From DGLAP to Non Linear Evolution

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Outline

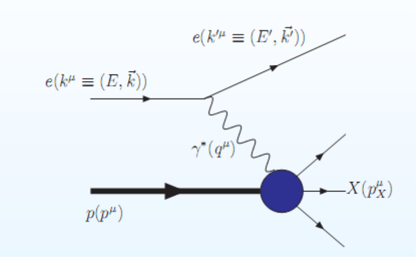
- Introduction
- Linear Evolution Equation
  - DGLAP
  - BFKL
- Non-Linear Evolution Equation
  - GLR
  - AGL
  - BK
  - JIMWLK
- Conclusion

Summary

- Deep Inelastic Scatterring
- Parton Model
- Quantum Chromodynamics
- Froissart Bound
- Unitarity
- DGLAP Equation
- DGLAP Solution
- BFKL Evolution Equation

Deep Inelastic Scaterring

- Deep Inelastic Scaterring (DIS) is a process characterized by eletromagnetic interaction between a lepton of high energy (ν, e^+, e^-) and a nucleon $(p, \bar{p}, n)$:



Deep Inelastic Scaterring

- Photon four-momentum ($q^{\mu} = k - k'$) defines the scale of the process.
- ep process in leading order $\Rightarrow$ e + p $\rightarrow$ e + X
- Squared momentum transferred is defined as boson's virtuality $Q^2 = -q^2 = -(k - k')^2$.
- From uncertainty principle $\Delta x \sim \frac{1}{\Delta p} = \frac{1}{\Delta Q}$ is defined the resolution with which the target is probed.

Deep Inelastic Scaterring

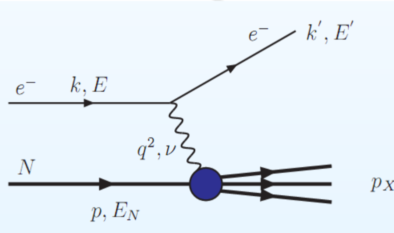
- Thinking the proton moving with very high momentum P.
- Proton featuring Lorentz contraction in longitudinal direction.
- Inclusive cross section averaged in spin in DIS lepton-hadron σ^{lh}.
- Expressed in terms of two invariant gauge functions that characterize the target structure, F_1 and F_2.
- For charged leptons scattering the process is mediated for photon virtual exchanged in the limit $Q^2 \ll M^2$.

$$\frac{d^2\sigma^{lh}}{dxdQ^2} = \frac{4\pi\alpha^2}{xQ^2}\left[xy^2 F_1\left(x, Q^2\right) + (1-y)\, F_2\left(x, Q^2\right)\right]$$

where the proton mass is neglected and $x = \frac{Q^2}{2p \cdot q}$ is the Bjorken scaling variable.

Deep Inelastic Scaterring

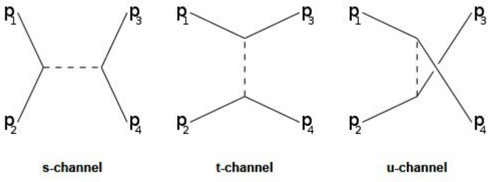
- Constant of electroweak coupling is α and y is the inelasticity.
- Inelasticity in the rest system of target proton can be written $y = 1 - \frac{E'}{E}$, where E and E' are the energies of initial and final state, respectively.
- General covariant case $\rightarrow \nu = \frac{(k'-k)\cdot p}{M} = \frac{p\cdot q}{M}$.
- Energy transfer $\rightarrow E' - E$.
- Energy scale of virtual photon is greater than proton.
- Resolution of proton constituents can be obtained.



Deep Inelastic Scaterring

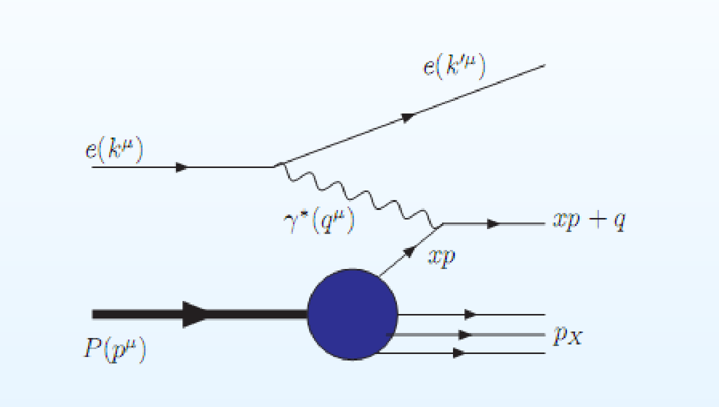
Mandelstam Variables

- s channel: $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$
- t channel: $t = (p_1 - p_3)^2 = (p_2 - p_4)^2$
- u channel: $s = (p_4 - p_1)^2 = (p_2 - p_3)^2$



Parton Model

- In the parton model the DIS can be viewed as the inelastic scattering off point particles (partons).



Parton Model

- With the collinear factorization and the parton model, the DIS cross section is written in the form

$$\left.\frac{d^2\sigma}{dxdQ^2}\right|_{ep\to eX} = \sum_i \int_0^1 dx f_i(x) \left.\frac{d^2\sigma}{dxdQ^2}\right|_{eq\to eq}$$

where $f_i(x)$ is the parton distribution function.

- The number of partons i inside the proton is obtained by

$$N_i = \int_0^1 f_i(x_i) dx_i$$

- Momentum conservation

$$\sum_i \int_0^1 x_i f_i(x_i) dx_i = 1$$

Parton Model

- Structure functions defined in the Bjorken limit F_1 and F_2 are relacioned in the parton model

$$F_2 = 2xF_1 = \sum_i e_i^2 x f_i(x)$$

- Experimental results show that

$$\sum_i \int_0^1 x_i f_i(x_i) dx_i \approx 0.5$$

- Then, neutral partons are necessary and they carry approximately 50% of the total momentum.
- Such particles are the gluons, mediators of the strong interaction.

