$$\vec{\Phi}^{\mathcal{G}_1\mathcal{G}_2}_{GR_1c} = \vec{C}_1\Bigg[-\ln\left(\frac{\vec{q}_2^{\,2}(\vec{q}_1-\vec{r}_1)^2}{\vec{r}_1^{\,2}\vec{k}^{\,2}}\right)\ln\left(\frac{\vec{q}_2^{\,2}(\vec{q}_1-\vec{r}_1)^2}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}}\right)-\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right)$$

$$-\frac{3}{4}\ln^2\left(\frac{\vec{k}^{\,2}\vec{r}_1^{\,2}}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}}\right)-2\frac{(\vec{k}^{\,2})^\epsilon}{\epsilon^2}+4\zeta(2)\Bigg]$$

$$+\frac{1}{4}\vec{C}_2\ln\left(\frac{\vec{q}_2^{\,2}(\vec{q}_1-\vec{r}_1)^2}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}}\right)\ln\left(\frac{\vec{q}_2^{\,4}(\vec{q}_1-\vec{r}_1)^4\vec{r}_1^{\,2}\vec{k}^{\,2}}{\vec{r}_2^{\,6}\vec{q}_1^{\,6}}\right)$$

$$+\frac{3}{2}\Bigg[\vec{C}_2\times\Bigg(\Big[\vec{r}_1\times\vec{r}_2\Big]I_{\vec{r}_1,\vec{r}_2}+\Big[\vec{q}_1\times\vec{r}_1\Big]I_{\vec{q}_1,-\vec{r}_1}+\Big[\vec{k}\times\vec{r}_2\Big]I_{\vec{k},\vec{r}_2}+\Big[\vec{q}_1\times\vec{k}\Big]I_{\vec{q}_1,-\vec{k}}\Bigg)\Bigg].$$

This expression gives us the NLO correction to the impact factor for Reggeon-gluon transition in the scheme with conformal kernel and energy evolution parameter, which were used for the calculation of the remainder function.

# The impact factor in the Möbius scheme

However, it is the impact factor for the full amplitude, not for the remainder function. To obtain the impact factor for the remainder function we have to take the impact factor with the correction $\Phi^{\mathcal{G}_1\mathcal{G}_2}_{GR_1c}$ and with the polarisation vector $\vec{e}^{\,*}$ of definite helicity, and to extract from it the piece included in the BDS ansatz.

Let us consider, for definiteness, the production of a gluon with positive helicity, $\vec{e}^{\,*} = (\vec{e}_x - i\vec{e}_y)/\sqrt{2}$. Then,

$$\vec{e}^{\,*}\vec{C}_1 = -\frac{q_1^- r_1^+}{\sqrt{2}(q_1 - r_1)^+}, \quad \vec{e}^{\,*}\vec{C}_2 = -\frac{q_1^- q_2^+}{\sqrt{2}k^+}, \quad \frac{\vec{e}^{\,*}\vec{C}_2}{\vec{e}^{\,*}\vec{C}_1} = 1 - z,$$

where the ratio $z = -q_1^+ r_2^+/(k^+ r_1^+)$ is conformal invariant, i.e. invariant with respect to Möbius transformations of complex variables $p_i$ such that
$r_1^+ = p_1 - p_2, r_2^+ = p_2 - p_3, -q_1^+ = p_3 - p_4, k_1^+ = p_4 - p_1)$.

# The impact factor in the Möbius scheme

As the result, after some algebra we obtain

$$\langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle = \langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle^{(B)}\left\{1+\frac{\bar{g}^2}{8}\left[(1-z)\right.\right.$$

$$\times\left(\ln\left(\frac{|1-z|^2}{|z|^2}\right)\ln\left(\frac{|1-z|^4}{|z|^6}\right)-3Li_2(z)+3Li_2(z^*)-\frac{3}{2}\ln|z|^2\ln\frac{1-z}{1-z^*}\right)$$

$$-4\ln|1-z|^2\ln\frac{|1-z|^2}{|z|^2}-3\ln^2|z|^2$$

$$\left.\left.-4\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right)-8\frac{(\vec{k}^{\,2})^\epsilon}{\epsilon^2}+16\zeta(2)\right]\right\}.$$

Finally, in order to move to the impact factor for calculation of the remainder function, one has to discard the terms $\bar{g}^2\left(-(1/2)\ln^2(\vec{q}_1^{\,2}/\vec{q}_2^{\,2})-(\vec{k}^{\,2})^\epsilon/\epsilon^2+2\zeta(2)\right)$, since they are already taken into account in the BDS ansatz.

The remainder is evidently conformal invariant.

For calculation of $\langle G_1 R_1 | e^{\hat{K}\ln\left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)} | G_2 R_3 \rangle$ it is convenient to transfer from the BFKL kernel $\hat{K}$ to the modified BFKL kernel $\hat{K}_m$ introduced in J. Bartels, L. N. Lipatov and A. Sabio Vera, 2009

,

$$\hat{K}_m = \hat{K} - \omega(t).$$

An evident advantage of this kernel is non-singular infrared behavior.

