The inclusion of fermions – J=1/2 particles

Weyl spinors

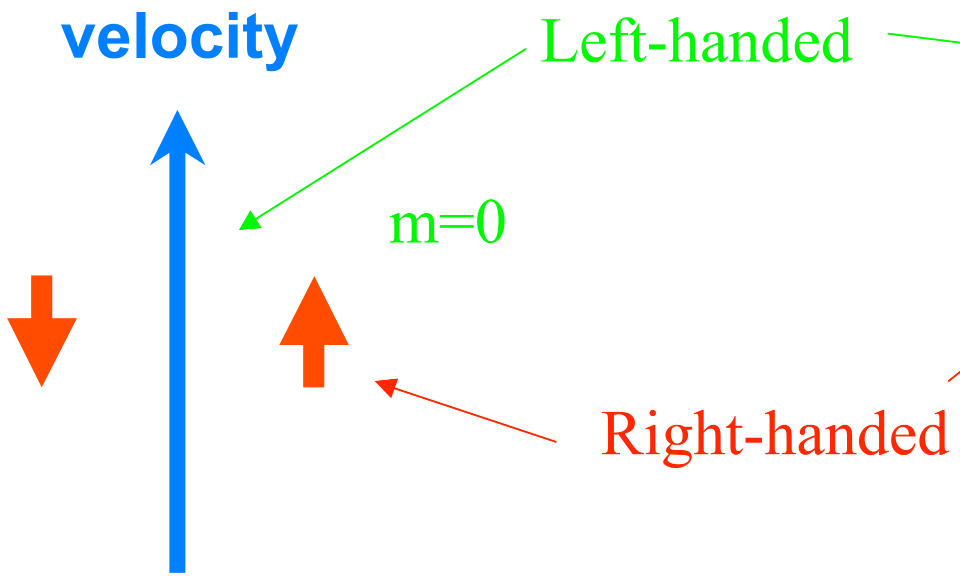
$(\frac{1}{2},0) \quad (0,\frac{1}{2})$

$\psi_L \qquad \psi_R$

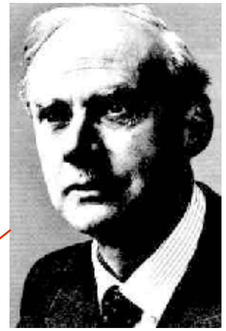
2-component spinors of SU(2) $\begin{pmatrix} a(x) \\ b(x) \end{pmatrix}$

Dirac spinor

$$\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$$



$$u_{e^-_L} \equiv u^c_{e^+_R}$$



The spinor structure follows from the representation structure of the Lorentz Group …

- **The Lorentz transformations form a group, G** $(g_1 g_2 \in G \quad if \quad g_1, g_2 \in G)$

Rotations

$$R(\theta) = e^{-i\mathbf{J}.\theta/\hbar}, \quad J_z = i\hbar(x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x})..$$

Angular momentum operator

$$(c.f.\, \mathbf{J} = \mathbf{r} \times \mathbf{p})$$

$$[J_i, J_j] = i\hbar \sum_{k=1}^{3} \varepsilon_{ijk} J_k$$

$$SO(3) \quad (SU(2))$$

$$e.g. \quad Spin \quad R(\theta)\begin{pmatrix}1\\0\end{pmatrix} = e^{-i\alpha.\sigma/2\hbar}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \quad \sigma_2 = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \quad \sigma_3 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

The Lorentz group

Rotations J_i Boosts K_i

$$[J_i, J_j] = i\varepsilon_{ijk} J_k$$
$$[J_i, K_j] = i\varepsilon_{ijk} K_k$$
$$[K_i, K_j] = -i\varepsilon_{ijk} J_k$$

} Generate the group SO(3,1)

$$\left(M_{\rho\sigma} = i\left(x_\rho \frac{\partial}{\partial x^\sigma} - x_\sigma \frac{\partial}{\partial x^\rho}\right) \qquad J_i = \tfrac{1}{2}\varepsilon_{ijk} M_{jk} \qquad K_i = M_{0i}\right)$$

