Impact Factors for Reggeon-Gluon Transitions

V.S Fadin

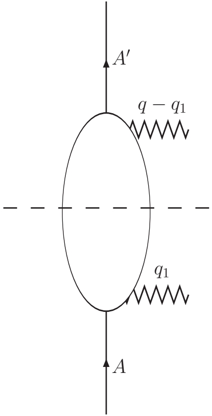
Budker Institute of Nuclear Physics
Novosibirsk

New Trends in High-Energy Physics and QCD
Federal University of Rio Grande do Norte
International Institute of Physics
Natal, Brasil, October 21 – November 06

Contents

Overview

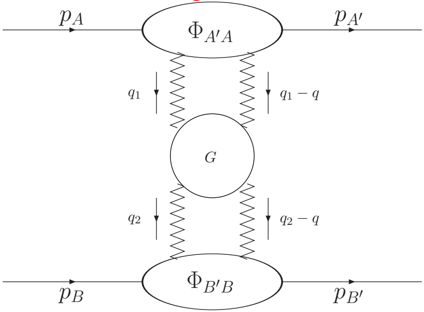
The notion of particle-particle impact factors i.e. impact factors $\Phi_{A'A}$ describing transitions of particles $A \to A'$ due to interaction with reggeized gluons (t-channel from left to right, s-channel from down to up) is well known.



It is represented by the figure on the right, where dashed line denotes discontinuity of the reggeon-particle scattering amplitude. In fact, the impact factor is given by the integral over squared invariant mass of particles on this discontinuity.

Overview

The particle-particle impact factors enter as essential part in description of scattering amplitudes in the BFKL approach (here s-channel from left to right, t-channel from up to down).



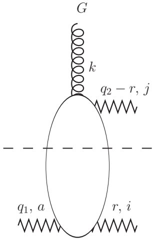
$$\hat{\mathcal{G}} = e^{Y\hat{\mathcal{K}}},$$

$\hat{\mathcal{K}}$ is the BFKL kernel, Y is the total rapidity $(Y = \ln(s/s_0))$, and the amplitude (more precisely, its imaginary part)

$$Im\,\mathcal{A}_{AB}^{A'B'} = \langle AA'|e^{Y\hat{\mathcal{K}}}|BB'\rangle$$

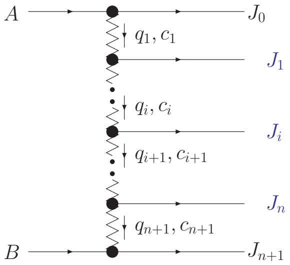
Overview

The impact factors for reggeon-gluon transitions are natural generalization of particle-particle ones. They describe transitions of Reggeons (reggeized gluons) into particles (ordinary gluons) due to interaction with reggeized gluons and are represented by the figure below (t-channel from left to right, s-channel from down to up), where dashed line denotes discontinuity of the reggeon $\rightarrow$ gluon scattering amplitude.



In fact, the impact factor is given by the integral over squared invariant mass of particles on this discontinuity.

These impact factors appeared firstly
V. S. F., 2002
J. Bartels, V. S. F., R. Fiore, 2003
in the proof of the multi-Regge form of QCD amplitudes.



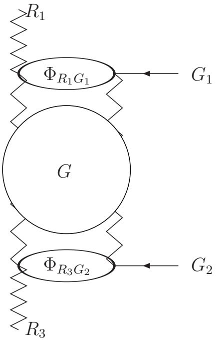
An idea of this form is the basis of the BFKL approach. It can be proved with the use of the *s*-channel unitarity.

Overview

Compatibility of the unitarity with the multi-Regge form leads to bootstrap relations connecting discontinuities of the amplitudes with products of their real parts and gluon trajectories. It turns out that fulfillment of an infinite set of these relations guarantees the multi–Regge form of scattering amplitudes. On the other hand, all bootstrap relations are fulfilled if several conditions imposed on the Reggeon vertices and the trajectory (bootstrap conditions) hold true. The most complicated condition, which includes the impact factors for Reggeon-gluon transition, was proved recently, both in QCD
M. G. Kozlov, A. V. Reznichenko, and V. S. F., 2011
M. G. Kozlov, A. V. Reznichenko, and V. S. F., 2012
and in its supersymmetric generalisation
M. G. Kozlov, A. V. Reznichenko, and V. S. F., 2013.

Recently, the Reggeon-gluon transition impact factors were used for the calculation of the high-energy behavior of the remainder factor to the BDS ansatz

It was shown
J. Bartels, L.N. Lipatov, A. Sabio Vera, 2009
that in the so called Mandelstam kinematical region the BDS amplitude $M_{2\to4}^{BDS}$ should be multiplied by the factor containing the contribution of the Mandelstam cut, and this contribution for the 6-point scattering amplitude was found in the leading logarithmic approximation (LLA)
J. Bartels, L.N. Lipatov, and A. Sabio Vera, 2010
and in the next-to-leading one (NLA)
L.N. Lipatov, A. Prygarin, 2010
J. Bartels, L.N. Lipatov, A. Prygarin, 2011
V.S. F. and L.N. Lipatov,1
In the BFKL approach this contribution is given by the convolution of the Green function of two interacting Reggeons with the impact factors for Reggeon-gluon transition.



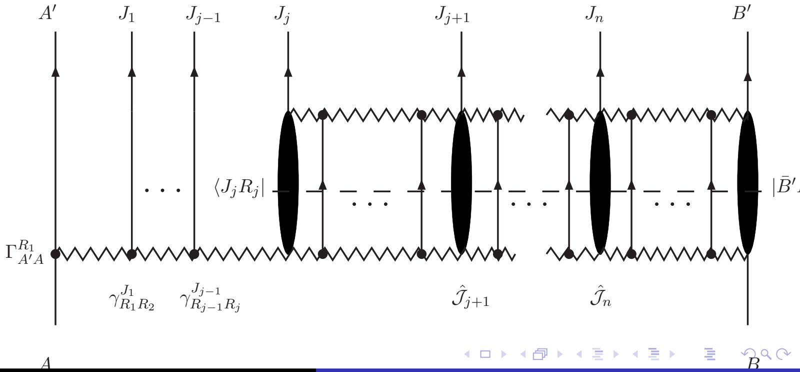
In the NLA the remainder function was calculated assuming the existence of conformal invariant) (in momentum space) representations of the modified (i.e. with the subtracted gluon trajectory depending on the total momentum transfer) BFKL kernel for the adjoint representation of the gauge group and impact factors for Reggeon-gluon transitions.

