New formulation of leading-order anisotropic hydrodynamics

Outline

- Second order viscous hydrodynamics
- Expansion around an anisotropic background
- Leading order in anisotropic hydrodynamics
- Extension of the model

11-th Polish Workshop on Relativistic Heavy-Ion Collisions

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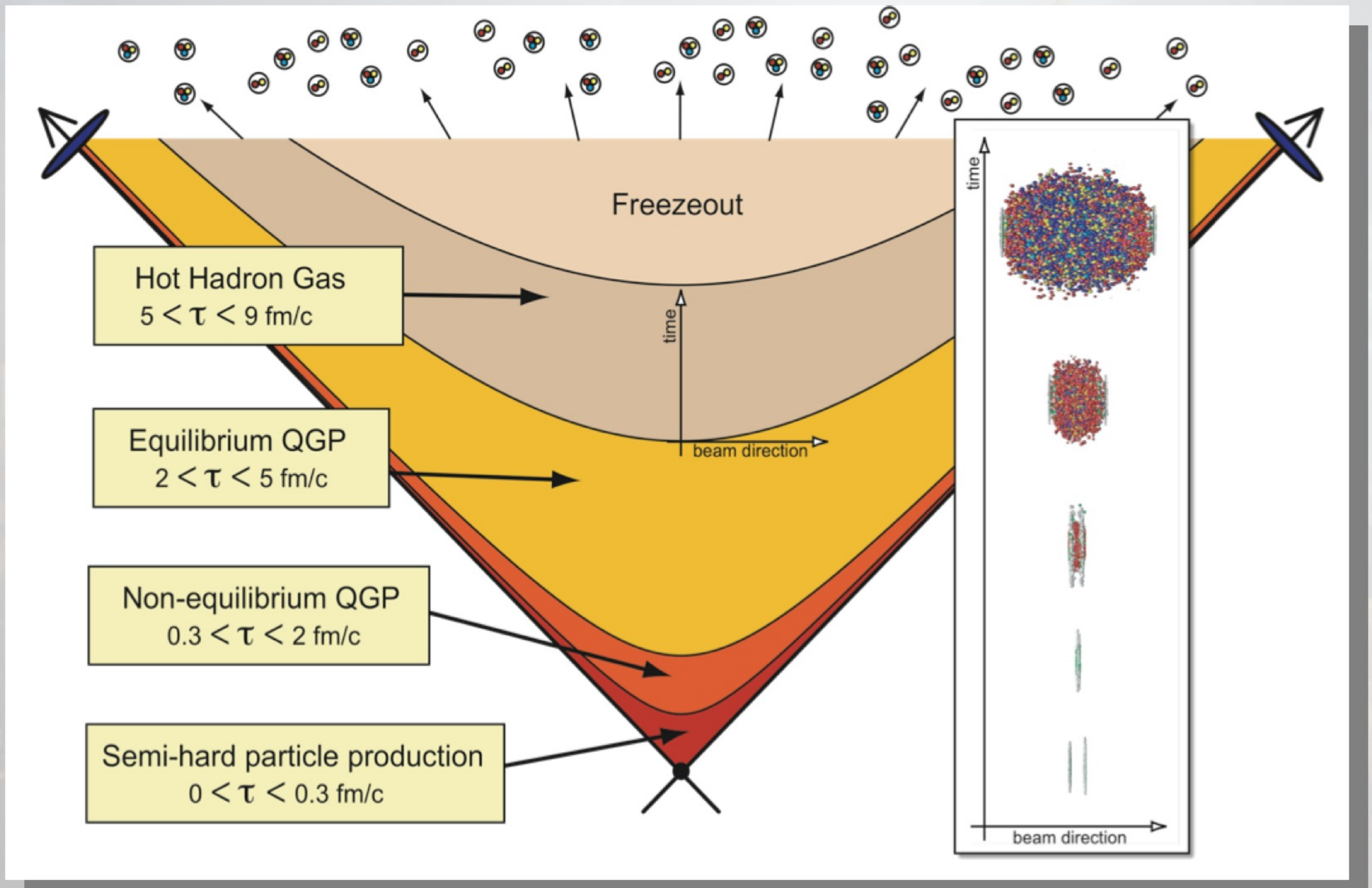
Motivations

Hydrodynamic modeling of heavy ion collisions (small viscosity)

Large gradients → *viscous corrections*



Strong longitudinal expansion,

pressure anisotropy (also AdS/CFT)

Large momentum anisotropy from microscopic models (pQCCD, CGC)

From kinetic theory to hydrodynamics

Relativistic Boltzmann equation:

$$p^\mu \, \partial_\mu f(x,p) = \mathcal{C}[f]$$

First moment:

$$\int dP \, p^\mu p^\nu \, \partial_\mu f = \partial_\mu T^{\mu\nu} =$$

$$= \int dP \, p^\nu \mathcal{C}[f] \left[= 0 \right]$$

- S. R. De Groot, W. A. van Leeuwen, Ch. G. van Weert, *Relativistic kinetic theory*, North Holland (1980).