To construct representations a more convenient (non-Hermitian) basis is

$$N_i = \tfrac{1}{2}(J_i + iK_i)$$

$$[N_i, N_j] = i\varepsilon_{ijk} N_k$$
$$[N_i^\dagger, N_j^\dagger] = i\varepsilon_{ijk} N_k^\dagger$$
$$[N_i, N_j^\dagger] = 0$$

} $SU(2) \otimes SU(2)$ *representation* (n, m)

The Lorentz group

Rotations J_i Boosts K_i

$$[J_i, J_j] = i\varepsilon_{ijk} J_k$$
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$$[N_i, N_j^\dagger] = 0$$

Representations $J_i = N_i + N_i^\dagger$

$$(n, m) \qquad J = n + m$$

$(0,0)$ scalar $J=0$

$(\tfrac{1}{2},0), (0,\tfrac{1}{2})$ LH and RH spinors $J=\tfrac{1}{2}$

$(\tfrac{1}{2},\tfrac{1}{2})$ vector $J=1$, etc

Weyl spinors

$$(\tfrac{1}{2},0) \quad (0,\tfrac{1}{2})$$

$$\psi_L \qquad \psi_R$$

2-component spinors of SU(2)

Rotations and Boosts $\quad \psi_{L(R)} \rightarrow S_{L(R)}\,\psi_{L(R)}$

$$S_{L(R)} = e^{i\frac{\alpha}{2}.\sigma} \quad : \text{ Rotations}$$

$$S_{L(R)} = e^{\pm\frac{\nu}{2}.\sigma} \quad : \text{ Boosts}$$

Dirac spinor

Can combine $\psi_L, \ \psi_R$ to form a 4-component "Dirac" spinor $\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$

Lorentz transformations $\quad \psi \rightarrow e^{i\omega\sigma}\psi, \qquad \omega\sigma = \omega^{\mu\nu}\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]\omega^{\mu\nu}$

where $\gamma_0 = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}, \quad \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & o \end{bmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

Weyl basis

$$\omega^{0i} \rightarrow boosts, \quad \omega^{ij} \rightarrow rotations \qquad i,j = 1,2,3$$

Weyl spinors

$(\frac{1}{2},0) \quad (0,\frac{1}{2})$

$\psi_L \quad \psi_R$

2-component spinors of SU(2)

Rotations and Boosts $\quad \psi_{L(R)} \to S_{L(R)}\, \psi_{L(R)}$

$S_{L(R)} = e^{i\frac{\alpha}{2}.\sigma}$: Rotations

$S_{L(R)} = e^{\pm\frac{v}{2}.\sigma}$: Boosts

†

Dirac spinor

Can combine $\psi_L, \; \psi_R$ to form a 4-component "Dirac" spinor $\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$

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(Dirac gamma matrices, …new 4-vector γ_μ)

$$\left\{\gamma^\mu, \gamma^\nu\right\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

Note : $\quad \psi_{L(R)} = \frac{1}{2}(1 \mp \gamma_5)\psi$

The Dirac equation

Fermions described by 4-cpt Dirac spinors ψ

- $\psi^{\dagger}\gamma^{0}\psi \equiv \overline{\psi}\psi = \overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L$ Lorentz invariant
- New 4-vector γ_μ

The Lagrangian

$$\mathfrak{L} = i\overline{\psi}\,\gamma_\mu\partial^\mu\,\psi - m\overline{\psi}\,\psi$$

Dimension?

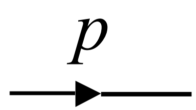
From Euler Lagrange equation obtain the Dirac equation

$$(i\gamma_\mu\partial^\mu - m)\psi = 0$$

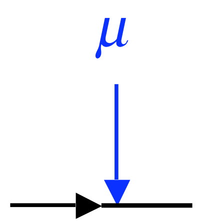
U(1) symmetry

$$\psi \to e^{i\alpha}\psi \quad \to \quad j^\mu = -e\overline{\psi}\,\gamma^\mu\psi$$