QCD properties

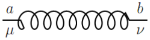
- Quantum Chromodynamics (QCD) $\rightarrow$ Strong interaction
- QCD Lagrangian

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^A_{\alpha\beta} F^{\alpha\beta}_A + \sum_{flavor} \bar{q}_a \left(i\hat{D} - m \right)_{ab} q_b + \mathcal{L}_{fix} + \mathcal{L}_{ghost}$$

- $F^A_{\alpha\beta} = \partial_\alpha G^A_\beta - \partial_\beta G^A_\alpha + g f^A_{\rho\gamma} G^\rho_\alpha G^\gamma_\beta$ are gluon field tensors.
- $f^A_{\rho\gamma}$ are the SU(3) structure constants
- Quarks are fermions with color charge $(q_a q_b)$
- Gluons carry color too
- $\mathcal{L}_{\text{fix}}$ is the gauge-fixing term
- $\mathcal{L}_{\text{ghost}}$ introduces the Fadeev-Popov ghosts
- They cancel unphysical degrees of freedom leading to anomalous terms.

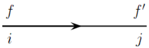
Feynman Rules for QCD

Gluon Propagator



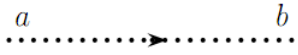
$$= \frac{-i\delta^{ab}}{k^2 + i0}\left(g^{\mu\nu} + (\xi - 1)\frac{k^\mu k^\nu}{k^2 + i0}\right)$$

Quark Propagator



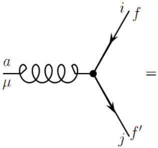
$$= \frac{i\delta^i_j \delta^f_{f'}}{\not p - m_f + i0}$$

Ghost Propagator



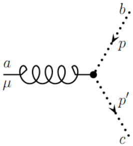
$$= \frac{i\delta^{ab}}{k^2 + i0}$$

Quark-Gluon Vertex



$$= -ig\gamma^\mu \times \delta^{f'}_f \times (t^a)^j_i$$

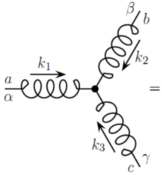
Ghost-Gluon Vertex



$$= -gf^{abc}p'^\mu$$

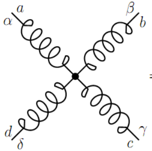
Feynman Rules for QCD

Three-Gluon Vertex



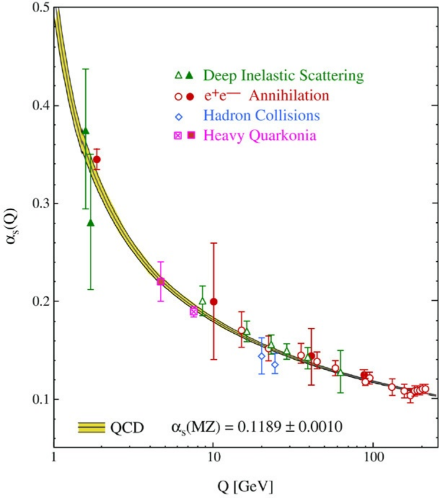
$$= -g f^{abc}\left[g^{\alpha\beta}(k_1 - k_2)^\gamma + g^{\beta\gamma}(k_2 - k_3)^\alpha + g^{\gamma\alpha}(k_3 - k_1)^\beta\right]$$

Four-Gluon Vertex



$$= -ig^2 \begin{bmatrix} f^{abe}f^{cde}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) \\ + f^{ace}f^{bde}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta}) \\ + f^{ade}f^{bce}(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta}) \end{bmatrix}$$

Asymptotic Freedom and Confinement



- Asymptotic freedom → high energies
- Confinement → low energies.

$$\alpha_s\left(Q^2\right) = \frac{12\pi}{(33 - 2n_f)\ \ln\left(\frac{Q^2}{\Lambda^2}\right)}$$

Froissart bound

- The Froissart bound is a limit for the cross section for the scattering of two hadrons.
- It is derived using Mandelstam representation and is based on two hypothesiss.
- First Froissart hypothesis: the strong interaction has finite range.
- This range is determined by the mass m_π

$$R \sim \frac{1}{m_\pi}$$

- This scale is nonperturbative.

Froissart bound

- Second Froissart hypothesis: S-matrix is unitary

$$SS^\dagger + S^\dagger S = 1$$

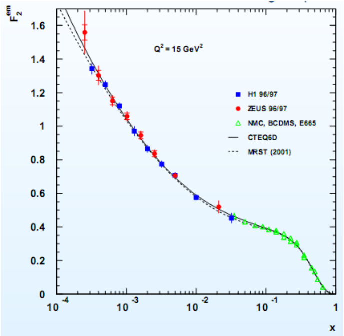
- The Froissart bound limits the total cross section for scattering of two hadrons:

$$\sigma_{TOT} \leq \frac{\pi}{m_\pi^2} (\ln s)^2$$

- It was derived for all regions of QCD (including pQCD and npQCD).
- However, the available data show no sign that Froissart bound is valid (or invalid).

GFPAE

Froissart bound



- F_2 structure function data from HERA collider and fixed target experiments.
- At high photon virtualities, the DIS structure function appears to increase very fast for a logarithm dependence.
- One hopes that with more exclusive processes (maybe diffraction) saturation can be observed.

Parton Model

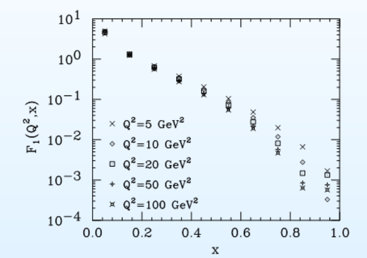
- How to obtain predictions for the structure functions in DIS through the QCD?
- Calculating the contributions for each order in the coupling through Feynman rules
- In dominant order, only the elastic scattering proton-quark contributes.
- This process is represented by $\gamma^* q \to q'$.
- Thus, the structure functions take the form

$$2F_1\left(x, Q^2\right) = \frac{1}{x} F_2\left(x, Q^2\right) = \sum_q e_q^2 \int d\omega q(\omega) \delta(x - \omega) = \sum_q e_q^2 q(x)$$

where ω is the momentum fraction carried by the scattered parton and $q(\omega)$ are the quarks distributions.

