

Particle Physics

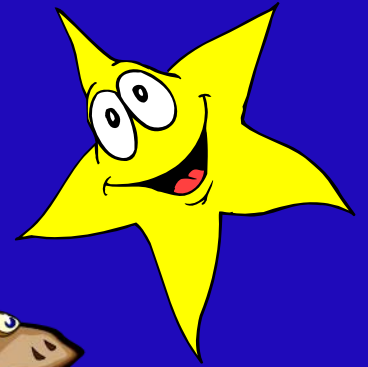
The Standard Model

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3. Gauge Invariance



- Field Theory
- Classical Electrodynamics
- Quantum Electrodynamics (QED)
- $SU(N)$ Gauge Theory
- Quantum Chromodynamics (QCD)

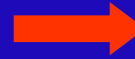
QUANTUM MECHANICS WAVE EQUATIONS

Non Relativistic:

Schrödinger equation

$$\vec{p} = -i\vec{\nabla} \quad ; \quad E = i\frac{\partial}{\partial t}$$

$$E = \frac{\vec{p}^2}{2m}$$



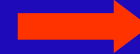
$$i\frac{\partial}{\partial t}\Psi = -\frac{\vec{\nabla}^2}{2m}\Psi$$

Relativistic:

Klein-Gordon equation

$$p^\mu = i\partial^\mu = i g^{\mu\nu} \frac{\partial}{\partial x^\nu}$$

$$E^2 = \vec{p}^2 + m^2$$



$$(\square + m^2)\phi = 0$$

$$\square \equiv \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

Spin $\frac{1}{2}$:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

Dirac equation

$$-(i\gamma^\nu \partial_\nu + m) [(i\gamma^\mu \partial_\mu - m)\psi] = 0 \equiv (\square + m^2)\psi$$



$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

spin matrices $(D=4)$

DIRAC ALGEBRA

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} ; \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} ; \quad [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

Particle
Antiparticle

Spinors

Chirality Projectors:

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} ; \quad (\gamma_5)^2 = I_4$$

$$P_R \equiv \frac{1+\gamma_5}{2} ; \quad P_L \equiv \frac{1-\gamma_5}{2}$$

$$P_R^2 = P_R ; \quad P_L^2 = P_L ; \quad P_R P_L = P_L P_R = 0$$

LAGRANGIAN FORMALISM

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\delta S = 0 \quad \longrightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi)} \right) = 0$$

Eq. Motion

Klein-Gordon: (spin 0)

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi \quad \longrightarrow \quad (\square + m^2) \phi = 0$$

Dirac: (spin $\frac{1}{2}$)

$$\bar{\psi} \equiv \psi^\dagger \gamma^0$$

$$\mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi \quad \longrightarrow \quad (i \gamma^\mu \partial_\mu - m) \psi = 0$$

CLASSICAL ELECTRODYNAMICS

Maxwell's Equations:

$$\vec{\nabla} \cdot \vec{E} = \rho \quad ; \quad \vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} \quad ; \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}$$

Potentials:

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} \quad ; \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

Covariant Notation:

$$A^\mu \equiv (V, \vec{A}) \quad ; \quad J^\mu \equiv (\rho, \vec{J})$$

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad \longrightarrow \quad \partial_\mu F^{\mu\nu} = J^\nu$$

Gauge Invariance:

$$A^\mu \quad \longrightarrow \quad A'^\mu \equiv A^\mu + \partial^\mu \Lambda$$

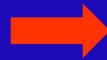
Same Physics described by many different A^μ

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\partial_\mu F^{\mu\nu} = \square A^\nu - \partial^\nu (\partial_\mu A^\mu) = J^\nu$$

CONSERVED ELECTROMAGNETIC CURRENT:

$$\partial_\mu \partial_\nu F^{\mu\nu} = 0$$



$$\partial_\mu J^\mu = 0$$

Lorentz Gauge $(\partial_\mu A^\mu = 0)$ and $J^\mu = 0$



$$\square A^\mu = 0$$

Klein-Gordon equation with $m = 0$

MASSLESS PHOTON

Residual Invariance:

$$A^\mu \longrightarrow A^\mu + \partial^\mu \Lambda \quad ; \quad \square \Lambda = 0$$

$\mu = 0, 1, 2, 3$; 2 arbitrary constraints



TWO PHOTON POLARIZATIONS

FREE Dirac fermion:

$$\mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi$$

Phase Invariance: $\psi \rightarrow \psi' = e^{iQ\theta} \psi$; $\bar{\psi} \rightarrow \bar{\psi}' = e^{-iQ\theta} \bar{\psi}$