Feynman rules



$$\frac{i}{(\not{p} - m)}$$



$$ie\gamma^\mu$$

Fermion masses

$$\psi^{\dagger}\gamma^{0}\psi \equiv \overline{\psi}\psi = \overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L$$

Lorentz invariant

$$\mathfrak{L} = m\left(\overline{e_L}e_R + \overline{e_R}e_L\right)$$

Dirac mass

$$\sigma_2\psi_R^{\dagger} \equiv \psi_L$$

$$\left(\psi_L^T\sigma_2\psi_L + hc\right)$$

Lorentz invariant

Only term allowed by charge conservation is

$$\mathfrak{L}' = m\left(\nu_L^T\sigma_2\nu_L + hc\right)$$

Majorana mass

Fermions and the weak interactions

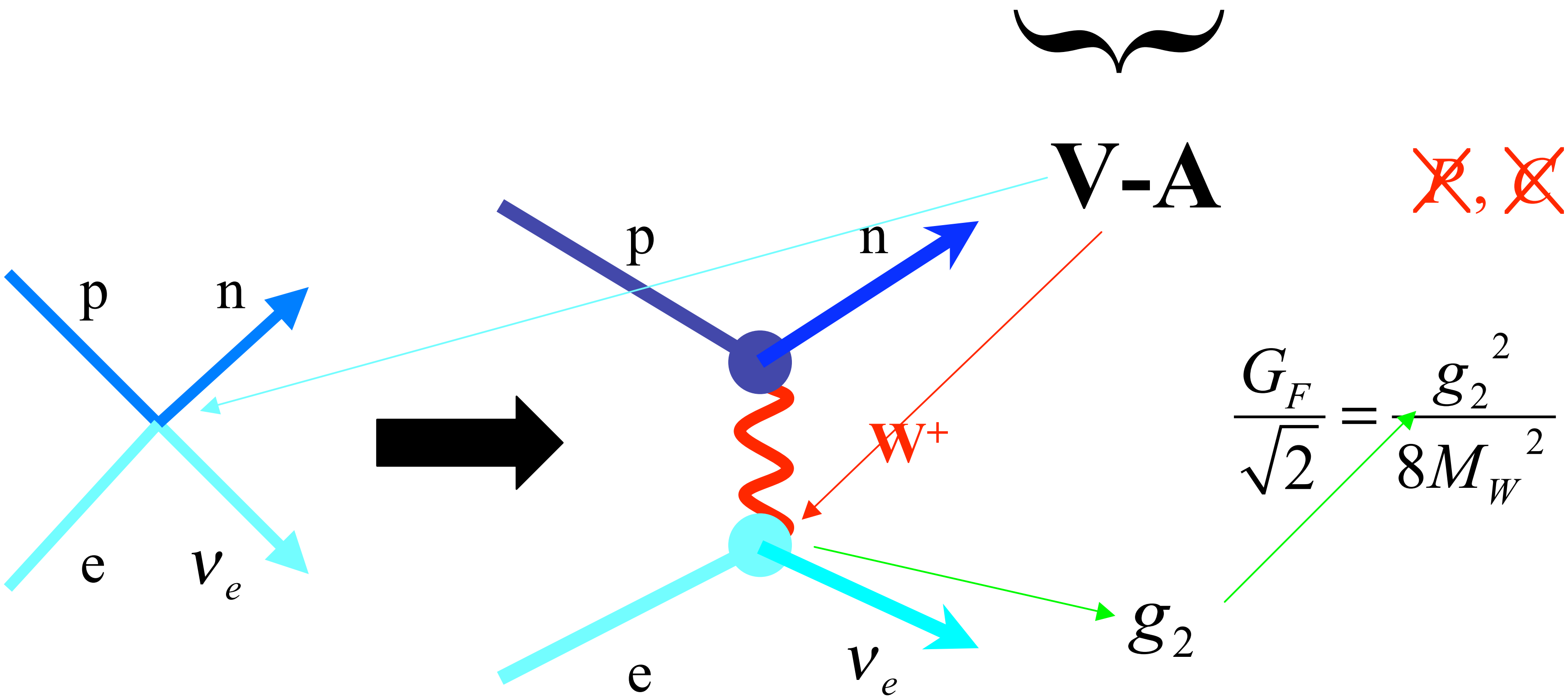
Fermi theory of β decay $\quad n \to pe\,\bar{\nu}_e$

$$\frac{G_F}{\sqrt{2}}\left[\bar{u}_n\gamma^\sigma(1-\gamma_5)u_p\right]\left[\bar{u}_{\nu_e}\gamma_\sigma(1-\gamma_5)u_e\right]$$