Overview

Later it was shown
V.S. F., R. Fiore, L. N. Lipatov, A. Papa, 2013
that indeed the modified BFKL kernel has the conformal invariant representation. As for the impact factors, actually not impact factors themselves, but their convolution was used. Moreover, the convolution used was not calculated in the framework of the BFKL approach, but was extracted
L. N. Lipatov, A. Prygarin, 2010
from the two-loop 6-point remainder function obtained
A. B. Goncharov, M. Spradlin, C. Vergu, A. Volovich, 2010
by simplification of the results
V. Del Duca, C. Duhr, and V. A. Smirnov, 2010.
In turn, in the derivation of these results it was supposed that remainder functions are given by expectation values of Wilson loops in N=4 SYM. All this makes important direct calculation of the impact factors in the BFKL framework and the investigation of their properties.

Dscontinuities of multi-Regge amplitudes

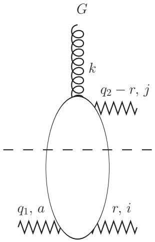
The *s*-channel discontinuities of elastic amplitudes are the simplest ones. They are expressed in terms of particle-particle impact factors and the BFKL kernel.
Evidently, discontinuities of multi-particle amplitudes are more involved.
Schematically they are presented by the picture:



Dscontinuities of multi-Regge amplitudes

They contain two additional ingredients: impact factors for reggeon-gluon transition and gluon production contribution to the discontinuity.

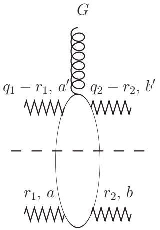
The impact factors for reggeon-gluon transition describe transition of reggeized gluon into ordinary ones due to interaction with reggeized gluons:



(*t*-channel from left to right, *s*-channel from down to up) dashed line denotes discontinuity of the reggeon $\rightarrow$ gluon scattering amplitude. The impact factor is given by the integral over squared invariant mass of particles on this discontinuity.

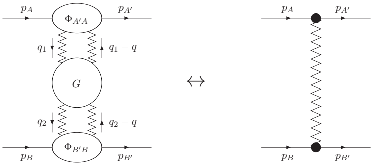
Dscontinuities of multi-Regge amplitudes

Te gluon production contribution to the discontinuity describe *t*-channel propagation of two reggeized gluon with production of ordinary one:



(*t*-channel from left to right, *s*-channel from down to up) dashed line denotes discontinuity of the reggeon $\to$ gluon scattering amplitude.
These ingredients are expressed through effective vertices describing interaction of reggeized gluons with gluons and quarks. Now all of them are known with the NLO accuracy.

The main idea of the proof is based on the s-channel unitarity. The requirement of compatibility of the s-channel unitarity with the Reggeized form of amplitudes leads to the bootstrap relations. These relations are quite simple in the elastic case:



$$\frac{1}{-\pi i}\mathrm{disc}_s \mathcal{A}_{AB}^{A'B'} = \frac{\partial}{\partial s}\, \Re \mathcal{A}_{AB}^{A'B'}/s$$

Bootstrap of the gluon Reggeization

It occurs that an infinite number of the bootstrap relations is fulfilled if:
1. The impact factors for scattering particles satisfy the conditions

$$|A'A\rangle = \frac{g}{2}\Gamma_{A'A}|R_\omega(q)\rangle$$

where $q = p_A - p_{A'}$ and
2. $|R_\omega(q)\rangle$ is the universal (process independent) eigenstate of the kernel $\hat{\mathcal{K}}$ with the eigenvalue $\omega(t)$

$$(\hat{\mathcal{K}} - \omega(t))|R_\omega(q)\rangle = 0$$

and the normalization

$$\frac{g^2 t N_c}{2(2\pi)^{D-1}}\langle R_\omega(q)|R_\omega(q)\rangle = \omega(t)\ , \quad t = q^2;$$

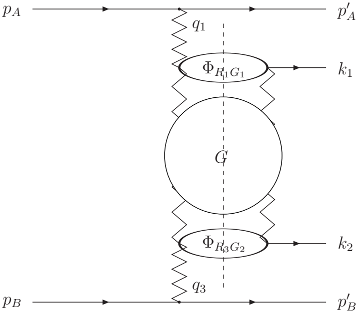
3. The Reggeon-gluon impact factors and the gluon production vertices satisfy the condition

$$\frac{gt_1}{2}\langle R_\omega(q_1)|\hat{\mathcal{G}} + \langle GR_1| = \frac{g}{2}\gamma^G(q_1, q_2)\langle R_\omega(q_2)|.$$

The conditions for the impact factors of scattering particles and the kernel were checked
M. Braun, G.P. Vacca,1999,
V.S.F., R. Fiore, M.I. Kotsky, A. Papa, 2000,
V.S.F, A. Papa, 2002
The last condition was checked recently, both in QCD
V.S.F., M.G. Kozlov, A. V. Reznichenko, 2011
V.S.F., M.G. Kozlov, A. V. Reznichenko, 2012
and in SYM
V.S.F., M.G. Kozlov, A. V. Reznichenko, 2013.