Absolute phases are not observable in Quantum Mechanics

GAUGE PRINCIPLE:

$$\theta = \theta(x)$$

Phase Invariance should hold **LOCALLY**

BUT

$$\partial_\mu \psi \rightarrow e^{iQ\theta} (\partial_\mu + iQ \partial_\mu \theta) \psi$$

SOLUTION:

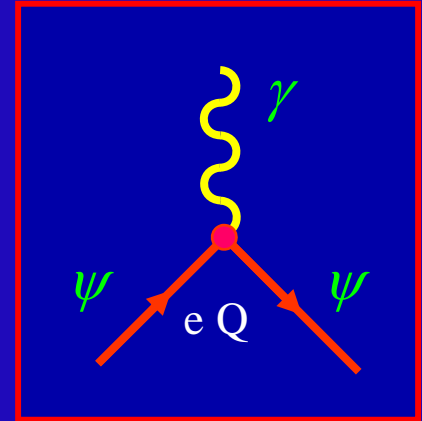
Covariant Derivative

$$D_\mu \psi \equiv (\partial_\mu + i e Q A_\mu) \psi \rightarrow e^{iQ\theta} D_\mu \psi$$

One needs a spin-1 field A_μ satisfying $A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \theta$

QUANTUM ELECTRODYNAMICS

$$\begin{aligned}\mathcal{L} &= \bar{\psi} (i \gamma^\mu D_\mu - m) \psi \\ &= \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi - e Q A_\mu (\bar{\psi} \gamma^\mu \psi)\end{aligned}$$



Kinetic term:

$$\mathcal{L}_K = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\longrightarrow \partial_\mu F^{\mu\nu} = e Q (\bar{\psi} \gamma^\nu \psi)$$

Maxwell

Mass term:

$$[\text{exp: } m_\gamma < 6 \cdot 10^{-17} \text{ eV}]$$

$$\mathcal{L}_M = \frac{1}{2} m_\gamma^2 A^\mu A_\mu$$

Not Gauge Invariant



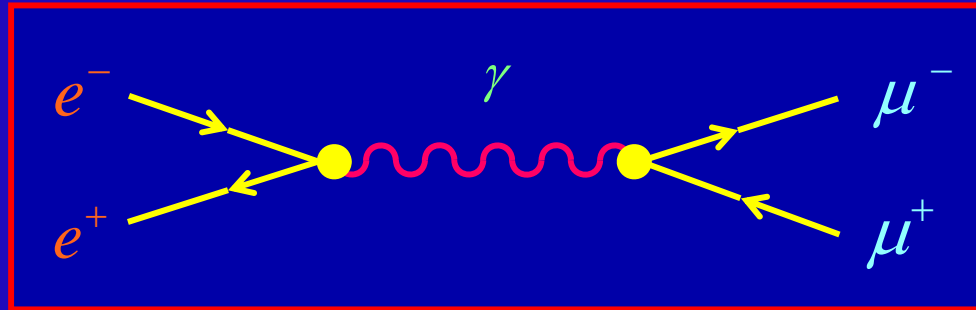
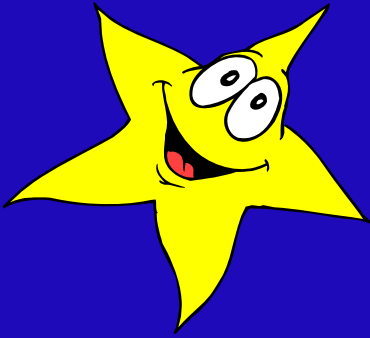
$$m_\gamma = 0$$