Fermions and the weak interactions

Fermi theory of β decay $\qquad n \rightarrow p e \bar{\nu}_e$

$$L = \frac{G_F}{\sqrt{2}}\left[\bar{u}_n \gamma^\sigma (1-\gamma_5) u_p\right]\left[\bar{u}_{\nu_e} \gamma_\sigma (1-\gamma_5) u_e\right]$$



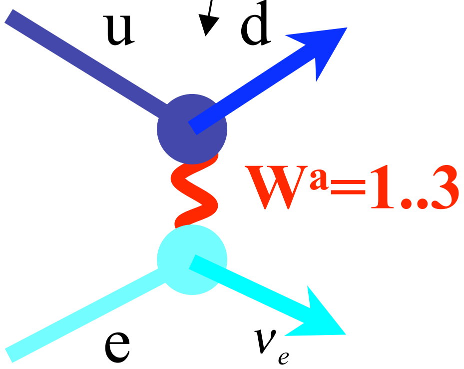
Weak Interactions

SU(2) local gauge theory

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \begin{pmatrix} \nu \\ e \end{pmatrix}_L$$

$$u_R, d_R, e_R$$

Weak coupling, α_2



Gauge boson (J=1)

$W^{\pm}, Z$

Neutral currents

Symmetry :

Local conservation of 2 weak isospin charges

$$\Psi_a \rightarrow \left(e^{ig_2\alpha(x).\tau}\right)^a_b \Psi_a$$

$$W^r_\mu \rightarrow W^r_\mu - \partial_\mu \alpha^r - f^{rst}\alpha^s W^t_\mu$$

$$\overline{\Psi}(\partial_\mu - ig_2\lambda_r W^r_\mu)\gamma^\mu \Psi$$

A non-Abelian (SU(2)) local gauge field theory

Massive vector propagator (W, Z bosons)

$$\left(g^{\nu\mu}(\partial^2 + M^2) - \partial^\nu \partial_\mu\right) B^\mu = j^\nu$$

$$(-g^{\mu\nu}(-p^2 + M^2) + p_\mu p_\nu)^{-1} = \frac{i(-g^{\mu\nu} + p^\mu p^\nu / M^2)}{p^2 - M^2}$$

$$B_\mu = \varepsilon_\mu e^{ip.x}$$

Free particle solution

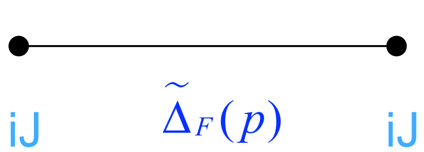
$$\varepsilon^{(\lambda=\pm 1)} = \mp(0, 1, \pm i, 0) / \sqrt{2}$$

$$\varepsilon^{(\lambda=0)} = (|\mathbf{p}|, 0, 0, E) / M$$

Helicity polarisation vectors

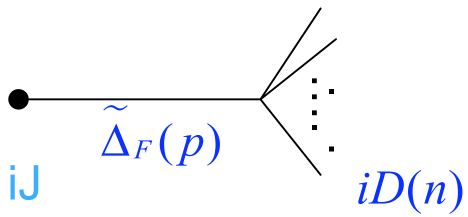
$$\sum_\lambda \varepsilon_\mu^{(\lambda)*} \varepsilon_\nu^{(\lambda)} = -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}$$

