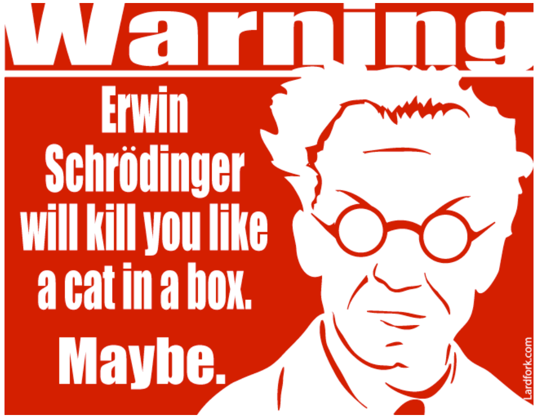
Quantum Mechanics

$$E - \frac{p^2}{2m} = 0$$

Classical – non relativistic

$$i\hbar\frac{\partial\phi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\phi = 0$$

Quantum Mechanical : Schrodinger eq



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$$i\hbar \frac{\partial \phi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \phi = 0$$

Quantum Mechanical : Schrodinger eq

$$E^2 - \mathbf{p}^2 = m^2$$

Classical – relativistic (Natural units $\hbar = c = 1$)

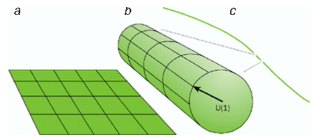
$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$

Quantum Mechanical - relativistic :

Klein-Gordon (Schrodinger) equation

Relativistic QM - The Klein Gordon equation (1926)

Scalar particle (field) (J=0) : $\phi(\mathbf{x})$



$$E^2 = \mathbf{p^2} + m^2 \quad \Rightarrow \quad -\frac{\partial^2 \phi}{\partial t^2} + \nabla^{\mathbf{2}} \phi = m^2 \phi$$

(natural units)

Energy eigenvalues $E = \pm(\mathbf{p^2} + m^2)^{1/2}$???

1927 Dirac tried to eliminate negative solutions by writing a relativistic equation linear in E (a theory of fermions)

1934 Pauli and Weisskopf revived KG equation with E<0 solutions as E>0 solutions for particles of opposite charge (antiparticles). Unlike Dirac’s hole theory this interpretation is applicable to bosons (integer spin) as well as to fermions (half integer spin).

As we shall see the antiparticle states make the field theory causal

Physical interpretation of Quantum Mechanics

Schrödinger equation (S.E.)

$$i\frac{\partial\phi}{\partial t} + \frac{1}{2m}\nabla^2\phi = 0$$

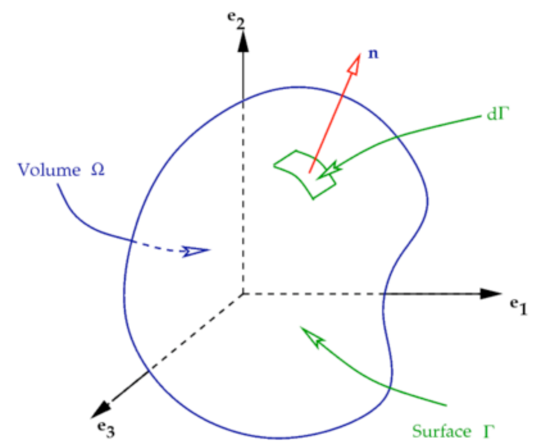
$$i\phi^*(S.E.) - i\phi(S.E.)^* \quad \Rightarrow \quad \frac{\partial\rho}{\partial t} + \nabla.\mathbf{j} = 0 \text{ continuity eq.}$$

$$\rho = |\phi|^2$$

"probability density"

$$\mathbf{j} = -\frac{i}{2m}\left(\phi^*\nabla\phi - \phi\nabla\phi^*\right)$$

"probability current"



Divergence Theorem

$$\iiint_\Omega \nabla.\mathrm{M}\ \mathrm{dV} = \iint_\Gamma M.n\ dS$$

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Klein Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2}+\nabla^2\phi=m^2\phi$$

$$\rho=2E|N|^2$$

Negative probability?