Parton Model

- In the structure functions, there is only x dependence (Bjorken scale)
- Relation between both structure functions (Callan-Gross): $F_2 = 2xF_1$

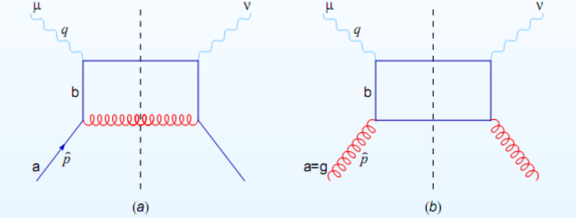


- F_1 structure function using MRS(A) partonic distributions.

Parton Model

- How does the presence of gluon radiation determines the Bjorken scale violation?
- Next order in perturbative expansion occurs gluon emission

$$\gamma^*(q) + q(P) \rightarrow q(p') + g(k)$$



Parton Model

- When structure functions of the hadronic vertex are extracted, a dependence on Q^2 is found.
- In terms of partonic densities which depend on Q^2, $q\left(\omega, Q^2\right)$:

$$\frac{1}{x} F_2\left(x, Q^2\right) \equiv \sum_q e_q^2 q\left(x, Q^2\right)$$

- $F_2 \neq 2xF_1$

Parton Model

- Its expression is written as

$$\frac{1}{x}F_2\left(x,Q^2\right) = \sum_q \frac{e_q^2\alpha_s}{2\pi}\int_x^1 \frac{dz}{z} q\left(\frac{x}{z}\right)\int_0^{Q^2/z} d\left(-\hat{t}\right) \times$$
$$\times \quad \frac{4}{3}\left[\frac{1}{-\hat{t}}\frac{1+z^2}{1-z} - \frac{z^2\left(\hat{t}+2Q^2\right)}{(1-z)\,Q^4}\right]$$

- Introducing the z variable

$$z = \frac{x}{\omega} = \frac{Q^2}{2p\cdot q} = \frac{Q^2}{\hat{s}+Q^2}$$

where the variables denoted with hats are the Mandelstam variables $\hat{s}$ e $\hat{t}$ (partonic level).

Parton Model

- The previous expression has singularities.
- There is a soft infrared singularity in z = 1, corresponding to

$$\hat{s} = Q^2 \frac{1-z}{z} = 0$$

in the limit where the momentum of gluon emitted is $k = 0$.

- These kind of singularities arise in theories containing a gauge field without mass (γ in QED and gluon in QCD)
- It is canceled when contributions of vertex corrections are considered.
- There is a cutoff $z_{soft} < 1$

Parton Model

- Another singularity is the mass singularity or collinear in $\hat{t}$.
- Related to the incident quark emitting a collinear gluon still on the mass shell
- These divergences take place when the non-massive field couples with another massless field (quarks without mass in QCD or gluons in QCD)
- Soft and collinear singularities are named "infrared divergences"
- In any process observed there is emission of an indefinite number of soft photons or gluons.
- Experimentally, the final state of a charged particle is not fully specified because there are soft photons and gluons with difficult detection.

Parton Model

- Considering the collinear divergences $\rightarrow$ regularizing for another cutoff $\hat{t} = -\mu^2_{col}$, which can be absorbed later in the redefinition of the initial quark distribution.
- Keeping the dominant logarithm $\ln\left(Q^2/\mu^2_{col}\right)$, the structure function F_2 can be written as,

$$\frac{1}{x}F_2\left(x, Q^2\right) = \sum_q \frac{e_q^2\alpha_s}{2\pi}\int_x^{z_{soft}}\frac{dz}{z}q\left(\frac{x}{z}\right)P_{qq}\left(z\right)\ln\left(\frac{Q^2}{\mu^2_{col}}\right)$$

- In general, the integration has non-logarithm terms absorbed $P_{qq}(z)\ \ln\ \left(\mu^2/\mu^2_{col}\right)$ changing the collinear cutoff for a different scale μ.
- In high Q^2, the effect of these terms are suppressed.

Parton Model

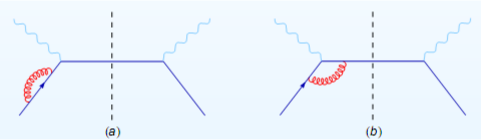
- Using this freedom of choice and defining μ as the renormalization scale in which the coupling is defined, the equation above is written as

$$\frac{1}{x}F_2\left(x,Q^2\right)=\sum_q \frac{e_q^2\alpha_s\left(\mu^2\right)}{2\pi}\int_x^{z_{soft}}\frac{dz}{z}q\left(\frac{x}{z}\right)P_{qq}\left(z\right)\ln\left(\frac{Q^2}{\mu^2}\right)$$

- The function P_{qq} (splitting function quark-quark) is included with dependence on z of the form

$$P_{qq}=\frac{4}{3}\left(\frac{1-z^2}{1-z}\right)$$

Parton Model



- This function is independent of the regularization prescription universal for different processes where a quark emerges as a quark with a gluon radiation.
- Cancellation of soft divergence can be understood using dimensional regularization t' Hooft and Veltman.
- Here, the Feynman diagrams are calculated in $4 - 2\varepsilon$ dimensions and the singularities are extracted as poles in ε.