Propagation of unstable scalar particle



$$= -iJ^2 \tilde{\Delta}_F(p)$$

No decay



$$= -iJ\tilde{\Delta}_F(p)D(n)$$

Particle decays into final state n

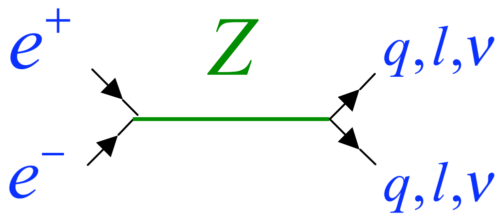
Optical theorem – conservation of probability, time evolution is unitary

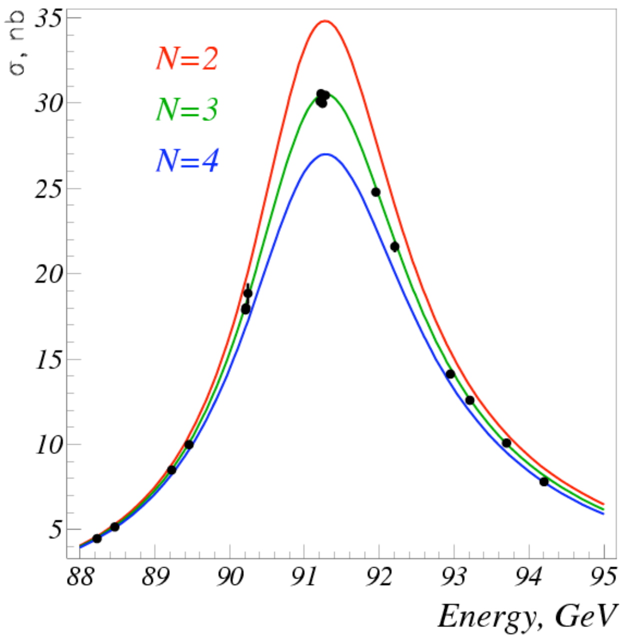
$$S^\dagger S = SS^\dagger = 1 \qquad S_{fi} = \delta_{fi} + iT_{fi}$$

$$\mathrm{Im}(T_{kk}) = \tfrac{1}{2}\sum_n |T_{nk}|^2$$

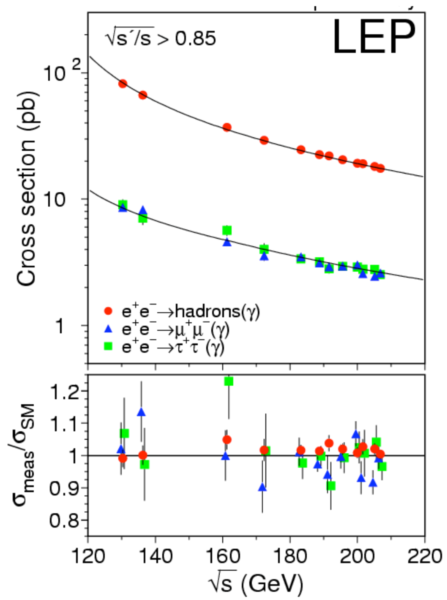
$$-J^2\,\mathrm{Im}(\tilde{\Delta}_F(p)) = \tfrac{1}{2}\sum_n \left|-iJ\tilde{\Delta}_F(p)D(n)\right|^2 = J^2\left|\tilde{\Delta}_F(p)\right|^2 \underbrace{\int \tfrac{1}{2}\sum_n |D(n)|^2 dQ}_{m\Gamma_{tot}}$$

$$\Rightarrow \quad \boxed{\tilde{\Delta}_F(p) = \frac{1}{p^2 - m^2 + im\Gamma_{tot}}} \qquad (\Delta_F(x) \propto e^{-m\Gamma_{tot}t})$$