Hydrodynamics

Ansatz for the relativistic Boltzmann distribution

Perfect fluid:

$$f \simeq f_{\text{eq.}} = k \exp\left[-\frac{p \cdot U(x)}{T(x)}\right]$$

$$T^{\mu\nu} = k \int dP\, p^\mu p^\nu \exp\left[-\frac{p \cdot U}{T}\right] = (\varepsilon + P)\, U^\mu U^\nu - g^{\mu\nu} P$$

$$T_{\text{L.R.F.}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Dissipative hydrodynamics

$$f = f_{\text{eq.}} + \delta f \quad \Rightarrow \quad T^{\mu\nu} = T^{\mu\nu}_{\text{eq.}} + \delta T^{\mu\nu}$$

$\delta f \Rightarrow \delta T^{\mu\nu}$ treated as, small, perturbations

Landau frame, massless particles

$$\delta T^{\mu\nu} = \pi^{\mu\nu}$$

$$U^{\mu}\pi_{\mu\nu} = 0 \qquad g_{\mu\nu}\pi^{\mu\nu} = 0$$

Four equations, five more degrees of freedom!

Entropy current and entropy source

Obtaining the remaining equations from the second principle of thermodynamics

$$\mathcal{S}^\mu = \mathcal{S}^\mu_{\rm eq.} + \delta\mathcal{S}^\mu \simeq \frac{p}{T}U^\mu + \frac{1}{T}T^{\mu\nu}U_\nu \ - \frac{\tau_\pi}{4\eta T}\pi^{\alpha\beta}\pi_{\alpha\beta}U^\mu$$

$$\partial_\mu \mathcal{S}^\mu \geq 0$$

$$\partial_\mu \mathcal{S}^\mu \simeq \frac{1}{T}\pi^{\mu\nu}\left[\sigma_{\mu\nu} \ - \frac{\tau_\pi}{2\eta}\Delta^\alpha_\mu\Delta^\beta_\nu D\pi_{\alpha\beta} - \frac{1}{2}T\pi_{\mu\nu}\partial\cdot\left(\frac{\tau_\pi}{2\eta T}U\right)\right]$$

$$\pi^{\mu\nu}\pi_{\mu\nu} \geq 0$$

$$\tau_\pi \Delta^\alpha_\mu\Delta^\beta_\nu D\pi_{\alpha\beta} + \ \pi_{\mu\nu} = 2\eta\sigma_{\mu\nu} \ - \pi_{\mu\nu}T\eta\,\partial\cdot\left(\frac{\tau_\pi}{2\eta T}U\right)$$

Example, Bjorken flow

0+1 dimensions: boost invariant in the longitudinal direction, homogeneous in the transverse plane

$$U = (\cosh\eta_{\|}, 0, 0, \sinh\eta_{\|}) \qquad \eta_{\|} = \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right) \qquad \tau = \sqrt{t^2 - z^2}$$

Gradients of the four velocity are proportional to $1/\tau$

In the Navier-Stokes limit $\pi_{\mu\nu} \simeq 2\eta\sigma_{\mu\nu}$

therefore

$$\frac{P_{\|}}{P_{\perp}} = \frac{P_{\text{eq.}} + \pi_{ZZ}}{P_{\text{eq.}} + \pi_{XX}} \simeq \frac{3T\tau - 16\bar{\eta}}{3T\tau + 8\bar{\eta}}$$

Anisotropic hydrodynamics

Reorganization of the hydrodynamic expansion

$$f = f_{\mathrm{eq.}} + \delta f$$

around an anisotropic background instead of the local equilibrium

$$f = f_{\mathrm{aniso.}} + \delta \tilde{f}$$

"Romatschke-Strickland" form:

$$f_{\mathrm{aniso.}} = k \exp\left[-\frac{\sqrt{\left(p \cdot U(x)\right)^2 + \xi(x)\left(p \cdot Z(x)\right)^2}}{\Lambda(x)} \right]$$