Contradiction with the BDS anzatz

There is almost evident contradiction of the expressions for s_i channel discontinuities with the BDS ansatz
Z. Bern, L.J. Dixon and V.A. Smirnov, 2005
for n gluon amplitudes with maximal helicity violation (MHV) in $N = 4$ supersymmetric Yang-Mills theory at $n \geq 6$. Indeed, let us consider the discontinuity of $A_{2\to4}$ in the s_2- channel (s_2 is the invariant mass square of produced gluons)



Contradiction with the BDS anzatz

Let us denote

$$M_{2\to 2+n}=\frac{A_{2\to 2+n}}{A^{(B)}_{2\to 2+n}}.$$

Then

$$4(2\pi)^4\delta(\vec{q}_1-\vec{k}_1-\vec{k}_2-\vec{q}_3)\Im\Delta_{s_2}\frac{M_{2\to 4}}{\Gamma_{BFKL}(t_1)\Gamma_{BFKL}(t_3)}=$$

$$=\left|\frac{s_1}{|\vec{q}_1||\vec{k}_1|}\right|^{\omega(t_1)}\left|\frac{s_3}{|\vec{k}_2||\vec{q}_3|}\right|^{\omega(t_3)}\frac{\langle G_1R_1|e^{\hat{K}\ln\left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)}|G_2R_3\rangle}{\gamma^{G_1(B)}_{R_1R_2}(\vec{q}_2^{\,2})^{-1}\gamma^{G_2(B)}_{R_2R_3}},$$

where $\Gamma_{BFKL}(t)$- gluon-gluon-reggeon vertex at NLO, $\gamma^{G_1(B)}_{R_1R_2}$ and $\gamma^{G_2(B)}_{R_2R_3}$ – Born vertices for gluon production in reggeon transitions, $\langle G_1R_1|$ and $|G_2R_3\rangle$ – impact factors for reggeon-gluon transition.

Contradiction with the BDS anzatz

In the MRK, energy dependence of the BDS ansatz for the $M_{2\to 2+n}$ is given by the Regge factors,
J. Bartels, L. N. Lipatov and A. Sabio Vera, 2009

$$M_{2\to 2+n} \sim |s_1|^{\omega(t_1)}\,|s_2|^{\omega(t_2)}\,|s_3|^{\omega(t_3)}\,.$$

Therefore, for agreement with the BDS one needs

$$\langle G_1 R_1|e^{\hat{K}\ln\left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)}|G_2 R_3\rangle \sim |s_2|^{\omega(t_2)}\,,$$

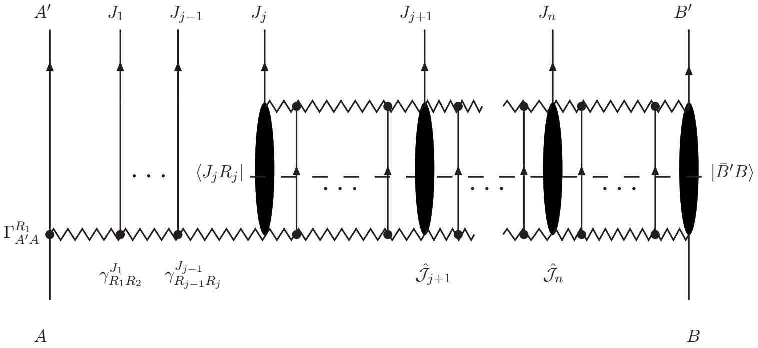
i.e. impact factor for reggeon-gluon transition must be the eigenfunction of the BFKL kernel with the eigenvalue equal to the gluon trajectory. But it follows from the bootstrap conditions

$$\hat{K}|A'A\rangle = \omega(t)|A'A\rangle$$

where $|A'A\rangle$ is the particle-particle impact factor, which evidently differs from the reggeon-particle impact factor.
This contradiction exists for $A_{2\to 2+n}$ at $n \geq 2$.

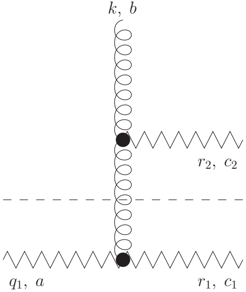
Impact factor in the bootstrap scheme

The impact factor for Reggeon-gluon transition appears as one of components of the last bootstrap condition in partial discontinuites of MRK amplitudes



Impact factor in the bootstrap scheme

In the Born approximation only gluon can be in the intermediate state



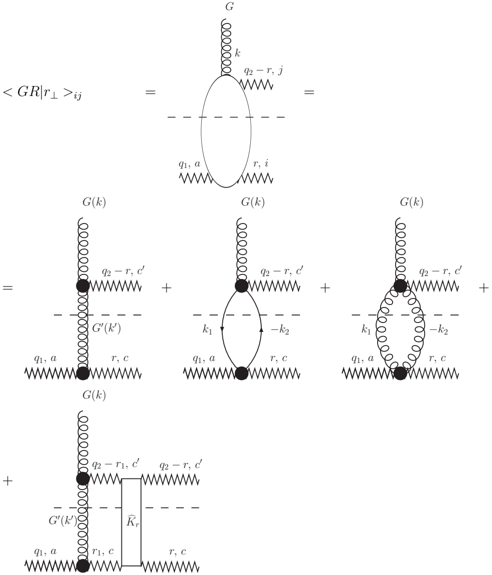
$$\langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle^{(B)} = 2g^2\delta(\vec{q}_1 - \vec{k} - \vec{r}_1 - \vec{r}_2)\left(T^aT^b\right)_{c_1c_2}\ \vec{e}^{\,*}\vec{C}_1,$$

$\vec{e}^{\,*}$ is the conjugated transverse part of the polarization vector $e(k)$ in the gauge $e(k)p_2 = 0$ with the lightcone vector p_2 close to the vector p_B,

$$\vec{C}_1 \ = \ \vec{q}_1 - (\vec{q}_1 - \vec{r}_1)\frac{\vec{q}_1^{\,2}}{(\vec{q}_1 - \vec{r}_1)^2}.$$

Impact factor in the bootstrap scheme

In the NLO



Impact factor in the bootstrap scheme

In N=4 SYM the NLO impact factor contains gluon, fermion and scalar contributions. These contributions were found
M.G. Kozlov, A.V. Reznichenko, V.S. F. 2012,
M.G. Kozlov, A.V. Reznichenko, V.S. F. 2013,
for Yang-Mills theories with any number of fermions and scalars in arbitrary representations of the gauge group.

In general, the impact factors contain two parts with different colour structure. In the planar limit only parts with Born colour structure remain, so that the impact factor can be written as

$$\langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle \;=\; g^2\delta(\vec{q}_1-\vec{k}-\vec{r}_1-\vec{r}_2)\left(T^aT^b\right)_{c_1c_2}\;\vec{e}^{\,*}\left[2\vec{C}_1+\bar{g}^2\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1}\right].$$

The results of
M.G. Kozlov, A.V. Reznichenko, V.S. F. 2012,
M.G. Kozlov, A.V. Reznichenko, V.S. F. 2013,
are in dimensional regularization, which differs from the dimensional reduction used in supersymmetric theories.