Gauge Symmetry



QED Dynamics

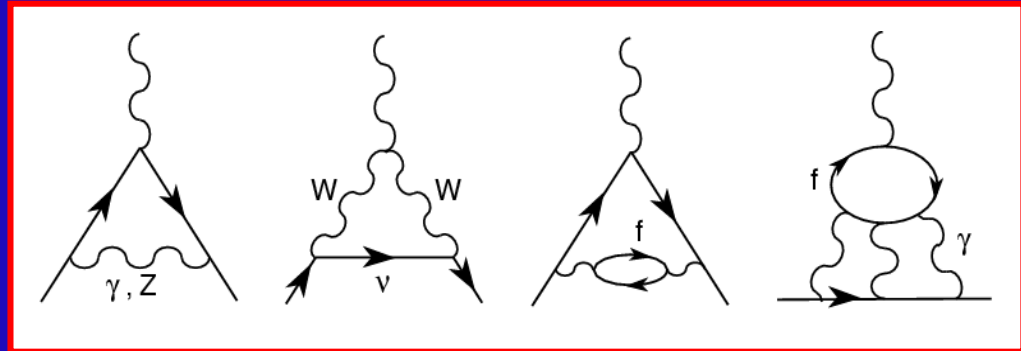
Successful Theory



Anomalous Magnetic Moment

$$\mu_l \equiv g_l \frac{e}{2m_l}$$

$$a_l \equiv \frac{1}{2} (g_l - 2)$$



$$a_e = (1\,159\,652\,180.85 \pm 0.76) \times 10^{-12} \quad \longrightarrow \quad \alpha^{-1} = 137.035\,999\,710 \pm 0.000\,000\,096$$

$$\longrightarrow \quad a_\mu^{\text{th}} = (11\,659\,188 \pm 10) \times 10^{-10} \quad [\text{Exp: } (11\,659\,208.0 \pm 6.3) \times 10^{-10}]$$

QUANTUM CHROMODYNAMICS

FREE QUARKS:

$$N_c = 3$$

$$\mathbf{q} \equiv \begin{pmatrix} q \\ q \\ q \end{pmatrix}$$

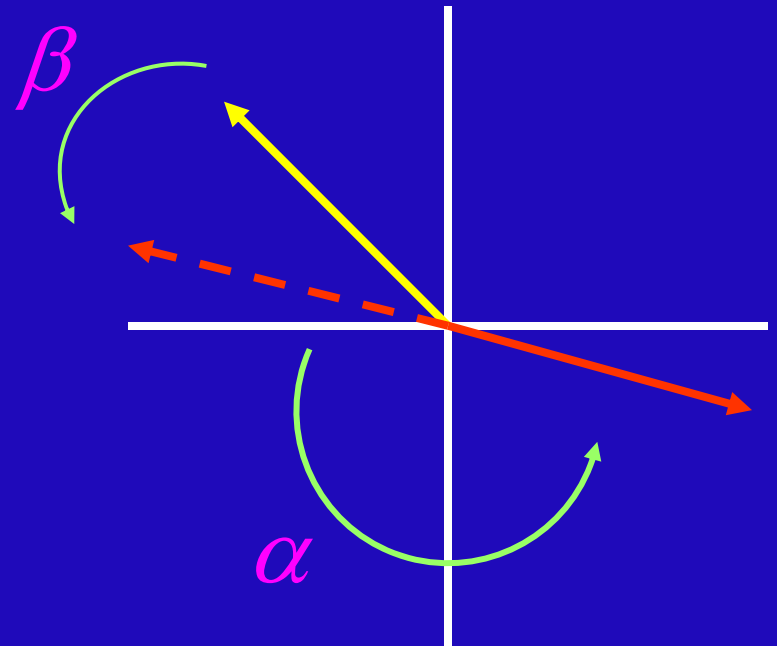
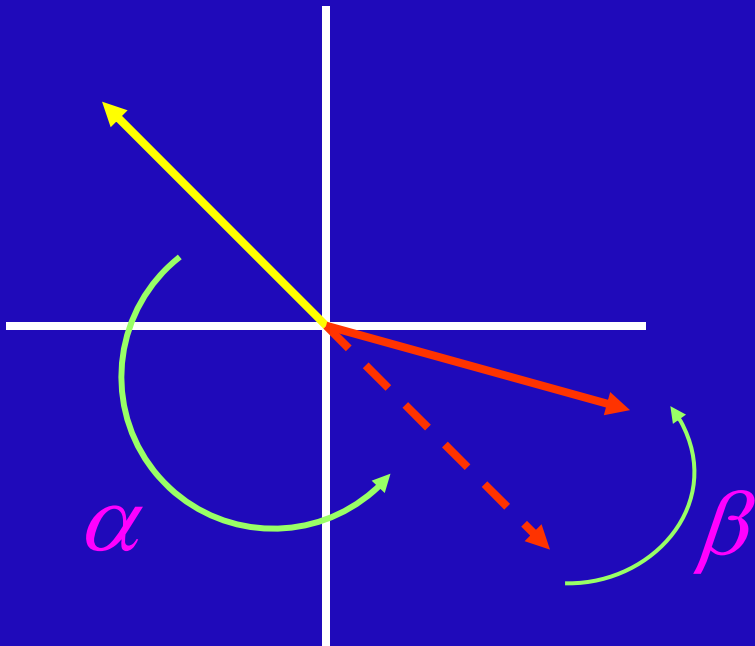
$$\mathcal{L} = \bar{\mathbf{q}} [i \gamma^\mu \partial_\mu - m] \mathbf{q}$$