$$\rho=i\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)$$

$$\mathbf{j}=-i(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

$$j^\mu=(\rho,\mathbf{j})$$

$$\phi=Ne^{-ip.x},\quad \rho=2E|N|^2$$

$$\int_V\rho\,dV=\int\rho\,d^3x=2E$$

$$f_p^\pm=e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}}$$

Normalised free particle solutions

Physical interpretation of Quantum Mechanics

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Klein Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2}+\nabla^2\phi=m^2\phi$$

$$\rho=i(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t})$$

$$\mathbf{j}=-i(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

$$j^\mu=(\rho,\mathbf{j})$$

Pauli and Weisskopf

$$j^\mu\rightarrow e(\rho,\mathbf{j})=j^\mu_{EM}$$

$$\frac{\partial\rho}{\partial t}+\nabla.\mathbf{j}=0=\partial^\mu j_\mu$$

Physical interpretation of Quantum Mechanics

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Klein Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = m^2\phi$$

$$\rho_{EM} = i\left(\phi^*\frac{\partial \phi}{\partial t} - \phi\frac{\partial \phi^*}{\partial t}\right)$$

$$\mathbf{j} = -i(\phi^*\nabla\phi - \phi\nabla\phi^*)$$

$$j^\mu = (\rho, \mathbf{j})$$

Pauli and Weisskopf

$$j^\mu \to e(\rho, \mathbf{j}) = j^\mu_{EM}$$

$$\frac{\partial \rho}{\partial t} + \nabla.\mathbf{j} = 0 = \partial^\mu j_\mu$$

Relativistic QM - The Klein Gordon equation (1926)

Scalar particle (field) (J=0) : $\phi(\mathbf{x})$

$$E^2 = \mathbf{p}^2 + m^2 \quad \Rightarrow \quad -\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$

(natural units)

Relativistic notation :

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

4 vector notation

$$A^{\mu} = (A^0, \underline{A}), \quad B^{\mu} = (B^0, \underline{B})$$ contravariant

$$A_{\mu} = (A^0, \underline{-A}), \quad B_{\mu} = (B^0, -\underline{B})$$ covariant

$$A_{\mu} = g_{\mu\nu} A^{\nu} \qquad A^{\mu} = g^{\mu\nu} A_{\nu}$$

$$A.B = A_{\mu} B^{\mu} = A^{\mu} B_{\mu} = A^0 B^0 - \underline{A}.\underline{B}$$

4 vectors

$$(ct, \underline{x}) \equiv x^{\mu} \qquad (\frac{E}{c}, \underline{p}) \equiv p^{\mu}$$

$$\partial^{\mu} = (\frac{\partial}{c\partial t}, -\underline{\nabla}) \qquad \partial_{\mu} = (\frac{\partial}{c\partial t}, \underline{\nabla})$$

$$p^{\mu} \to i\hbar\partial^{\mu} \qquad p_{\mu} p^{\mu} = E^2 - \underline{p}^2 \quad \to \quad -\Box^2 \equiv \partial_{\mu}\partial^{\mu}$$

Field theory of $\pi^{\pm}$

Scalar particle – satisfies KG equation

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

- Classical electrodynamics, motion of charge –e in EM potential $\mathrm{A}^\mu = (\mathrm{A}^0, \mathbf{A})$

is obtained by the substitution : $p^\mu \to p^\mu + eA^\mu$

- Quantum mechanics : $i\partial^\mu \to i\partial^\mu + eA^\mu$

The Klein Gordon equation becomes:

$$(\partial_\mu \partial^\mu + m^2)\phi = -V\phi \quad where \quad V = -ie(\partial_\mu A^\mu + A^\mu \partial_\mu) - e^2 A^2$$

The smallness of the EM coupling, $\alpha_{em} = \frac{e^2}{4\pi} \sim \frac{1}{137}$, means that it is sensible to