Impact factor in the bootstrap scheme

To take into account this difference one has to take the number of the scalar fields n_S equal to $6-2\epsilon$ ($\epsilon=(D-4)/2,\ \ D$ is the space-time dimension).

With account of this, in the planar N=4 SYM

$$\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1*}=\vec{C}_1\left(\ln\left(\frac{(\vec{q}_1-\vec{r}_1)^2}{\vec{k}^{\,2}}\right)\ln\left(\frac{\vec{r}_2^{\,2}}{\vec{k}^{\,2}}\right)+\ln\left(\frac{(\vec{q}_1-\vec{r}_1)^2\vec{q}_1^{\,2}}{\vec{k}^{\,4}}\right)\ln\left(\frac{\vec{r}_1^{\,2}}{\vec{q}_1^{\,2}}\right)\right.$$

$$\left.-4\frac{(\vec{k}^{\,2})^{\epsilon}}{\epsilon^2}+6\zeta(2)\right)+\vec{C}_2\left(\ln\left(\frac{\vec{k}^{\,2}}{\vec{r}_2^{\,2}}\right)\ln\left(\frac{(\vec{q}_1-\vec{r}_1)^2}{\vec{r}_2^{\,2}}\right)\right.$$

$$\left.+\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,2}}{\vec{q}_2^{\,2}}\right)\right)-2\left[\vec{C}_1\times\left(\left[\vec{k}\times\vec{r}_2\right]I_{\vec{k},\vec{r}_2}-\left[\vec{q}_1\times\vec{r}_1\right]I_{\vec{q}_1,-\vec{r}_1}\right)\right]$$

$$\left.+2\left[\vec{C}_2\times\left(\left[\vec{k}\times\vec{r}_2\right]I_{\vec{k},\vec{r}_2}+\left[q_1\times\vec{k}\right]I_{\vec{q}_1,-\vec{k}}\right)\right]\right\}\,.$$

Impact factor in the bootstrap scheme

Here $\bar{g}^2 = g^2\Gamma(1-\epsilon)/(4\pi)^{2+\epsilon}$,

$$\vec{C}_2 \;=\; \vec{q}_1 - \vec{k}\frac{\vec{q}_1^{\,2}}{\vec{k}^{\,2}},$$

$\Gamma(x)$ is the Euler gamma-function, $\zeta(n)$ is the Riemann zeta-function ($\zeta(2) = \pi^2/6$), $\left[\vec{a} \times c\left[\vec{b} \times \vec{c}\right]\right]$ is a double vector product, and

$$I_{\vec{p},\vec{q}} = \int_0^1 \frac{dx}{(\vec{p}+x\vec{q})^2} \ln\left(\frac{\vec{p}^{\,2}}{x^2\vec{q}^{\,2}}\right) \;, \quad I_{\vec{p},\vec{q}} = I_{-\vec{p},-\vec{q}} = I_{\vec{q},\vec{p}} = I_{\vec{p},-\vec{p}-\vec{q}} \,.$$

The NLO correction $\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1*}$ is obtained after huge cancellations between gluon, fermion and scalar contributions. In particular, solely due to these cancellations only two vector structures ($\vec{C}_1$ and $\vec{C}_2$) remain; each of the contributions separately contains three independent vector structures.

Impact factor in the bootstrap scheme

As it is known, NLO corrections are scheme dependent. The scheme used in the derivation of $\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1 *}$ was adjusted simplifying the verification of the bootstrap conditions (we call it bootstrap scheme). It is different from the standard scheme defined in

V.S. F., R. Fiore, M.G. Kozlov, A.V. Reznichenko, 2006.

The impact factors in these schemes are connected by the transformation

$$\langle GR_1|_* = \langle GR_1|_s - \langle GR_1|^{(B)}\widehat{\mathcal{U}}_k,$$

where subscript s means the standard scheme and the the operator $\widehat{\mathcal{U}}_k$ is defined by the matrix elements

$$\langle \mathcal{G}_1'\mathcal{G}_2'|\widehat{\mathcal{U}}_k|\mathcal{G}_1\mathcal{G}_2\rangle = \frac{1}{2}\ln\Big(\frac{\vec{k}^{\,2}}{(\vec{r}_1 - \vec{r}_1{}')^2}\Big)\langle \mathcal{G}_1'\mathcal{G}_2'|\widehat{\mathcal{K}}_r^B|\mathcal{G}_1\mathcal{G}_2\rangle.$$

Here $\widehat{\mathcal{K}}_r^B$ is the part of the LO BFKL kernel related with the real gluon production:

Transformation to the standard scheme

$$\langle \mathcal{G}_1' \mathcal{G}_2' | \widehat{\mathcal{K}}_r^B | \mathcal{G}_1 \mathcal{G}_2 \rangle = \delta(\vec{r}_1{}' + \vec{r}_2{}' - \vec{r}_1 - \vec{r}_2) \frac{g^2}{(2\pi)^{D-1}} T^i_{c_1 c_1'} T^i_{c_2' c_2}$$

$$\times \left(\frac{\vec{r}_1{}^2 \vec{r}_2{}'^{\,2} + \vec{r}_2{}^2 \vec{r}_1{}'^{\,2}}{\vec{l}^{\,2}} - \vec{q}_2{}^2 \right),$$

where $\vec{l} = \vec{r}_1 - \vec{r}_1{}' = \vec{r}_2{}' - \vec{r}_2, \quad \vec{q}_2 = \vec{r}_1 + \vec{r}_2 = \vec{r}_1{}' + \vec{r}_2{}'$.
At large N_c we can write

$$\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1 s} = \vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1 *} + \vec{\mathcal{I}}_1,$$

$$\vec{\mathcal{I}}_1 = \int \frac{d\vec{l}}{\Gamma(1-\epsilon)\pi^{1+\epsilon}} \vec{C}_1{}' \frac{1}{\vec{r}_1{}'^{\,2} \vec{r}_2{}'^{\,2}} \left(\frac{\vec{r}_1{}^2 \vec{r}_2{}'^{\,2} + \vec{r}_2{}^2 \vec{r}_1{}'^{\,2}}{\vec{l}^{\,2}} - \vec{q}_2{}^2 \right) \ln \left(\frac{\vec{k}^{\,2}}{\vec{l}^{\,2}} \right),$$

where

$$\vec{C}_1{}' = \vec{q}_1 - \vec{q}_1{}^2 \frac{(\vec{q}_1 - \vec{r}_1{}')}{(\vec{q}_1 - \vec{r}_1{}')^2}.$$