0+1 dimensions

Boost invariant in the longitudinal direction, homogeneous in the transverse plane

$$T_{\mathrm{L.R.F.}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P_\perp & 0 & 0 \\ 0 & 0 & P_\perp & 0 \\ 0 & 0 & 0 & P_\parallel \end{pmatrix}$$

No negative pressure already at the leading order

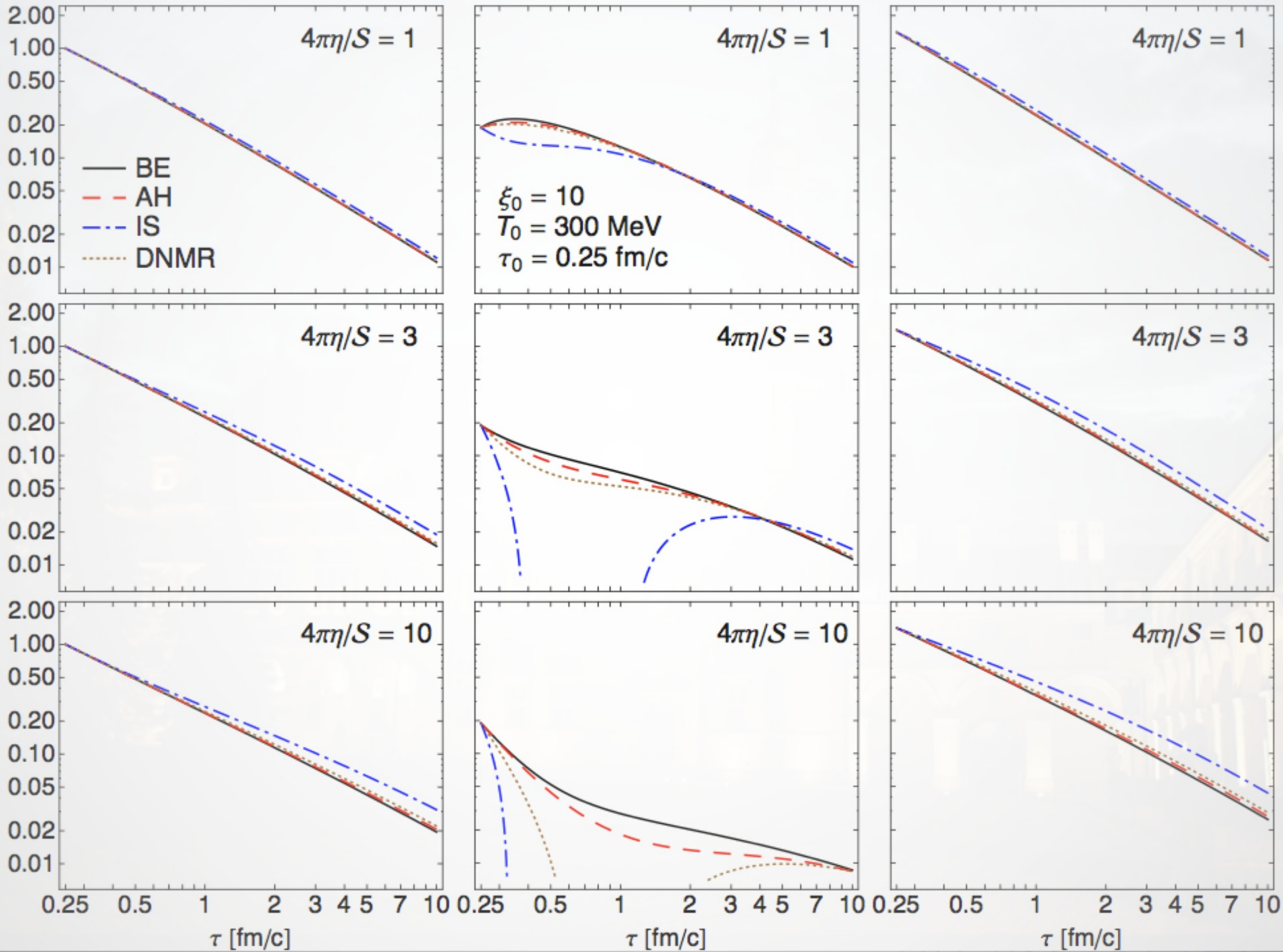
$$\left(P_\perp, P_\parallel \right) = \int dP \left(\frac{1}{2} p_\perp^2, p_\parallel^2 \right) f_{\mathrm{RS}} > 0$$