Make a "perturbation" expansion of V in powers of α_{em}

$$V \simeq -ie(\partial_\mu A^\mu + A^\mu \partial_\mu)$$

The pion propagator

Want to solve :

$$(\partial_\mu \partial^\mu + m^2)\psi = -V\psi$$

Solution :

$$\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x)V(x')\psi(x')$$

where

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

and

$$(\partial_\mu \partial^\mu + m^2)\Delta_F(x'-x) = \delta^4(x'-x)$$

Feynman propagator

Dirac Delta function

$$\int d^4x' \delta^4(x'-x) f(x') = f(x)$$

The pion propagator

Want to solve : $$(\partial_\mu \partial^\mu + m^2)\psi = -V\psi$$

Solution : $$\boxed{\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x) V(x')\psi(x')}$$

where $$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

and $$(\partial_\mu \partial^\mu + m^2)\Delta_F(x'-x) = \delta^4(x'-x)$$

Simplest to solve for propagator in momentum space by taking Fourier transform

$$\frac{1}{(2\pi)^2}\int e^{-ip.(x'-x)} \quad (\partial_\mu \partial^\mu + m^2)\Delta_F(x'-x) d^4(x'-x) = \frac{1}{(2\pi)^2}\int e^{-ip.(x'-x)}\delta^4(x'-x) d^4(x'-x)$$

$$\Rightarrow \quad (-p^2 + m^2)\Delta_F(p) = \frac{1}{(2\pi)^2}$$

$$\boxed{\tilde{\Delta}_F(p) = \frac{1}{(2\pi)^2}\frac{1}{-p^2 + m^2 + i\varepsilon}, \qquad \Delta_F(x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.x}\frac{1}{p^2 - m^2 - i\varepsilon}}$$

The Born series

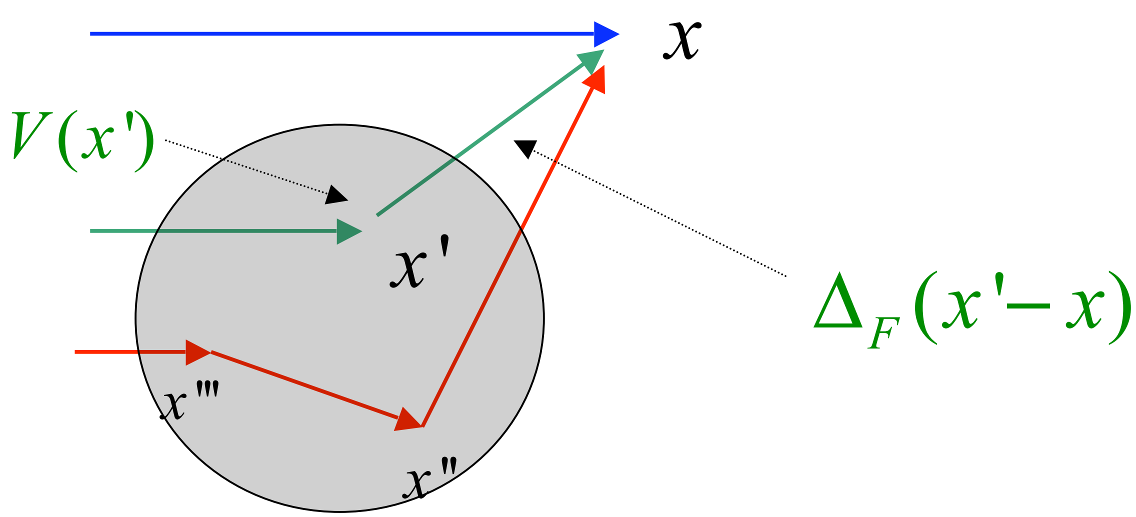
$$\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x) V(x') \psi(x')$$

Since V(x) is small can solve this equation iteratively :

$$\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x) V(x') \phi(x')$$

$$+ \int d^4x'' \int d^4x''' \Delta_F(x''-x) V(x'') \Delta_F(x'''-x'') V(x''') \phi(x''') + \ldots$$

Interpretation :