Parton Model

- Introducing the low order vertexes with the diagram with virtual gluon
- Sum of Born term and contributions from virtual gluon is

$$\frac{1}{x}F_2\left(x,Q^2\right) = \sum_q e_q^2 \int_x^1 q\left(\frac{x}{z}\right)\left[\delta(1-z)+\frac{\alpha_s\left(\mu^2\right)}{2\pi}P_{qq}\left(z\right)\times \right.$$
$$\left. \times \left[\ln\left(\frac{Q^2}{\mu^2}\right)-\frac{1}{\varepsilon}\right]+\alpha_s\left(\mu^2\right)f\left(z\right)\right]$$

Parton Model

- Soft singularities are canceled by virtual corrections.
- Splitting function is modified to remove the original singularity

$$P_{qq}(z) = \frac{4}{3}\frac{\left(1+z^2\right)}{\left(1-z\right)_+} + 2\delta\left(1-z\right)$$

where $+$ is

$$\int_0^1 dz \frac{g(z)}{(1-z)_+} \equiv \int_0^1 dz \frac{g(z)-g(1)}{1-z}$$

Parton Model

- Now, the absorption of these collinear singularities can be defined in a renormalized distribution of quarks

$$q_{\mathcal{R}}(x) \equiv q(x) + \int_x^1 \frac{dz}{z} q\left(\frac{x}{z}\right) \left[\alpha_s\left(\mu^2\right) f(z) - \frac{\alpha_s\left(\mu^2\right)}{2\pi} P_{qq}(z) \frac{1}{z}\right]$$

- Notation: $q_{\mathcal{R}} \rightarrow$ renormalized distribution
- Combining these results, the structure function F_2 is

$$\frac{1}{x} F_2\left(x, Q^2\right) = \sum_q e_q^2 \int_x^1 \frac{d\omega}{\omega} q(\omega) \left[\delta(1 - \tfrac{x}{\omega}) + \frac{\alpha_s(\mu^2)}{2\pi} P_{qq}\left(\tfrac{x}{\omega}\right) \ln\left(\frac{Q^2}{\mu^2}\right)\right]$$

Parton Model

- A redefinition of quark distributions at high Q^2

$$\frac{1}{x}F_2\left(x,Q^2\right)=\sum_q e_q^2 q\left(x,Q^2\right)=\sum_q e_q^2\left[q(x)+\delta q(x,Q^2)\right]$$

where

$$\delta q\left(x,Q^2\right)=\frac{\alpha_s(\mu^2)}{2\pi}\ln\left(\frac{Q^2}{\mu^2}\right)\int\frac{d\omega}{\omega}q(\omega)P_{qq}\left(\frac{x}{\omega}\right)$$

DGLAP

- Effect of high orders in the expansion $\rightarrow$ sum of terms for order $\propto \left[\alpha_s\left(\mu\right)\ \ln\ Q^2/\mu^2\right]^n$
- These terms are important at high Q^2
- Sum can be made through an integro-differential equation,

$$\frac{\partial q\left(x, Q^2\right)}{\partial \ln\ Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\omega}{\omega} q\left(\omega, Q^2\right) P_{qq}\left(\frac{x}{\omega}\right) + \mathcal{O}\left(\alpha_s\left(\mu^2\right)\ \ln\ Q^2\right)$$

- This equation considers the ladder diagrams, summing the contributions of collinear emission of n gluons with quark distribution

DGLAP

- One uses the coupling in the renormalization fixed scale μ^2
- Using $\alpha_s\left(Q^2\right)$ and inserting the propagator in the diagrams
- For high Q^2

$$\frac{\partial q\left(x,Q^2\right)}{\partial \ln\ Q^2}=\frac{\alpha_s}{2\pi}\int_x^1\frac{d\omega}{\omega}q\left(\omega,Q^2\right)P_{qq}\left(\frac{x}{\omega}\right)$$

- This evolution equation considers the case when the photon is absorbed by one quark originated by an initial quark with momentum fraction $\omega < x$
- Since this quark originates in a gluon, the splitting function of quark is

$$P_{qg}(z)=\frac{1}{2}\left[z^2+(1-z)^2\right]$$

DGLAP

- Evolution for quarks becomes

$$\frac{\partial q_i\left(x,Q^2\right)}{\partial \ \ln \ Q^2} = \frac{\alpha_s\left(Q^2\right)}{2\pi} \int_x^1 \frac{d\omega}{\omega} \left[q_i\left(\omega, Q^2\right) P_{qq}\left(\frac{x}{\omega}\right) + g\left(\omega, Q^2\right) P_{qg}\left(\frac{x}{\omega}\right)\right]$$

where the collinear singularity ε^{-1} is absorbed in the gluon distribution (quarks)

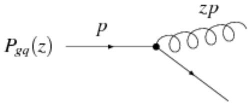
- Evolution has validity for any massless quark or antiquark q_i
- Additional contribution of equations Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) $\rightarrow$ correspondent expression for gluons distribution

$$\frac{\partial g\left(x,Q^2\right)}{\partial \ \ln \ Q^2} = \frac{\alpha_s\left(Q^2\right)}{2\pi} \int_x^1 \frac{d\omega}{\omega} \left[\sum_i q_i\left(\omega, Q^2\right) P_{gq}\left(\frac{x}{\omega}\right) + g\left(\omega, Q^2\right) P_{gg}\left(\frac{x}{\omega}\right)\right]$$

DGLAP

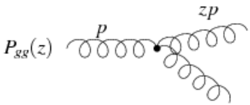
- Quark-gluon and gluon-gluon splitting functions

$$P_{gq}(z) = \frac{4}{3}\left[\frac{1+(1-z)^2}{z}\right]$$



Splitting function $P_{gq} \rightarrow$ probability of an initial quark to emit a gluon.

$$P_{gg}(z) = 6\left[\frac{(1-z)}{z} + \frac{z}{(1-z)_+} + z(1-z) + \left(\frac{11}{12} - \frac{n_f}{18}\right)\delta(1-z)\right]$$

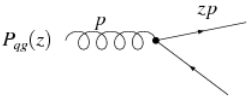


Splitting function $P_{gg} \rightarrow$ probability of a gluon in the initial state to emit a gluon.