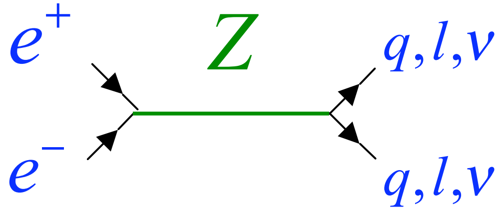


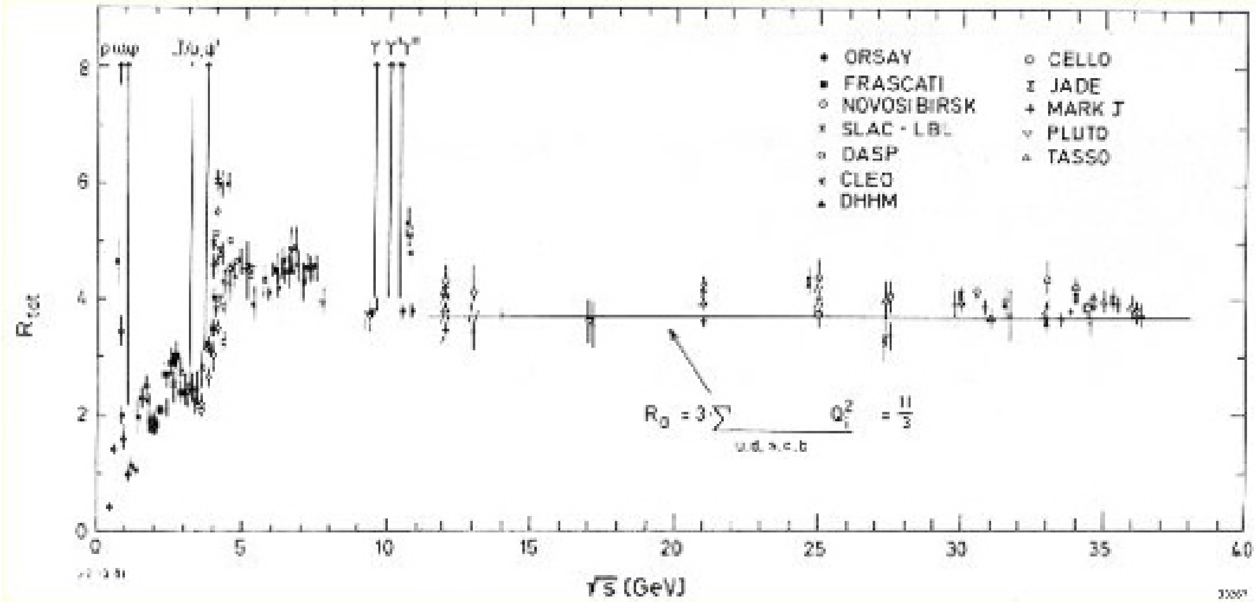


(N is no. of light - wrt M_Z - neutrinos)



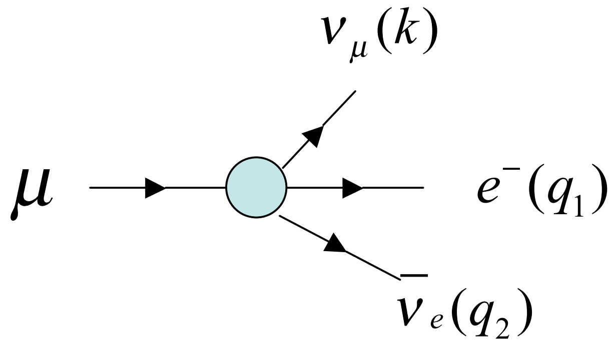
$$\sigma(e^+e^- \rightarrow hadrons) = \frac{12\pi\,\Gamma(Z \rightarrow ee)\Gamma(Z \rightarrow hadrons)}{(E^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2}$$





$$\frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_{f=1}^{n_f} Q_f^2$$

μ *decay*



Fermi theory ('40s)

$$M = \frac{G_F}{\sqrt{2}} \bar{u}(k)\gamma^\mu (1-\gamma_5) u(p)\, \bar{u}(q\ )\gamma_\mu (1-\gamma_5) v(p)$$