The integration gives

$$\vec{\mathcal{I}}_1 = \frac{1}{2}\vec{C}_1\left[\ln\left(\frac{\vec{r}_2^{\,2}}{\vec{k}^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,4}}{(\vec{q}_1-\vec{r}_1)^2\vec{r}_2^{\,2}}\right)+\ln\left(\frac{\vec{r}_1^{\,2}}{\vec{k}^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,2}\vec{q}_1^{\,2}}{(\vec{q}_1-\vec{r}_1)^2\vec{r}_1^{\,2}}\right)\right.$$

$$\left.-\ln\left(\frac{\vec{k}^{\,2}}{(\vec{q}_1-\vec{r}_1)^2}\right)\ln\left(\frac{\vec{k}^{\,2}\vec{q}_1^{\,2}}{(\vec{q}_1-\vec{r}_1)^4}\right)+4\frac{(\vec{k}^{\,2})^{\epsilon}}{\epsilon^2}-4\zeta(2)\right]$$

$$-\frac{1}{2}\vec{C}_2\left[\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{r}_2^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,4}}{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}\right)+\ln\left(\frac{(\vec{q}_1-\vec{r}_1)^2}{\vec{q}_1^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,4}}{\vec{r}_1^{\,2}(\vec{q}_1-\vec{r}_1)^2}\right)\right]$$

$$+\left[\vec{C}_1\times\left(\left[\vec{k}\times\vec{r}_2\right]I_{\vec{k},\vec{r}_2}-\left[\vec{q}_1\times\vec{r}_1\right]I_{\vec{q}_1,-\vec{r}_1}\right)\right]$$

$$+\left[\vec{C}_2\times\left(\left[\vec{r}_1\times\vec{r}_2\right]I_{\vec{r}_1,\vec{r}_2}+\left[\vec{q}_1\times\vec{r}_1\right]I_{\vec{q}_1,-\vec{r}_1}\right)\right],$$

Transformation to the standard scheme

and the one-loop correction to the impact factor in the standard scheme

$$\vec{\Phi}^{\mathcal{G}_1\mathcal{G}_2}_{GR_1(s)} = \frac{1}{2}\vec{C}_1\left[\ln\left(\frac{\vec{q}_1^{\,2}}{\vec{r}_1^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,2}\vec{r}_1^{\,2}}{\vec{q}_1^{\,4}}\right)+\ln\left(\frac{\vec{r}_2^{\,2}}{\vec{k}^{\,2}}\right)\ln\left(\frac{(\vec{q}_1-\vec{r}_1)^2}{\vec{r}_2^{\,2}}\right)\right.$$

$$\left.+\ln\left(\frac{(\vec{q}_1-\vec{r}_1)^2}{\vec{k}^{\,2}}\right)\ln\left(\frac{\vec{k}^{\,4}\vec{r}_1^{\,2}}{(\vec{q}_1-\vec{r}_1)^4\vec{q}_1^{\,2}}\right)-4\frac{(\vec{k}^{\,2})^{\epsilon}}{\epsilon^2}+8\zeta(2)\right]$$

$$+\left[\vec{C}_1\times\left(\left[\vec{q}_1\times\vec{r}_1\right]I_{\vec{q}_1,-\vec{r}_1}-\left[\vec{k}\times r_2\right]I_{\vec{k},\vec{r}_2}\right)\right]$$

$$+\frac{1}{2}\vec{C}_2\left[\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}}\right)\ln\left(\frac{\vec{r}_1^{\,2}\vec{r}_2^{\,2}}{\vec{q}_2^{\,4}}\right)+\ln\left(\frac{\vec{r}_2^{\,2}}{(\vec{q}_1-\vec{r}_1)^2}\right)\ln\left(\frac{\vec{r}_2^{\,2}\vec{q}_1^{\,2}}{\vec{r}_1^{\,2}(\vec{q}_1-\vec{r}_1)^2}\right)\right]$$

$$+\left[\vec{C}_2\times\left(\left[\vec{r}_1\times\vec{r}_2\right]I_{\vec{r}_1,\vec{r}_2}+\left[\vec{q}_1,\vec{r}_1\right]I_{\vec{q}_1,-\vec{r}_1}+2\left[\vec{k},\vec{r}_2\right]I_{\vec{k},\vec{r}_2}+2\left[\vec{q}_1,\vec{k}\right]I_{\vec{q}_1,-\vec{k}}\right)\right].$$

The impact factor in the Möbius scheme

This correction corresponds to the standard kernel and the energy scale $|\vec{k}_1||\vec{k}_2|$, where $\vec{k}_{1,2}$ are the transverse momenta of produced gluons in the two impact factors connected by the Green function of the two interacting Reggeons (BFKL ladder). Our goal is the impact factor in the Möbius scheme, that means the impact factor for Reggeon-gluon transition which can be used for the calculation of the remainder function with conformal invariant kernel and energy evolution parameter. Let us remind here that the kernel used for the calculation of the remainder function (which is called modified kernel $\hat{\mathcal{K}}_m$) is the BFKL kernel in $N = 4$ SYM for the adjoint representation of the gauge group with subtracted gluon trajectory depending on the total momentum transfer, $\hat{\mathcal{K}}_m = \hat{\mathcal{K}} - \omega(t)$ (the subtraction is made to avoid double counting of terms included in the BDS ansatz).