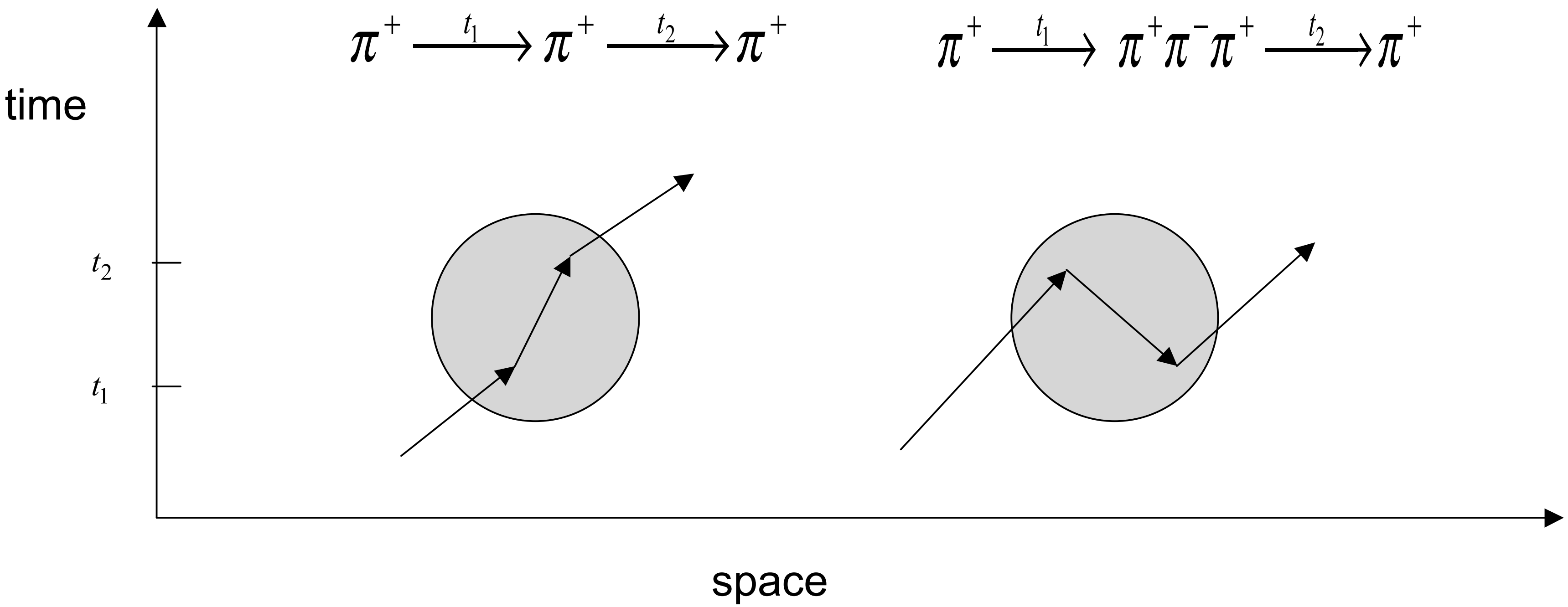
But energy eigenvalues $E = \pm(\mathbf{p}^2 + m^2)^{1/2}$???