Positive by construction

Exact solution! Test for viscous and anisotropic hydrodynamics

Some plots

$\varepsilon(\tau)/\varepsilon(\tau_0)$ $3P_{\parallel}(\tau)/\varepsilon(\tau_0)$ $3P_{\perp}(\tau)/\varepsilon(\tau_0)$



Radial flow

1+1 dimensions, boost invariance in the longitudinal direction, rotation invariance in the transverse plane

Four-velocity

$$U = \gamma \begin{pmatrix} 1 \\ v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \cosh\theta_\perp \cosh\eta_{||} \\ \sinh\theta_\perp \cos\phi \\ \sinh\theta_\perp \sin\phi \\ \cosh\theta_\perp \sinh\eta_{||} \end{pmatrix}$$

Another unknown function

$$\tanh\theta_\perp(\tau, r) = \cosh\eta_{||}\sqrt{v_x^2 + v_y^2}$$

Orthonormal basis

$$X = \begin{pmatrix} \sinh\theta_\perp \cosh\eta_{||} \\ \cosh\theta_\perp \cos\phi \\ \cosh\theta_\perp \sin\phi \\ \sinh\theta_\perp \sinh\eta_{||} \end{pmatrix} \quad Y = \begin{pmatrix} 0 \\ -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \quad Z = \begin{pmatrix} \sinh\eta_{||} \\ 0 \\ 0 \\ \cosh\eta_{||} \end{pmatrix}$$

- **W Florkowski and R Ryblewski, Phys. Rev. C 85, 044902 (2012)**

Improving the starting point of the expansion

For a conformal system

$$f_{\text{aniso.}} = k \exp\left[-\frac{\sqrt{(1+\xi_X)\left(p\cdot X\right)^2 + (1+\xi_Y)\left(p\cdot Y\right)^2 + (1+\xi_Z)\left(p\cdot Z\right)^2}}{\lambda(x)} \right]$$

$$\sum_I \xi_I = 0$$

Pressure not isotropic in the transverse plane

$$I \in \{X, Y, Z\}$$

$$T_{\text{L.R.F.}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P_X & 0 & 0 \\ 0 & 0 & P_Y & 0 \\ 0 & 0 & 0 & P_Z \end{pmatrix}$$

Still two trivial equations from the local four-momentum conservation and four unknown scalar function: $\theta_\perp, \lambda$ and two independent anisotropy parameter

- L Tinti and W Florkowski , Phys. Rev. C 89, 034907 (2014)

How to match with second order hydrodynamics

Using a close to equilibrium expansion

$$f \simeq f_{\rm eq}\left(1 + \frac{\lambda - T}{T^2}(p\cdot U) - \frac{\xi_X (p\cdot X)^2 + \xi_Y (p\cdot Y)^2 + \xi_Z (p\cdot Z)^2}{2T(p\cdot U)}\right)$$

We can find the pressure corrections $\left(T^{\mu\nu} = \int dP p^\mu p^\nu f\right)$ $\pi^{\mu\nu} = \sum_I \pi_I I^\mu I^\nu$

$$T^{\mu\nu} \simeq T^{\mu\nu}_{\rm eq} - \frac{32\pi}{5} k T^4 \left(\xi_X X^\mu X^\nu + \xi_Y Y^\mu Y^\nu + \xi_Z Z^\mu Z^\nu\right)$$

We can verify the matching if the equations for the anisotropy parameters correspond to the second order hydrodynamics equations for the pressure corrections

$$\xi_I \simeq -\frac{15}{4}\frac{\pi_I}{\varepsilon}$$

The remaining equations

Looking at the second moment of the Boltzmann equation, in relaxation time approximation, to close the system.

$$p^\lambda \partial_\lambda f = (p\cdot U)\frac{f_{\rm eq.} - f}{\tau_{\rm eq.}} \quad \Rightarrow \quad \int dP\; p^\mu p^\nu p^\lambda \partial_\lambda f = \frac{U_\lambda}{\tau_{\rm eq.}} \int dP \left(p^\mu p^\nu p^\lambda f_{\rm eq.} - p^\mu p^\nu p^\lambda f\right)$$

$$\partial_\lambda \Theta^{\lambda\mu\nu} = \frac{1}{\tau_{\rm eq}}\left(U_\lambda \Theta^{\lambda\mu\nu}_{\rm eq} - U_\lambda \Theta^{\lambda\mu\nu}\right)$$

$$\Theta^{\lambda\mu\nu} = \int dP\; p^\lambda p^\mu p^\nu f \qquad \Theta^{\lambda\mu\nu}_{\rm eq} = \int dP\; p^\lambda p^\mu p^\nu f_{\rm eq.}$$