To obtain the impact factor in the Möbius scheme we have to perform two transformations, to reconcile the impact factor with

The impact factor in the Möbius scheme

As it was shown, the conformal invariant $\hat{\mathcal{K}}_c$ and standard $\hat{\mathcal{K}}_m$ forms of the modified kernel are connected by the similarity transformation

$$\hat{\mathcal{K}}_c = \hat{\mathcal{K}}_m + \frac{1}{4}\left[\hat{\mathcal{K}}^B, \left[\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right), \hat{\mathcal{K}}^B\right]\right] ,$$

where $\hat{\mathcal{K}}^B$ is the usual LO kernel and $\hat{\vec{q}}_{1,2}$ are the operators of Reggeon momenta. Note that in the commutator there is no difference between usual and modifided kernels, so that $\hat{\mathcal{K}}^B$ is taken instead of $\hat{\mathcal{K}}^B_m$. The corresponding transformation for the impact factor is

$$\langle GR_1|_t = \langle GR_1|_s - \frac{1}{4}\langle GR_1|^{(B)}\left[\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right), \hat{\mathcal{K}}^{(B)}\right] ,$$

where the subscript t means transformed to fit the conformal kernel.

The impact factor in the Möbius scheme

For the NLO correction we obtain

$$\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1 t} = \vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1 s} + \vec{\mathcal{I}}_2,$$

$$\vec{\mathcal{I}}_2 = \int \frac{d\vec{l}}{\Gamma(1-\epsilon)\pi^{1+\epsilon}} \vec{C}_1{}' \frac{1}{\vec{r}_1{}'^{\,2}\vec{r}_2{}'^{\,2}} \left(\frac{\vec{r}_1{}^2\vec{r}_2{}'^{\,2} + \vec{r}_2{}^2\vec{r}_1{}'^{\,2}}{\vec{l}^{\,2}} - \vec{q}_2^{\,2} \right) \ln\left(\frac{\vec{r}_1{}^2\vec{r}_2{}^2}{\vec{r}_1{}'^{\,2}\vec{r}_2{}'^{\,2}} \right).$$

This integral is infrared finite and can be calculated in two-dimensional space, with the help of the decomposition used before, the decomposition

$$\frac{1}{\vec{r}_1{}'^{\,2}\vec{r}_2{}'^{\,2}} \left(\frac{\vec{r}_1{}^2\vec{r}_2{}'^{\,2} + \vec{r}_2{}^2\vec{r}_1{}'^{\,2}}{\vec{l}^{\,2}} - \vec{q}_2^{\,2} \right) = -\left(\frac{1}{(r_1-l)^+} + \frac{1}{l^+} \right)$$

$$\times \left(\frac{1}{(r_2+l)^-} - \frac{1}{l^-} \right) - \left(\frac{1}{(r_1-l)^-} + \frac{1}{l^-} \right) \left(\frac{1}{(r_2+l)^+} - \frac{1}{l^+} \right).$$

The impact factor in the Möbius scheme

The transformed impact factor takes the form:

$$\vec{\Phi}^{\mathcal{G}_1\mathcal{G}_2}_{GR_1t} = \vec{C}_1 \left[\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}}\right) \ln\left(\frac{\vec{q}_2^{\,4}(\vec{q}_1-\vec{r}_1)^4}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}\vec{k}^{\,2}\vec{r}_1^{\,2}}\right) - \ln\left(\frac{\vec{q}_2^{\,2}(\vec{q}_1-\vec{r}_1)^2}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}}\right)\right.$$

$$\left. \times \ln\left(\frac{\vec{q}_2^{\,2}(\vec{q}_1-\vec{r}_1)^2}{\vec{k}^{\,2}\vec{r}_1^{\,2}}\right) \frac{3}{4}\ln^2\left(\frac{\vec{k}^{\,2}\vec{r}_1^{\,2}}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}}\right) - \ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right) - 2\frac{(\vec{k}^{\,2})^\epsilon}{\epsilon^2} + 4\zeta(2)\right]$$

$$+\frac{1}{4}\vec{C}_2 \ln\left(\frac{\vec{q}_2^{\,2}(\vec{q}_1-\vec{r}_1)^2}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}}\right) \ln\left(\frac{(\vec{q}_1-\vec{r}_1)^4\vec{q}_1^{\,2}\vec{k}^{\,2}\vec{r}_1^{\,2}}{\vec{r}_2^{\,6}\vec{q}_2^{\,4}}\right)$$

$$+\frac{3}{2}\left[\vec{C}_2 \times \left(\left[\vec{r}_1\times\vec{r}_2\right] I_{\vec{r}_1,\vec{r}_2} + \left[\vec{q}_1\times\vec{r}_1\right] I_{\vec{q}_1,-\vec{r}_1} + \left[\vec{k}\times\vec{r}_2\right] I_{\vec{k},\vec{r}_2} + \left[\vec{q}_1\times\vec{k}\right] I_{\vec{q}_1,-\vec{k}} \right)\right] .$$

The impact factor in the Möbius scheme

The Möbius invariant kernel was used for the calculation of the NLO remainder function with the Möbius invariant convolution of the NLO BFKL impact factors (which was called for brevity simply impact factor) obtained from direct two-loop calculations and with the energy scale s_0 chosen in such a way that the ratio (energy evolution parameter) $s/s_0 = s\vec{q}_2^{\,2}/\sqrt{\vec{q}_1^{\,2}\vec{q}_3^{\,2}\vec{k}_1^{\,2}\vec{k}_2^{\,2}}$ is Möbius invariant. This energy scale differs from the energy scale used in the standard definition of the impact factor which is equal $|\vec{k}_1||\vec{k}_2|$. To adjust the impact factor to the energy scale $\sqrt{\vec{q}_1^{\,2}\vec{q}_3^{\,2}\vec{k}_1^{\,2}\vec{k}_2^{\,2}}/\vec{q}_2^{\,2}$, we need to perform an additional transformation:

$$\langle GR_1|_t \to \langle GR_1|_c = \langle GR_1|_t - \frac{1}{2}\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}}\right)\langle GR_1|^{(B)}\hat{\mathcal{K}}_m^{(B)}|\mathcal{G}_1\mathcal{G}_2\rangle\,,$$

The impact factor in the Möbius scheme

where the subscript c means transformed to fit the conformal energy scale and $\hat{\mathcal{K}}_m^{(B)}$ is the modified LO kernel. Let

$$\vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1 c} = \vec{\Phi}^{\mathcal{G}\mathcal{G}_2}_{GR_1 t} + \vec{\mathcal{I}}_3,$$

then the integral for $\vec{\mathcal{I}}_3$ can be written as

$$\vec{\mathcal{I}}_3 = -\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}_1^{\,2}}\right)\int \frac{d\vec{l}}{\pi}\left(\vec{C}_1' - \vec{C}_1\right)\frac{1}{\vec{r}_1^{\,\prime\,2}\vec{r}_2^{\,\prime\,2}}\left(\frac{\vec{r}_1^{\,2}\vec{r}_2^{\,\prime\,2} + \vec{r}_2^{\,2}\vec{r}_1^{\,\prime\,2}}{\vec{l}^{\,2}} - \vec{q}_2^{\,2}\right),$$

where instead of $\vec{C}_1'$ the difference $\left(\vec{C}_1' - \vec{C}_1\right)$ is taken and instead of the full modified kernel only its part related to real gluon production is kept. Moreover, the integral is written in two-dimensional transverse space.