Not so evident, but even more important is conformal invariance in the momentum space. In the LO this invariance is almost obvious J. Bartels, L. N. Lipatov and A. Sabio Vera, 2009

. Existence of the conformal invariant representation of the NLO kernel was proved recently

V. S. F. , R. Fiore, L. N. Lipatov and A. Papa, 2013.

# The NLO discontinuity of the $A_{2\to 4}$ amplitude

The impact factors for reggeon-gluon transitions in this scheme with conformal invariant energy evolution parameter $\frac{s_2 \vec{q}_2^{\,2}}{|\vec{q}_1||\vec{q}_3||\vec{k}_1||\vec{k}_2|}$ was also recently found
V. S. F. , R. Fiore, 2014.
They can be written with the NLO accuracy as

$$\langle G_1 R_1 | \mathbf{G}_1 \mathbf{G}_2 \rangle = -\sqrt{2} g^2 \delta(\vec{q}_1 - \vec{k}_1 - \vec{r}_1 - \vec{r}_2) \frac{q_1^- r_1^+}{(q_1 - r_1)^+} [1 + \bar{g}^2 I(z_1)]$$
$$\times \left[ 1 + \bar{g}^2 \left( -\frac{1}{2} \ln^2 \left( \frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}} \right) - \frac{(\vec{k}^{\,2})^\epsilon}{\epsilon^2} + 2\zeta(2) \right) \right],$$

$$\langle \mathbf{G}_1' \mathbf{G}_2' | G_2 R_3 \rangle = \sqrt{2} g^2 \delta(\vec{r}_1{}' + \vec{r}_2{}' - \vec{q}_3 - \vec{k}_2) \frac{q_3^+ r_1'^-}{(r_1' - q_3)^-} [1 + \bar{g}^2 I^*(z_2)]$$
$$\times \left[ 1 + \bar{g}^2 \left( -\frac{1}{2} \ln^2 \left( \frac{\vec{q}_3^{\,2}}{\vec{q}_2^{\,2}} \right) - \frac{(\vec{k}^{\,2})_2^\epsilon}{\epsilon^2} + 2\zeta(2) \right) \right],$$

where $z_1 = -q_1^+ r_2^+ / (k_1^+ r_1^+)$, $z_2 = q_3^+ r_2'^+ / (k_2^+ r_1'^+)$

$$I(z)=(1-z)\left(\ln\left(\frac{|1-z|^2}{|z|^2}\right)\ln\left(\frac{|1-z|^4}{|z|^6}\right)-6Li_2(z)+6Li_2(z^*)\right.$$
$$\left.-3\ln|z|^2\ln\frac{1-z}{1-z^*}\right)-4\ln|1-z|^2\ln\frac{|1-z|^2}{|z|^2}-3\ln^2|z|^2\,.$$

In terms of eigenstates $|\nu,n\rangle$ and eigenvalues $\omega(\nu,n)$ of $\hat{K}_m$

$$\langle G_1R_1|e^{\hat{K}\ln\left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)}|G_2R_3\rangle=\left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)^{\omega(t_2)}$$
$$\times\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\nu\,\langle G_1R_1|\nu,n\rangle e^{\omega(\nu,n)\ln\left(\frac{s_2\vec{q}_2^{\,2}}{|\vec{q}_1||\vec{q}_3||\vec{k}_1||\vec{k}_2|}\right)}\langle\nu,n|G_2R_3\rangle\,.$$

The eigenfunctions are well known. The eigenvalues $\omega(\nu,n)$ with the NLO accuracy were found two years ago V. S. F. , L. N. Lipatov, 2012.

# The NLO discontinuity of the $A_{2\to 4}$ amplitude

In an explicit form

$$\langle G_1 R_1 | e^{\hat{K}\ln\left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)} | G_2 R_3 \rangle = \delta(\vec{q}_1 - \vec{k}_1 - \vec{k}_2 - \vec{q}_3) g^2 \gamma_{R_1 R_2}^{G_1(B)} \gamma_{R_2 R_3}^{G_2(B)}$$

$$\times \left[ 1 + \bar{g}^2 \left( -\frac{1}{2}\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right) - \frac{(\vec{k}_1^{\,2})^\epsilon}{\epsilon^2} - \frac{1}{2}\ln^2\left(\frac{\vec{q}_3^{\,2}}{\vec{q}_2^{\,2}}\right) - \frac{(\vec{k}_2^{\,2})^\epsilon}{\epsilon^2} + 4\zeta(2) \right) \right]$$

$$\times \frac{1}{2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu\, e^{\omega(\nu,n)\ln\left(\frac{s_2 \vec{q}_2^{\,2}}{|\vec{q}_1||\vec{q}_3||\vec{k}_1||\vec{k}_2|}\right)} w^{\frac{n}{2}+i\nu} (w^*)^{-\frac{n}{2}+i\nu}$$

$$\int \frac{dz_1}{\pi |z_1|^2} \frac{1}{1-z_1} \left(1 + \bar{g}^2 I(z_1)\right) z_1^{\frac{n}{2}+i\nu} (z_1^*)^{-\frac{n}{2}+i\nu}$$

$$\int \frac{dz_2}{\pi |z_2|^2} \frac{1}{1-z_2^*} \left(1 + \bar{g}^2 I^*(z_2)\right) (z_2^*)^{\frac{n}{2}-i\nu} z_2^{-\frac{n}{2}-i\nu}$$

where $w = k_2^+ q_1^+ / (k_1^+ q_3^+)$.

# Summary

- The impact factors for Reggeon-gluon transition are an integral part of the discontinuities of multi-Regge amplitudes.
- Formal expressions for $s_i$-channel discontinuities of MRK amplitudes in the NLA are known since 2006.
- Fulfilment of all bootstrap conditions is proved.
- The LLA discontinuities are in an evident contradiction with the BDS anzatz for $2 \to 2 + n$ amplitudes at $n \geq 2$ even in the LLA.
- The NLO impact factors are known now in Yang-Mills theories with any number of fermions and scalars in arbitrary representations of the gauge group.
- The discontinuity in invariant mass of two produced gluons is calculated in the NLA in planar $N = 4$ SYM.
- It's compatibility with the BDS ansatz corrected by the remainder factor is under consideration.