SU(3) Colour Symmetry:

$$\mathbf{q} \rightarrow \mathbf{U} \mathbf{q} \quad ; \quad \bar{\mathbf{q}} \rightarrow \bar{\mathbf{q}} \mathbf{U}^\dagger$$

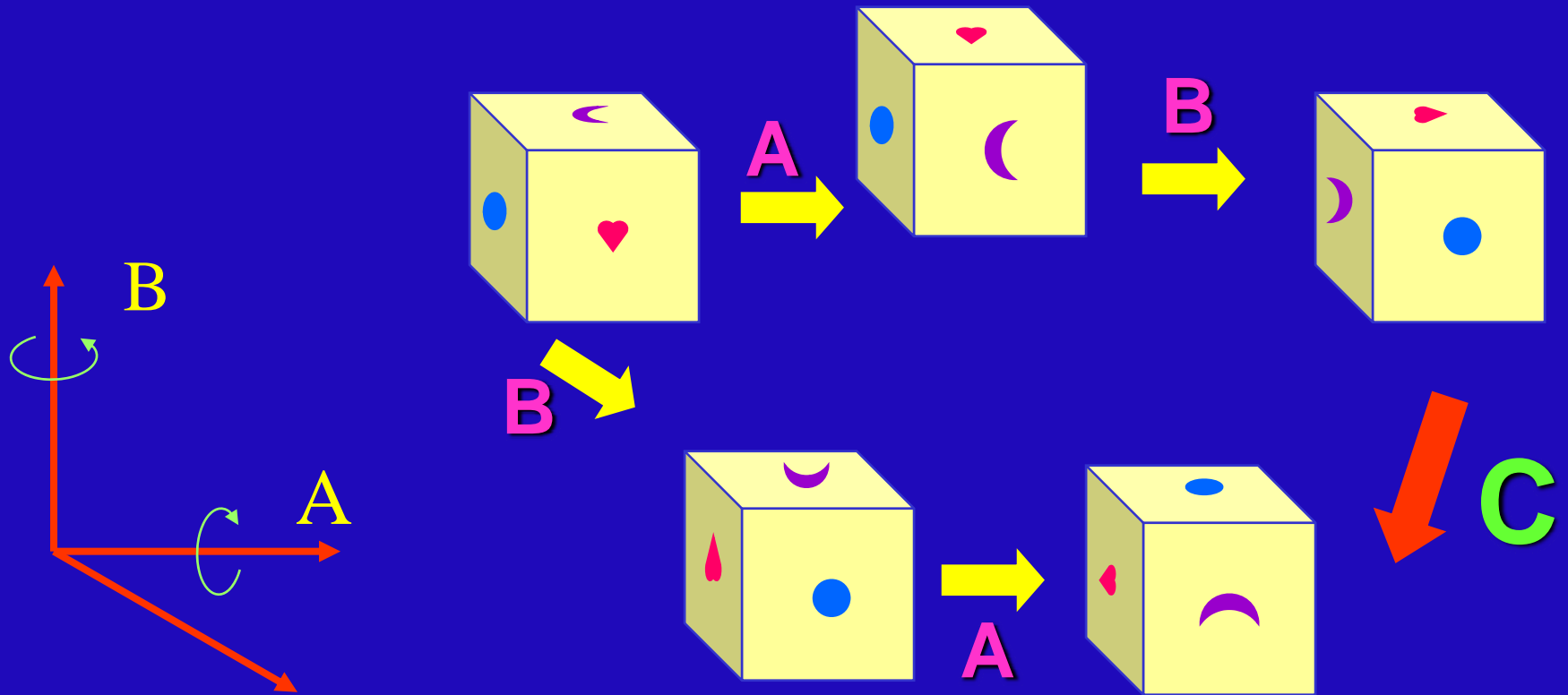
$$\mathbf{U} \mathbf{U}^\dagger = \mathbf{U}^\dagger \mathbf{U} = \mathbf{1} \quad ; \quad \det \mathbf{U} = 1$$