The impact factor in the Möbius scheme

This result gives

$$\vec{\Phi}^{\mathcal{G}_1\mathcal{G}_2}_{GR_1c} = \vec{C}_1\left[-\ln\left(\frac{\vec{q}_2^{\,2}(\vec{q}_1-\vec{r}_1)^2}{\vec{r}_1^{\,2}\vec{k}^{\,2}}\right)\ln\left(\frac{\vec{q}_2^{\,2}(\vec{q}_1-\vec{r}_1)^2}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}}\right)-\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right)\right.$$

$$\left.-\frac{3}{4}\ln^2\left(\frac{\vec{k}^{\,2}\vec{r}_1^{\,2}}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}}\right)-2\frac{(\vec{k}^{\,2})^{\epsilon}}{\epsilon^2}+4\zeta(2)\right]$$

$$+\frac{1}{4}\vec{C}_2\ln\left(\frac{\vec{q}_2^{\,2}(\vec{q}_1-\vec{r}_1)^2}{\vec{q}_1^{\,2}\vec{r}_2^{\,2}}\right)\ln\left(\frac{\vec{q}_2^{\,4}(\vec{q}_1-\vec{r}_1)^4\vec{r}_1^{\,2}\vec{k}^{\,2}}{\vec{r}_2^{\,6}\vec{q}_1^{\,6}}\right)$$

$$+\frac{3}{2}\left[\vec{C}_2\times\left(\left[\vec{r}_1\times\vec{r}_2\right]I_{\vec{r}_1,\vec{r}_2}+\left[\vec{q}_1\times\vec{r}_1\right]I_{\vec{q}_1,-\vec{r}_1}+\left[\vec{k}\times\vec{r}_2\right]I_{\vec{k},\vec{r}_2}+\left[\vec{q}_1\times\vec{k}\right]I_{\vec{q}_1,-\vec{k}}\right)\right].$$

This expression gives us the NLO correction to the impact factor for Reggeon-gluon transition in the scheme with conformal kernel and energy evolution parameter, which were used for the calculation of the remainder function.

The impact factor in the Möbius scheme

However, it is the impact factor for the full amplitude, not for the remainder function. To obtain the impact factor for the remainder function we have to take the impact factor with the correction $\Phi^{\mathcal{G}_1\mathcal{G}_2}_{GR_1c}$ and with the polarisation vector $\vec{e}^{\,*}$ of definite helicity, and to extract from it the piece included in the BDS ansatz.

Let us consider, for definiteness, the production of a gluon with positive helicity, $\vec{e}^{\,*} = (\vec{e}_x - i\vec{e}_y)/\sqrt{2}$. Then,

$$\vec{e}^{\,*}\vec{C}_1 = -\frac{q_1^- r_1^+}{\sqrt{2}(q_1 - r_1)^+}, \quad \vec{e}^{\,*}\vec{C}_2 = -\frac{q_1^- q_2^+}{\sqrt{2}k^+}, \quad \frac{\vec{e}^{\,*}\vec{C}_2}{\vec{e}^{\,*}\vec{C}_1} = 1 - z,$$

where the ratio $z = -q_1^+ r_2^+/(k^+ r_1^+)$ is conformal invariant, i.e. invariant with respect to Möbius transformations of complex variables p_i such that
$r_1^+ = p_1 - p_2, r_2^+ = p_2 - p_3, -q_1^+ = p_3 - p_4, k_1^+ = p_4 - p_1)$.

The impact factor in the Möbius scheme

As the result, after some algebra we obtain

$$\langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle = \langle GR_1|\mathcal{G}_1\mathcal{G}_2\rangle^{(B)}\left\{1+\frac{\bar{g}^2}{8}\left[(1-z)\right.\right.$$

$$\times\left(\ln\left(\frac{|1-z|^2}{|z|^2}\right)\ln\left(\frac{|1-z|^4}{|z|^6}\right)-3Li_2(z)+3Li_2(z^*)-\frac{3}{2}\ln|z|^2\ln\frac{1-z}{1-z^*}\right)$$

$$-4\ln|1-z|^2\ln\frac{|1-z|^2}{|z|^2}-3\ln^2|z|^2$$

$$\left.\left.-4\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right)-8\frac{(\vec{k}^{\,2})^\epsilon}{\epsilon^2}+16\zeta(2)\right]\right\}.$$

Finally, in order to move to the impact factor for calculation of the remainder function, one has to discard the terms $\bar{g}^2\left(-(1/2)\ln^2\left(\vec{q}_1^{\,2}/\vec{q}_2^{\,2}\right)-(\vec{k}^{\,2})^\epsilon/\epsilon^2+2\zeta(2)\right)$, since they are already taken into account in the BDS ansatz.

The remainder is evidently conformal invariant.

The NLO discontinuity of the $A_{2\to 4}$ amplitude

For calculation of $\langle G_1 R_1 | e^{\hat{K}\ln\left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)} | G_2 R_3 \rangle$ it is convenient to transfer from the BFKL kernel $\hat{K}$ to the modified BFKL kernel $\hat{K}_m$ introduced in
J. Bartels, L. N. Lipatov and A. Sabio Vera, 2009

,

$$\hat{K}_m = \hat{K} - \omega(t).$$

An evident advantage of this kernel is non-singular infrared behavior.
Not so evident, but even more important is conformal invariance in the momentum space. In the LO this invariance is almost obvious J. Bartels, L. N. Lipatov and A. Sabio Vera, 2009
. Existence of the conformal invariant representation of the NLO kernel was proved recently
V. S. F. , R. Fiore, L. N. Lipatov and A. Papa, 2013.