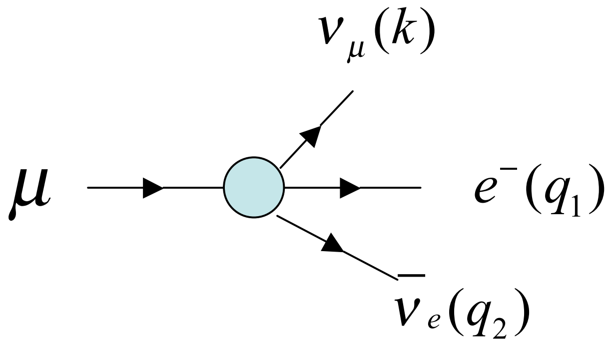
Dimensional analysis

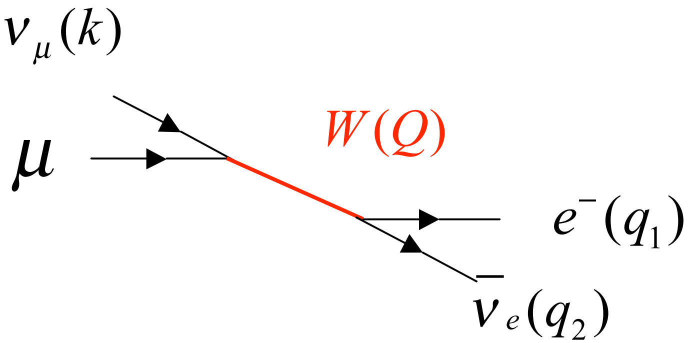
$$\Gamma_{tot} = \frac{1}{192\pi^3} m_\mu^5 G_F^2$$

The hard part!

$$\tau_\mu^{\exp t} = \frac{1}{\Gamma} = 2.19703(4)\, 10^{-6}\ \text{sec} \qquad \Rightarrow \qquad G_F = 1.16637(1)\, 10^{-5}\ GeV^{-2}$$

μ decay





$$\frac{Q^\mu Q^\nu}{M_W^2} \sim \frac{m_\mu m_e}{M_W^2} \sim 0$$

$$M = i g_W \, \bar{u}(k)\gamma^\mu(1-\gamma_5)u(p) \quad \frac{g_{\mu\nu} - \frac{Q_\mu Q_\nu}{M_W^2}}{Q^2 - M_W^2 + i\varepsilon} \quad g_W \, \bar{u}(q\ )\gamma_\mu(1-\gamma_5)v(p)$$

In μ decay $\quad Q^2 \leq O(m_\mu^2) << M_W^2$

$$\frac{g_W^2}{Q^2 - M_W^2} \simeq \frac{-g_W^2}{M_W^2}$$

$$G_F = 1.16637(1)\, 10^{-5}\ GeV^{-2}$$

$$\frac{g_W^2}{M_W^2} = \frac{G_F}{\sqrt{2}}$$

$\sim 80 GeV$

Fundamental principles of particle physics

- Introduction - Fundamental particles and interactions
- Symmetries I - Relativity
- Quantum field theory - Quantum Mechanics + relativity
- Theory confronts experiment - Cross sections and decay rates
- Symmetries II – Gauge symmetries, the Standard Model
- Fermions and the weak interactions

⇨ The Standard Model and Beyond

Have Fun!

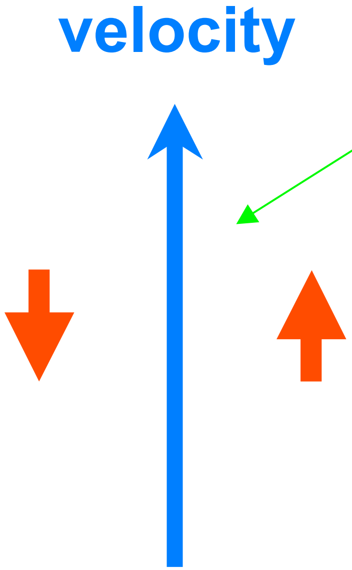
UNIVERSITY OF OXFORD

DOMINUS ILLUMINATIO MEA

$$u_{e_L} = \tfrac{1}{2}(1-\gamma_5)u_e$$

$$u_{e_R} = \tfrac{1}{2}(1+\gamma_5)u_e$$

$$u_{e^-_L} \equiv u^c_{e^+_R}$$



Left-handed

m=0

Right-handed