ABELIAN ROTATIONS



NON ABELIAN ROTATIONS

$$AB - BA = C$$



SU(N) ALGEBRA

$N \times N$ matrices: $\mathbf{U} \mathbf{U}^\dagger = \mathbf{U}^\dagger \mathbf{U} = \mathbf{1}$; $\det \mathbf{U} = 1$

$$\mathbf{U} = \exp \left\{ i \mathbf{T}^a \theta_a \right\} ; \quad \mathbf{T}^a = \mathbf{T}^{a\dagger} ; \quad \text{Tr}(\mathbf{T}^a) = 0 ; \quad a = 1, \dots, N^2 - 1$$

Commutation Relation: $[\mathbf{T}^a, \mathbf{T}^b] = i f^{abc} \mathbf{T}^c$

Structure Constants f^{abc} real, antisymmetric

Fundamental Representation: $\mathbf{T}_F^a = \frac{1}{2} \lambda^a$ $N \times N$

Adjoint Representation: $(\mathbf{T}_A^a)_{bc} = -i f^{abc}$ $(N^2 - 1) \times (N^2 - 1)$

$$\text{Tr}(\mathbf{T}_F^a \mathbf{T}_F^b) = T_F \delta_{ab} ; \quad \text{Tr}(\mathbf{T}_A^a \mathbf{T}_A^b) = C_A \delta_{ab} ; \quad (\mathbf{T}_F^a \mathbf{T}_F^a)_{\alpha\beta} = C_F \delta_{\alpha\beta}$$

$$T_F = \frac{1}{2} ; \quad C_A = N ; \quad C_F = \frac{N^2 - 1}{2N}$$

SU(2)

2×2 matrices: $\mathbf{U} \mathbf{U}^\dagger = \mathbf{U}^\dagger \mathbf{U} = \mathbf{1}$; $\det \mathbf{U} = 1$

$\mathbf{U} = \exp \{ i \mathbf{T}^a \theta_a \}$; $\mathbf{T}^a = \mathbf{T}^{a\dagger}$; $\text{Tr}(\mathbf{T}^a) = 0$; $a = 1, \dots, 3$

Commutation Relation: $[\mathbf{T}^a, \mathbf{T}^b] = i \varepsilon^{abc} \mathbf{T}^c$

Fundamental Representation:

$$\mathbf{T}_F^a = \frac{1}{2} \sigma^a$$

Pauli

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

SU(3)

$$[T^a, T^b] = i f^{abc} T^c$$

Fundamental Representation:

$$T_F^a = \frac{1}{2} \lambda^a$$

Gell-Mann

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} ; \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ; \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} ; \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\frac{1}{2} f^{123} = f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{\sqrt{3}} f^{458} = \frac{1}{\sqrt{3}} f^{678} = \frac{1}{2}$$

QUANTUM CHROMODYNAMICS

FREE QUARKS:

$$N_c = 3$$

$$\mathbf{q} \equiv \begin{pmatrix} q \\ q \\ q \end{pmatrix}$$

$$\mathcal{L} = \bar{\mathbf{q}} [i \gamma^\mu \partial_\mu - m] \mathbf{q}$$

SU(3) Colour Symmetry:

$$\mathbf{q} \rightarrow \mathbf{U} \mathbf{q} = \exp \left\{ i \frac{\lambda^a}{2} \theta_a \right\} \mathbf{q}$$

Gauge Principle:

Local Symmetry

$$\theta_a = \theta_a(x)$$

$$\mathbf{D}^\mu \mathbf{q} \equiv (\mathbf{I}_3 \partial^\mu + i g_s \mathbf{G}^\mu) \mathbf{q} \rightarrow \mathbf{U} \mathbf{D}^\mu \mathbf{q}$$

$$\mathbf{D}^\mu \rightarrow \mathbf{U} \mathbf{D}^\mu \mathbf{U}^\dagger ;$$

$$\mathbf{G}^\mu \rightarrow \mathbf{U} \mathbf{G}^\mu \mathbf{U}^\dagger + \frac{i}{g_s} (\partial^\mu \mathbf{U}) \mathbf{U}^\dagger$$

$$[\mathbf{G}^\mu]_{\alpha\beta} \equiv \frac{1}{2} (\lambda^a)_{\alpha\beta} G_a^\mu(x)$$

8 Gluon Fields

Infinitesimal SU(3) Transformation:

$$\delta q^\alpha = i \delta\theta_a \left(\frac{\lambda^a}{2} \right)_{\alpha\beta} q^\beta \quad ; \quad \delta G_a^\mu = -\frac{1}{g_s} \partial^\mu (\delta\theta_a) - f^{abc} \delta\theta_b G_c^\mu$$