The NLO discontinuity of the $A_{2\to 4}$ amplitude

The impact factors for reggeon-gluon transitions in this scheme with conformal invariant energy evolution parameter $\frac{s_2\vec{q}_2^{\,2}}{|\vec{q}_1||\vec{q}_3||\vec{k}_1||\vec{k}_2|}$ was also recently found
V. S. F. , R. Fiore, 2014.
They can be written with the NLO accuracy as

$$\langle G_1 R_1|\mathbf{G}_1\mathbf{G}_2\rangle = -\sqrt{2}g^2\delta(\vec{q}_1-\vec{k}_1-\vec{r}_1-\vec{r}_2)\frac{q_1^- r_1^+}{(q_1-r_1)^+}[1+\bar{g}^2 I(z_1)]$$

$$\times\left[1+\bar{g}^2\left(-\frac{1}{2}\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right)-\frac{(\vec{k}^{\,2})^\epsilon}{\epsilon^2}+2\zeta(2)\right)\right],$$

$$\langle \mathbf{G}_1'\mathbf{G}_2'|G_2 R_3\rangle = \sqrt{2}g^2\delta(\vec{r}_1{}'+\vec{r}_2{}'-\vec{q}_3-\vec{k}_2)\frac{q_3^+ r_1'^-}{(r_1'-q_3)^-}[1+\bar{g}^2 I^*(z_2)]$$

$$\times\left[1+\bar{g}^2\left(-\frac{1}{2}\ln^2\left(\frac{\vec{q}_3^{\,2}}{\vec{q}_2^{\,2}}\right)-\frac{(\vec{k}^{\,2})_2^\epsilon}{\epsilon^2}+2\zeta(2)\right)\right],$$

where $z_1 = -q_1^+ r_2^+/(k_1^+ r_1^+)$, $z_2 = q_3^+ r_2'^+/(k_2^+ r_1'^+)$

The NLO discontinuity of the $A_{2\to4}$ amplitude

$$I(z) = (1-z)\left(\ln\left(\frac{|1-z|^2}{|z|^2}\right)\ln\left(\frac{|1-z|^4}{|z|^6}\right) - 6Li_2(z) + 6Li_2(z^*)\right.$$

$$\left.-3\ln|z|^2\ln\frac{1-z}{1-z^*}\right) - 4\ln|1-z|^2\ln\frac{|1-z|^2}{|z|^2} - 3\ln^2|z|^2 \,.$$

In terms of eigenstates $|\nu, n\rangle$ and eigenvalues $\omega(\nu, n)$ of $\hat{K}_m$

$$\langle G_1R_1|e^{\hat{K}\ln\left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)}|G_2R_3\rangle = \left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)^{\omega(t_2)}$$

$$\times\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\nu\,\langle G_1R_1|\nu, n\rangle e^{\omega(\nu,n)\ln\left(\frac{s_2\vec{q}_2^{\,2}}{|\vec{q}_1||\vec{q}_3||\vec{k}_1||\vec{k}_2|}\right)}\langle\nu, n|G_2R_3\rangle \,.$$

The eigenfunctions are well known. The eigenvalues $\omega(\nu, n)$ with the NLO accuracy were found two years ago V. S. F. , L. N. Lipatov, 2012.

In an explicit form

$$\langle G_1 R_1 | e^{\hat{K}\ln\left(\frac{s_2}{|\vec{k}_1||\vec{k}_2|}\right)} | G_2 R_3 \rangle = \delta(\vec{q}_1 - \vec{k}_1 - \vec{k}_2 - \vec{q}_3) g^2 \gamma_{R_1 R_2}^{G_1(B)} \gamma_{R_2 R_3}^{G_2(B)}$$

$$\times \left[1 + \bar{g}^2 \left(-\frac{1}{2}\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}}\right) - \frac{(\vec{k}_1^{\,2})^\epsilon}{\epsilon^2} - \frac{1}{2}\ln^2\left(\frac{\vec{q}_3^{\,2}}{\vec{q}_2^{\,2}}\right) - \frac{(\vec{k}_2^{\,2})^\epsilon}{\epsilon^2} + 4\zeta(2) \right) \right]$$

$$\times \frac{1}{2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu \, e^{\omega(\nu,n)\ln\left(\frac{s_2 \vec{q}_2^{\,2}}{|\vec{q}_1||\vec{q}_3||\vec{k}_1||\vec{k}_2|}\right)} w^{\frac{n}{2}+i\nu} (w^*)^{-\frac{n}{2}+i\nu}$$

$$\int \frac{dz_1}{\pi |z_1|^2} \frac{1}{1 - z_1} \left(1 + \bar{g}^2 I(z_1)\right) z_1^{\frac{n}{2}+i\nu} (z_1^*)^{-\frac{n}{2}+i\nu}$$

$$\int \frac{dz_2}{\pi |z_2|^2} \frac{1}{1 - z_2^*} \left(1 + \bar{g}^2 I^*(z_2)\right) (z_2^*)^{\frac{n}{2}-i\nu} z_2^{-\frac{n}{2}-i\nu}$$

where $w = k_2^+ q_1^+ / (k_1^+ q_3^+)$.

Summary

- The impact factors for Reggeon-gluon transition are an integral part of the discontinuities of multi-Regge amplitudes.
- Formal expressions for s_i-channel discontinuities of MRK amplitudes in the NLA are known since 2006.
- Fulfilment of all bootstrap conditions is proved.
- The LLA discontinuities are in an evident contradiction with the BDS anzatz for $2 \to 2 + n$ amplitudes at $n \geq 2$ even in the LLA.
- The NLO impact factors are known now in Yang-Mills theories with any number of fermions and scalars in arbitrary representations of the gauge group.
- The discontinuity in invariant mass of two produced gluons is calculated in the NLA in planar $N = 4$ SYM.
- It's compatibility with the BDS ansatz corrected by the remainder factor is under consideration.