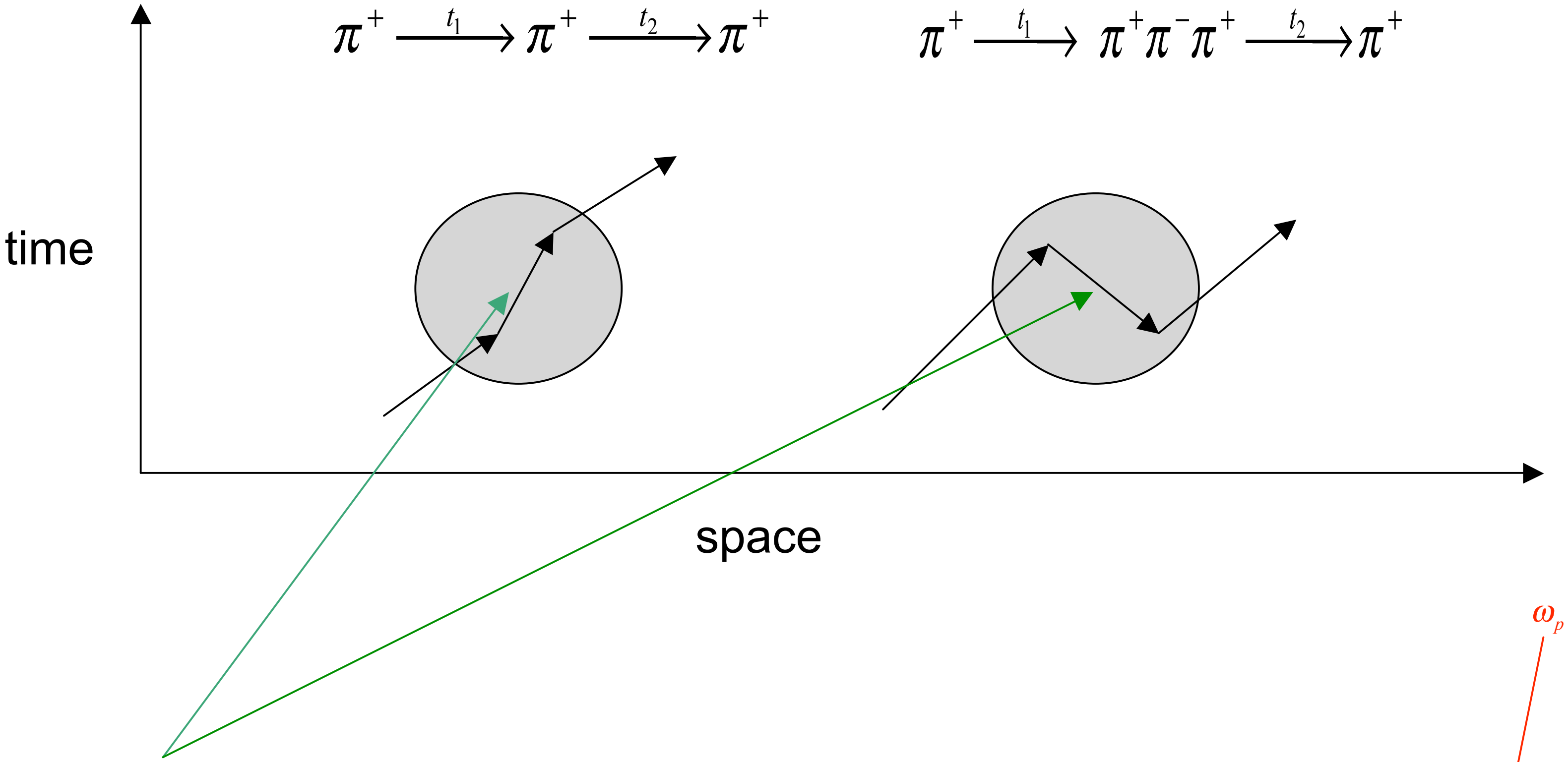
Feynman – Stuckelberg interpretation

t ↑ $\qquad \pi^+(E > 0)$ ↑ $\qquad \equiv \qquad \pi^-(E < 0)$ ↓

$$e^{-iEt} \qquad\qquad e^{-i(-E)(-t)}$$

Two different time orderings giving same observable event :





$$\omega_p = \left(\underline{p}^2 + m^2\right)^{1/2}$$

$$\Delta_F(x'-x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.(x'-x)}\,\frac{1}{p^2-m^2-i\varepsilon} = -i\int \frac{d^3p}{(2\pi)^3\, 2\omega_p}\, e^{-i\omega_p|t'-t|-i\mathbf{p}.(\mathbf{x'}-\mathbf{x})}$$

(p^0 integral most conveniently evaluated using contour integration via Cauchy's theorem)

$$\Delta_F(x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.x}\,\frac{1}{p^2-m^2-i\varepsilon}$$

$$p^2+m^2-i\varepsilon \quad\Rightarrow\quad p_o^2 = \underline{p}^2+m^2-i\varepsilon \quad\Rightarrow\quad p_0 = \pm\left(\underline{p}^2+m^2\right)^{1/2} \mp i\delta = \pm\omega_p \mp i\delta$$

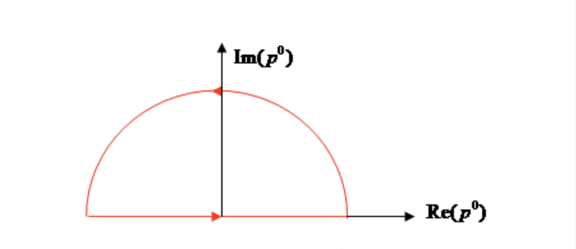
$$\Delta_F(x'-x) = -\frac{1}{(2\pi)^4}\int d^3p\, e^{-i\underline{p}.(\underline{x}'-\underline{x})}\underbrace{\int dp_0 \frac{e^{-ip_0(t'-t)}}{\left(p_0-(\omega_p-i\delta)\right)\left(p_0-(-\omega_p+i\delta)\right)}}_{I}$$