DGLAP

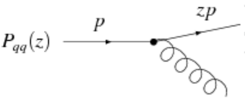
- gluon-quark and quark-quark splitting functions

$$P_{qg}(z) \quad = \quad \frac{1}{2}\left[z^2 + (1-z)^2\right]$$



Splitting function P_{qg} $\rightarrow$ probability of an initial gluon to emit a quark.

$$P_{qq}(z) = \frac{4}{3}\left[\frac{(1+z^2)}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]$$



Splitting function P_{qq} $\rightarrow$ probability of a quark in the initial state to emit a quark.

DGLAP

- Derivation made is in leading-order (LO) for the DGLAP formalism
- Splitting functions can be obtained as a perturbative expansion in α_s

$$P_{ab}\left(x, Q^2\right) = P_{ab}^{LO}(x) + \alpha_s(Q^2) P_{ab}^{NLO}(x) + ...$$

- Truncate after the first two terms leaves DGLAP evolution in next-to-leading-order (NLO).
- Beyond leading-order the splitting functions dependence on factorization scale.

DGLAP

- In next-to-leading-order (NLO) the Callan-Gross relation is no more satisfied → longitudinal structure function

$$F_L\left(x, Q^2\right) = \left(1 + \frac{4M^2x^2}{Q^2}\right) F_2\left(x, Q^2\right) - 2xF_1\left(x, Q^2\right)$$

  M is the proton mass
- Function $F_L = F_2 - 2xF_1(Q^2 \to \infty)$
- $F_L \ll F_2$ is the confirmation that quarks have spin $1/2$

DGLAP Solution

- Knowledge of the solution, knowledge of the evolution of the partonic distributions in Q^2.
- Initial value in any initial scale Q_0^2 sufficiently high to guarantee the use of perturbation theory.
- Condition $Q_0^2 \gg \Lambda^2$, where $Q^2 = \Lambda^2$ is the Landau pole for QCD:

$$\alpha_s\left(Q^2\right) = \frac{1}{b \ln \frac{Q^2}{\Lambda^2}} \equiv \frac{1}{bt}$$

- Introducing variable t, in contrast of Q and b

$$P_{qq} \otimes q_i \equiv \int_x^1 \frac{d\omega}{\omega} P_{qq}\left(\frac{x}{\omega}\right) q_i\left(\omega, t\right)$$

DGLAP Solution

- DGLAP equations are written as

$$\frac{dq_i(x,t)}{dt} = \frac{\alpha_s(t)}{2\pi}\left[P_{qq}\otimes q_i + P_{qg}\otimes g\right]$$

$$\frac{dg(x,t)}{dt} = \frac{\alpha_s(t)}{2\pi}\left[P_{gq}\otimes \sum_i q_i + P_{gg}\otimes g\right]$$

DGLAP Solution

- Simplify this equation $\rightarrow$ symmetry combination of flavor $SU(n_f)$ singlet and non-singlet of partonic distributions.
- Singlet combination is given by

$$q^S(x,t) = \sum_i [q_i(x,t) + \bar{q}_i(x,t)]$$

summing all over active flavors.

- Combinations non-singlet $q^{NS}(x,t)$ are given by $u - \bar{u}$, $d - \bar{d}$,...

DGLAP Solution

- Combinations satisfy the equations

$$\frac{dq^{NS}(x,t)}{dt} = \frac{\alpha_s(t)}{2\pi} P_{qq} \otimes q^{NS}$$

$$\frac{dq^{S}(x,t)}{dt} = \frac{\alpha_s(t)}{2\pi} \left[P_{qq} \otimes q^{S} + 2n_f P_{qg} \otimes g \right]$$

$$\frac{dg(x,t)}{dt} = \frac{\alpha_s(t)}{2\pi} \left[P_{gq} \otimes q^{S} + P_{gg} \otimes g \right]$$

DGLAP Solution

- Convenient transformation $\rightarrow$ considering the distributions of partonic momenta and splitting functions
- Evaluate by Mellin transform in a conjugate space j,

$$q_j(t) = \int_0^1 dx x^{j-1} q(x,t)$$

$$\gamma_j = \int_0^1 dx x^{j-1} P(x)$$

where this second equation is the definition of anomalous dimension.

DGLAP Solution

- Then, the equations can be expressed by

$$\frac{dq_j^{NS}(x,t)}{dt} = \frac{1}{2\pi bt}\gamma_j^{qq} q_j^{NS}$$
$$\frac{dq_J^{S}(x,t)}{dt} = \frac{1}{2\pi bt}\left[\gamma_j^{qq} q_j^{S} + 2n_f \gamma_j^{qq} g_j\right]$$
$$\frac{dg_j(x,t)}{dt} = \frac{1}{2\pi bt}\left[\gamma_j^{gq} q_j^{S} + \gamma_j^{gg} g_j\right]$$

DGLAP Solution

- Solution for non-singlet momenta $\rightarrow$ not the same

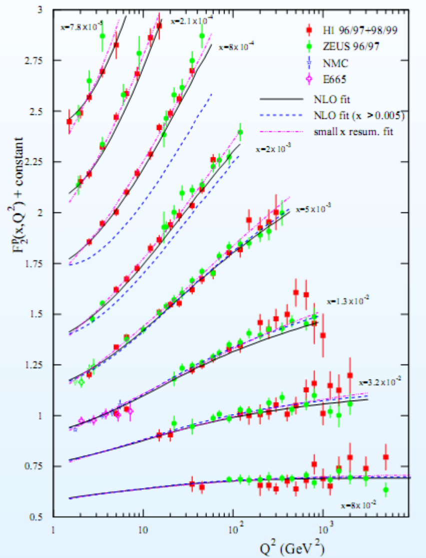
$$q_j^{NS}(t) = q_j^{NS}(t_0)\left(\frac{\alpha_s(t_0)}{\alpha_s(t)}\right)^{\frac{\gamma_j^{qq}}{2\pi b}}$$

- Distributions in x space are given by

$$q^{NS}(x,t) = \frac{1}{2\pi i}\int_C dj x^{-j} q_j^{NS}(t)$$

- Contour of integration in the complex plane with j parallel to the imaginary axis and with integer right singularities.
- Singlet momenta can be evaluated with a similar calculation.