Non Abelian Group:

$$f^{abc} \neq 0$$

- δG_a^μ depends on G_a^μ
- Universal g_s
- No Colour Charges

Kinetic Term:

$$\mathbf{G}^{\mu\nu} \equiv -\frac{i}{g_s} [\mathbf{D}^\mu, \mathbf{D}^\nu] = \partial^\mu \mathbf{G}^\nu - \partial^\nu \mathbf{G}^\mu + i g_s [\mathbf{G}^\mu, \mathbf{G}^\nu] \rightarrow \mathbf{U} \mathbf{G}^{\mu\nu} \mathbf{U}^\dagger$$

$$\mathbf{G}^{\mu\nu} \equiv \frac{\lambda^a}{2} G_a^{\mu\nu} \quad ; \quad G_a^{\mu\nu} = \partial^\mu G_a^\nu - \partial^\nu G_a^\mu - g_s f^{abc} G_b^\mu G_c^\nu$$

$$\mathcal{L}_K = -\frac{1}{2} \text{Tr} (\mathbf{G}^{\mu\nu} \mathbf{G}_{\mu\nu}) = -\frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a$$

Mass Term:

$$\mathcal{L}_M = \frac{1}{2} m_G^2 G_a^\mu G_\mu^a$$

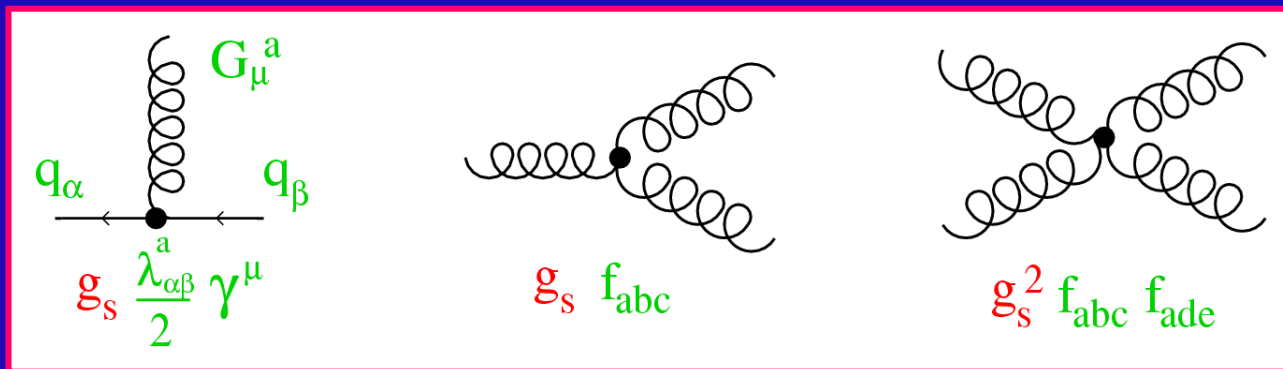
Not Gauge Invariant



$$m_G = 0$$

Massless Gluons

$$\begin{aligned}
\mathcal{L}_{\text{QCD}} &= -\frac{1}{2} \text{Tr} (\mathbf{G}^{\mu\nu} \mathbf{G}_{\mu\nu}) + \bar{\mathbf{q}} [i \gamma^\mu \mathbf{D}_\mu - m_q] \mathbf{q} \\
&= -\frac{1}{4} (\partial^\mu G_\nu^a - \partial^\nu G_\mu^a) (\partial_\mu G_\nu^a - \partial_\nu G_\mu^a) + \sum_q \bar{q}_\alpha [i \gamma^\mu \partial_\mu - m_q] q_\alpha \\
&\quad - \frac{1}{2} g_s G_\mu^a \sum_q [\bar{q}_\alpha (\lambda^a)_{\alpha\beta} \gamma^\mu q_\beta] \\
&\quad + \frac{1}{2} g_s f_{abc} (\partial_\mu G_\nu^a - \partial_\nu G_\mu^a) G_b^\mu G_c^\nu - \frac{1}{4} g_s^2 f_{abc} f_{ade} G_b^\mu G_c^\nu G_\mu^d G_\nu^e
\end{aligned}$$



▪ **Gluon Self – interactions**

\mathbf{G}^3 , \mathbf{G}^4

▪ **Universal Coupling** g_s

(No Colour Charges)