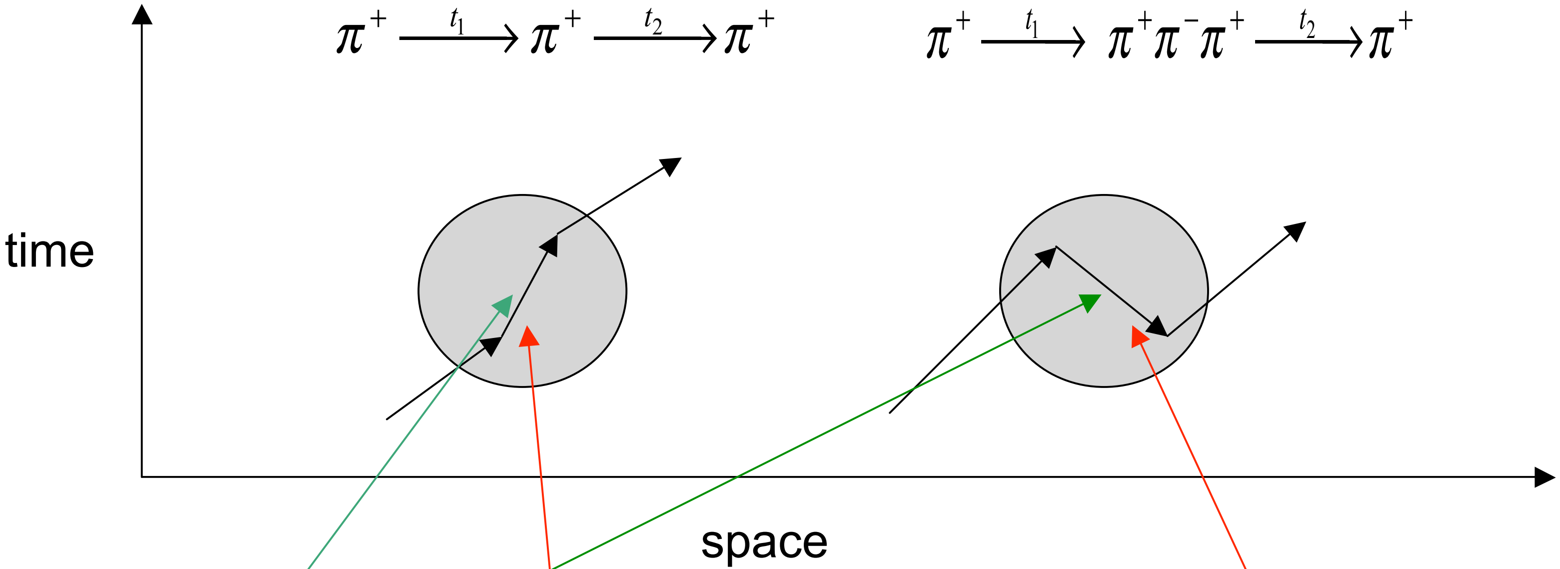
- If $t'-t>0$, choose contour such that $p_0 = -ip_I \quad (p_I\ +\text{ve}) \Rightarrow e^{-ip_0(t'-t)} = e^{-p_I(t'-t)}$

$$I = -\frac{\pi i}{\omega_p}e^{-i\omega_p(t'-t)}\theta(t'-t)$$

- If $t'-t<0$, choose contour such that $p_0 = +ip_I \quad (p_I\ +\text{ve}) \Rightarrow e^{+ip_0(t'-t)} = e^{-p_I(t'-t)}$

$$I = -\frac{\pi i}{\omega_p}e^{+i\omega_p(t'-t)}\theta(t-t')$$





$$\Delta_F(x'-x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.(x'-x)} \frac{1}{p^2-m^2-i\varepsilon} = -i\int \frac{d^3p}{(2\pi)^3\, 2\omega_p} e^{-i\omega_p|t'-t|-ip.(x'-x)}$$

$$\Delta_F(x-x') = -i\int d^3p\, f_p^+(x')\, f_p^{+*}(x)\,\theta(t'-t) - i\int d^3p\, f_p^-(x')\, f_p^{-*}(x)\,\theta(t-t')$$

where $f_p^{\pm} = e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}}$ are positive and negative energy solutions to free KG equation