Comparing with the data

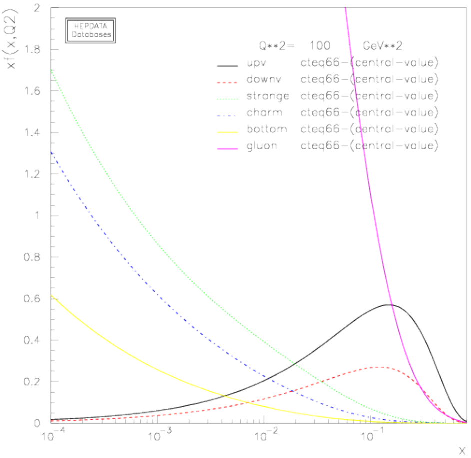


- F_2 structure function with fixed x, compared with global fit using DGLAP evolution elaborated by MRST group.

DGLAP Solution What is this good for?

- To extract partonic distributions from the data, consider an initial parametrization in the behavior of variable function x for different partonic distributions at low Q_0^2.
- Using the DGLAP evolution equations to evolve the partonic distributions for any larger Q^2 where observables are measured.
- Fit (choice) of parameters used in parametrization of initial conditions.
- Choice of nonperturbative parameters.
- Contribution from sea quarks (originated by quark-antiquark pairs produced in a gluonic splitting) $g \to q\bar{q} \to$ growth in small x.
- Leading distribution in this region is the gluonic distribution.

DGLAP Solution



- Partonic distributions determined by fit to ZEUS data.

DGLAP Solution

- For the DGLAP equations the contributions are proportional to

$$\left[\alpha_s\left(Q^2\right)\ \ln\left(\frac{Q^2}{Q_0^2}\right)\right]^n$$

- Strong ordering on the transverse momenta of parton in the partonic cascades and corresponding to leading logarithm approximation (LLA)
- Validity in the limit

$$\alpha_s\left(Q^2\right)\ \ln\left(\frac{1}{x}\right) \ll \alpha_s\left(Q^2\right)\ \ln\left(\frac{Q^2}{Q_0^2}\right) < 1$$

DGLAP Solution

- At small x the gluon distribution dominates.
- Divergence in the splitting functions P_{gq} and P_{gg}
- DGLAP in small x limit is written as

$$\frac{dq_i\left(x,Q^2\right)}{d\ \ln\ Q^2} \approx \frac{\alpha_s\left(Q^2\right)}{2\pi}\int_x^1 \frac{dz}{z}P_{qg}\left(z\right)g\left(\frac{x}{z},Q^2\right) \equiv \frac{\alpha_s\left(Q^2\right)}{2\pi}P_{qg}\otimes g$$
$$\frac{dg\left(x,Q^2\right)}{d\ \ln\ Q^2} \approx \frac{\alpha_s\left(Q^2\right)}{2\pi}\int_x^1 \frac{dz}{z}P_{gg}\left(z\right)g\left(\frac{x}{z},Q^2\right) \equiv \frac{\alpha_s\left(Q^2\right)}{2\pi}P_{gg}\otimes g$$

DGLAP Solution in Small-x

- Due to the Q^2 dependence of the parton distribution, the structure function in DIS, F_2, can be written as

$$F_2\left(x, Q^2\right) = x\sum_i e_i^2\left[q_i\left(x, Q^2\right) + \bar{q}_i\left(x, Q^2\right)\right] + \mathcal{O}\left(\alpha_\int^\epsilon\right)$$

- Described by the DGLAP evolution equations
- Higher orders in α_s are evaluated by the following substitution in the DGLAP evolution equations

$$\frac{\alpha_s\left(Q^2\right)}{2\pi}P_{ij}^0 \rightarrow \frac{\alpha_s\left(Q^2\right)}{2\pi}P_{ij}^0 + \left(\frac{\alpha_s\left(Q^2\right)}{2\pi}\right)^2 P_{ij}^1 + \ldots$$

DGLAP Solution in Small-x

- At small x $\rightarrow$ gluon dominance.

$$P_{gg}^{0}(z) \sim \frac{2N_c}{z} \Rightarrow xg\left(x, Q^2\right) \sim x^{-\lambda}, \quad \lambda > 0$$

- Parton distribution at small x

$$xp_i\left(x, Q_0^2\right) \sim const.$$

where Q_0^2 is the initial condition.

$$xp_i \sim exp\left\{\sqrt{\xi\left(Q^2\right)\ \ln\ (1/x)}\right\}$$

DGLAP Solution in Small-x

- This is the Double Logarithm approximation, where

$$\xi\left(Q^2\right) = \int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \frac{N_c \alpha_s q^2}{\pi}$$

- There is an increase in the gluon and quark distributions and in the structure function F_2
- Momentum fraction x decreases

DGLAP Solution in Small-x

- To understand the interaction in the strong regime for small x $\rightarrow$ challenge in QCD.
- Structure functions in the region of small x and study of transition between perturbative and nonperturbative regime is made by DIS.
- DGLAP equations for partonic distributions have good description of the physics of scaling violation.
- Several questions still remain:
  - Where begins the a small x regime?
  - For which values of momentum fraction x where DGLAP formalism for evolution of structure functions becomes unappropriated?
- With current data $\rightarrow$ F_2 growth in the $x = 0$ limit
- This growth is larger with increasing Q^2.

DGLAP Solution in Small-x

- Basic question: this behavior can be understood in terms of QCD?
- Splitting structure are evaluated with a series expansion with powers of coupling on the constant
- NLO terms are known
- Gluon-gluon splitting function $P_{gg}\left(x, \alpha_s\left(Q_2\right)\right)$ in fixed order singularity in small-x
- Behavior $\approx \alpha_s/x$
- Singularity growth of F_2 to small-x
- What is the dynamics of this growth?

DGLAP Solution in Small-x

- Naive version of evolution equation can be solution to this question.
- Evolution for the gluonic distribution (dominant parton in $x \to 0$ and singular term P_{gg},

$$\frac{dg_j}{dt} = \frac{\alpha_t(t)}{2\pi}\gamma_j^{gg} g_j$$

- Limit $x \to 0$, partonic distributions give the behavior of anomalous dimension γ_j^{gg} for $j \approx 1$ (N_c being the number of colors)

$$\gamma_j^{gg} \simeq \frac{2N_c}{j-1}$$

DGLAP Solution in Small-x

- Solution for the momenta of gluon distribution in this limit is

$$g_j(t) = g_j\left(t_0\right) exp \frac{N_c \eta(t)}{j-1}$$

with η function defined by

$$\eta(t) = \int_{t_0}^{t} dt' \alpha_s(t')$$

- Back to x space, consider the Mellin inverse transform

$$\begin{aligned} g(x,t) &= \frac{1}{2\pi i} \int dj x^{-j} g_j(t) \\ &= \frac{1}{2\pi i} \int dj g_j(t_0) exp\left(j \ln \frac{1}{x} + \frac{N_c \eta(t)}{j-1} \right) \end{aligned}$$

DGLAP Solution in Small-x

- When Q_2 is high and x is small
- Expansion close to saddle point of exponential j_{saddle}

$$j_{\text{saddle}} = 1 + \sqrt{\frac{N_c \eta(t)}{\ln \frac{1}{x}}}$$

- Solution is expressed in the original variable

$$g\left(x, Q^2\right) \sim \frac{1}{x} \exp \sqrt{\frac{N_c}{\pi b} \ln \left[\left(\ln \frac{Q^2}{\Lambda^2}\right) / \left(\ln \frac{Q_0^2}{\Lambda^2}\right)\right] \ln \frac{1}{x}}$$

DGLAP Solution in Small-x

- This result is Double Logarithm Approximation domain (DLA), where the logarithms are summed by

$$\left[\alpha_s\left(Q^2\right)\ \ln\left(\frac{Q^2}{Q_0^2}\right)\ \ln\left(\frac{1}{x}\right)\right]^n$$

- Solution is valid when one uses a soft initial distribution
- In this case, x asymptotic dependence is dominated by splitting perturbative function.
- A question arises: what happens if one chooses a stronger initial dependence, for example,

$$g\left(x, t_0\right) = \mathcal{C}x^{-j_0}$$

DGLAP Solution in Small-x

- In this case, one must be more careful $\rightarrow$ singularities on the right of the saddle point

$$g_j\left(t_0\right) = \frac{\mathcal{C}}{j - j_0}$$

- Solution can be found at small-x $\rightarrow$ choice of contour of integration close to the pole of the initial distribution.

DGLAP Solution in Small-x

- Integral value is

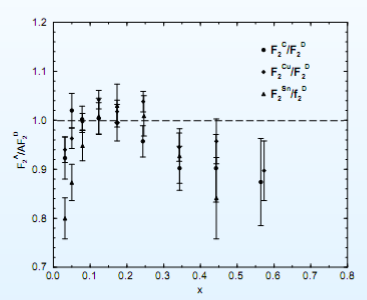
$$g\left(x,t\right) = \mathcal{C}x^{-j_0}\,\exp\,\frac{N\eta(t)}{j_0 - 1}$$

and the original behavior in x remains beyond the evolution in Q^2

- Behavior as asymptotic dependence of non-perturbative parameter j_0
- Presence of this non-perturbative term $\rightarrow$ starts as singular initial condition
- This is a way to produce perturbative growth in small-x

Nuclear effects

- Ratio between nuclear structure functions and nucleon, normalized to A shows some modication of the nuclear structure function
- First result → modication in the nuclear structure function was the EMC effect (Arneodo PR240, 301 (1994)).



Nuclear effects

- What is wrong with the parton model?
- Theoretical expectation

$$F_2^A\left(x,Q^2\right) = AF_2^p\left(x,Q^2\right)$$

- Then

$$R = \frac{F_2^A\left(x,Q^2\right)}{AF_2^p\left(x,Q^2\right)} = 1$$

- $R \neq 1$, and there is A dependence of the nuclear effects

Unitarity

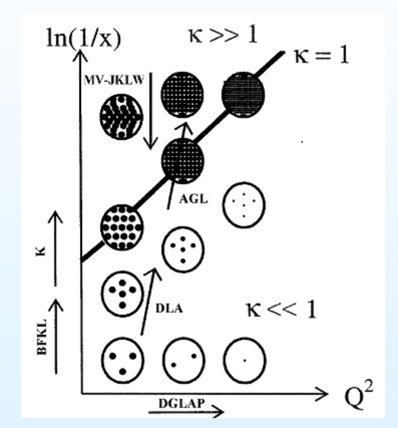
- Observed that F_2 increases for smaller values of x $\rightarrow$ violation of the unitarity
- Unitarity limit is the Froissart limit $\rightarrow$ shows that the cross section cannot be larger than $\sigma \leq cte \ln^2 s$
- Motivation - to restore the unitarity
- Small x region (high energy) is the interface between non-perturbative QCD (npQCD) and perturbative QCD
- In this interface the coupling constant α_s is still small
- $x < 10^{-2} \rightarrow$ dynamical (collective) effects

Unitarity

Problem:

- Analytically separate the perturbative and nonperturbative aspects.
- Small and large distances.
- Contributions to high energy.
- Amplitudes in a properly gauge invariant formalism.

Possible Solutions



BFKL

- Takes into account diagrams that contribute with terms of order $[(\alpha_s)\ \ln(1/x)]^n$ with $\alpha_s \ln\left(Q^2/Q_0^2\right) \ll 1$ e $\alpha_s \ln(1/x) \approx 1$.
- Ordering in transverse moments becomes moderate.
- Included the integration over the whole phase space formed by the transverse components of the moments of the emitted partons.
- In this kinematic region, the DGLAP evolution equations are no longer valid.
- A new dynamic is needed to describe the partonic distributions.

BFKL

- Proposal: Y. Balitski, V. Fadin, E. Kuraev e L. Lipatov (BFKL).
- Equation that describes the evolution in the Bjorken variable x.
- It is written in terms of non-integrated function of gluons $\phi\left(x, k_\perp^2\right)$, which gives the probability of finding a gluon in the nucleon with transverse momentum $k_\perp^2$ and fraction of longitudinal moment x.
- This function is related to the usual function of gluons by

$$xg\left(x, Q^2\right) = \int^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \phi\left(x, k_\perp^2\right)$$

BFKL

- Resummation of leading $\log(1/x)$ - terms $(\alpha_s \ln(1/x))^n$ is performed in the BFKL approach
- Based on the gluon Reggeization, property of QCD very important for description of high energy process.
- In the leading logarithmic approximation (LLA) it predicts $\sigma \sim \left(\frac{1}{x}\right)^{\omega_P}$ where the Pomeron intercept (with subtracted 1)

$$\omega_P = 4N_c \frac{\alpha_s}{\pi} \ln 2$$

- The gluon Reggeization hypothesis is proved in the NLA.

BFKL

- The differential form of BFKL equation is given by

$$\frac{\partial\phi\left(x,k_\perp^2\right)}{\partial\ \ln(1/x)}=\frac{3\alpha_s}{\pi}k_\perp^2\int_0^\infty\frac{dk_\perp'^2}{k_\perp'^2}\left\{\frac{\phi\left(x,k_\perp'^2\right)-\phi\left(x,k_\perp^2\right)}{\left|k_\perp'^2-k_\perp^2\right|}+\frac{\phi\left(x,k_\perp^2\right)}{\sqrt{4k_\perp'^4+k_\perp^4}}\right\}$$

- It is valid for sufficiently small values of x_0, such that

$$\alpha_s\ll 1,\quad \alpha_s\ln\left(Q^2/Q_0^2\right)\ll 1,\qquad \ln\left(1/x\right)\approx 1$$

BFKL

- In the high energy limit, the gluons distribuition dominates the evolution.
- The BFKL equation can be represented as a ladder diagram effective, with strong ordering in the longitudinal moments and without ordering in the transverse moments,

$$x \ll x_{i+1} \ll ... \ll x_1 \ll 1,$$
$$Q^2 \approx k_{\perp i+1} \approx ... \approx k_{\perp 1} \approx Q_0^2$$

BFKL Solution

- From Mellin transform of the function $\phi\left(x, k^2\right)$ in the variable k^2, is possible to obtain an analytical solution to the BFKL equation in the form

$$\frac{\partial \phi\left(x, \bar{\gamma}\right)}{\partial \ln(1/x)} = \bar{K}(\bar{\gamma})\phi\left(x, \bar{\gamma}\right)$$

whose solution is

$$\phi\left(x, \bar{\gamma}\right) = \phi\left(x, \bar{\gamma}\right)\left(\frac{x}{x_0}\right)^{-\bar{K}(\bar{\gamma})}$$

where $\bar{\gamma}$ is the conjugate variable of k^2, $\phi\left(x, \bar{\gamma}\right)$ is the transformed function and $\bar{K}$ is the transformed kernel.

BFKL Solution

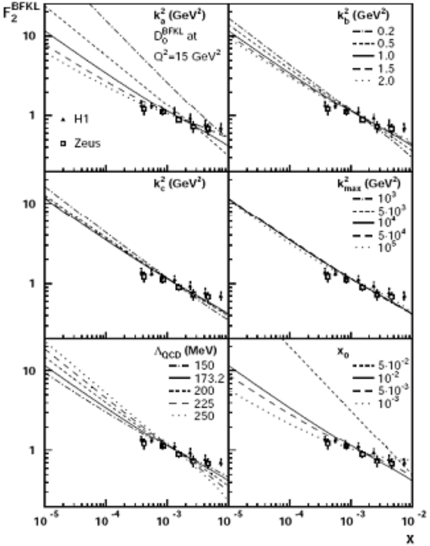
- Using an explicit form for the transformed functions, one obtains the non integrated gluon function

$$\phi\left(x,k^2\right)=\left(\frac{x}{x_0}\right)^{-\lambda}\frac{\sqrt{k^2}\phi\left(x_0,\bar{\gamma}=1/2\right)}{\left(6\alpha_s 28\zeta(3)\ \ln\left(x/x_0\right)\right)^{1/2}}\ \exp\left\{\frac{-\ \ln\left(k^2/\bar{k}^2\right)}{6\frac{\alpha_s}{\pi}28\zeta(3)\ \ln\left(x/x_0\right)}\right\}$$

where $\zeta(x)$ is the Riemann Zeta function.

- The first term produces the behavior $x^{-\lambda}$ for the non integrated gluon distribution, characteristic of the BFKL formalism.
- The BFKL dynamics predicts a rapid growth of the cross section $\sigma(\gamma^* N)$ with the energy.

Predictions for F_2



- BFKL prescription for F_2 compared with HERA data.

Limitations of BFKL

- Solutions obtained for α_s independent of Q^2.
- This limits the validity of the equation to a small range Q^2, where the behavior of the coupling constant can be approximated.
- The prediction of a large increase in the number of gluons violates the Froissart bound.

$$\sigma_{tot} < const\,(\ln s)^2$$

- To include new radiative corrections at any fixed order does not solve the problem.
- Proposal: Non linear evolution equations!