Many of the projections of these equations along the vectors of the basis are trivial, but there is a useful combination of the remaining ones

$$\frac{D\xi_I}{1+\xi_I} + 2\sigma_I + \frac{\xi_I}{\tau_{\rm eq.}}\left(\frac{T}{\lambda}\right)^5 \sqrt{\prod_J (1+\xi_J)} - \frac{1}{3}\sum_J \frac{D\xi_J}{1+\xi_J} = 0$$

Only two of the three equations are independent, if two of them are fulfilled all of them are

Small anisotropy limit

$$D\xi_I + 2\sigma_I + \frac{\xi_I}{\tau_{\rm eq.}} \simeq 0$$

Using the first moment equations and after some calculations

$$\tau_{\rm eq.} D\pi_I + \pi_I = 2\left(\frac{4}{15}\,\varepsilon\,\tau_{\rm eq.}\right)\sigma_I - \pi_I\left[\frac{4}{15}\,\varepsilon\tau_{\rm eq.} T\partial\cdot\left(\frac{15}{8\varepsilon T}U\right)\right]$$

Consistent with Israel-Stewart

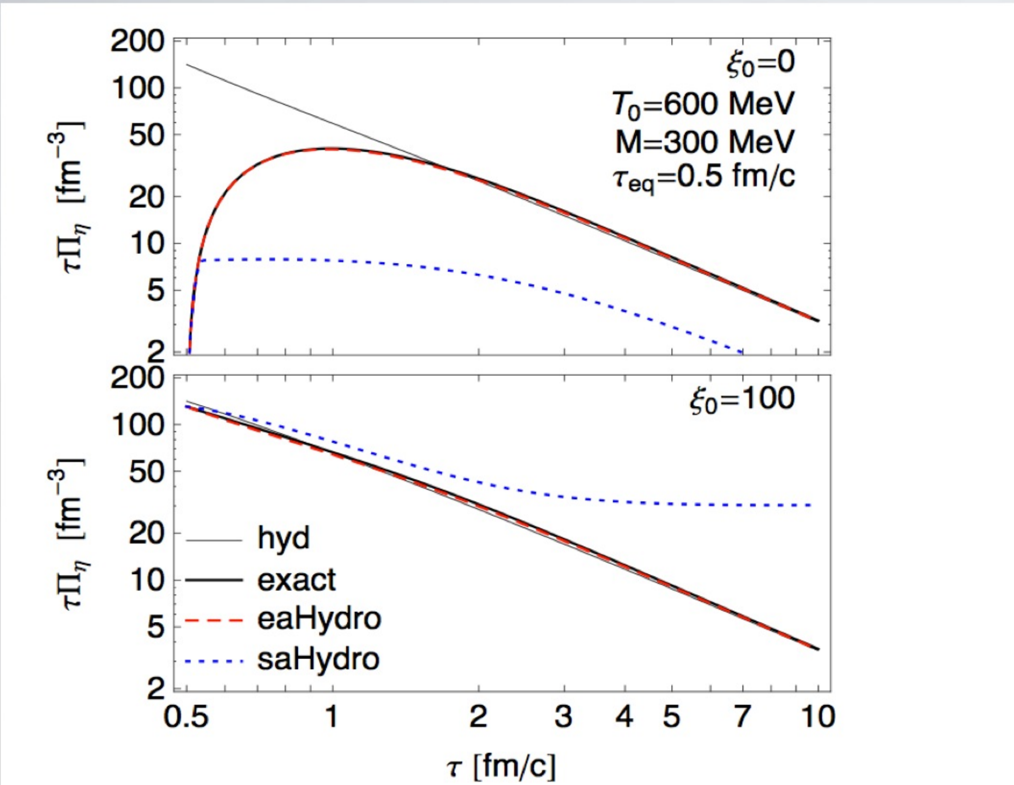
$$\tau_\pi\, D\pi_I + \pi_I = 2\eta\sigma_I - \pi_I\left[\eta T\partial_\lambda\left(\frac{\tau_\pi}{2\eta T}U^\lambda\right)\right]$$

provided

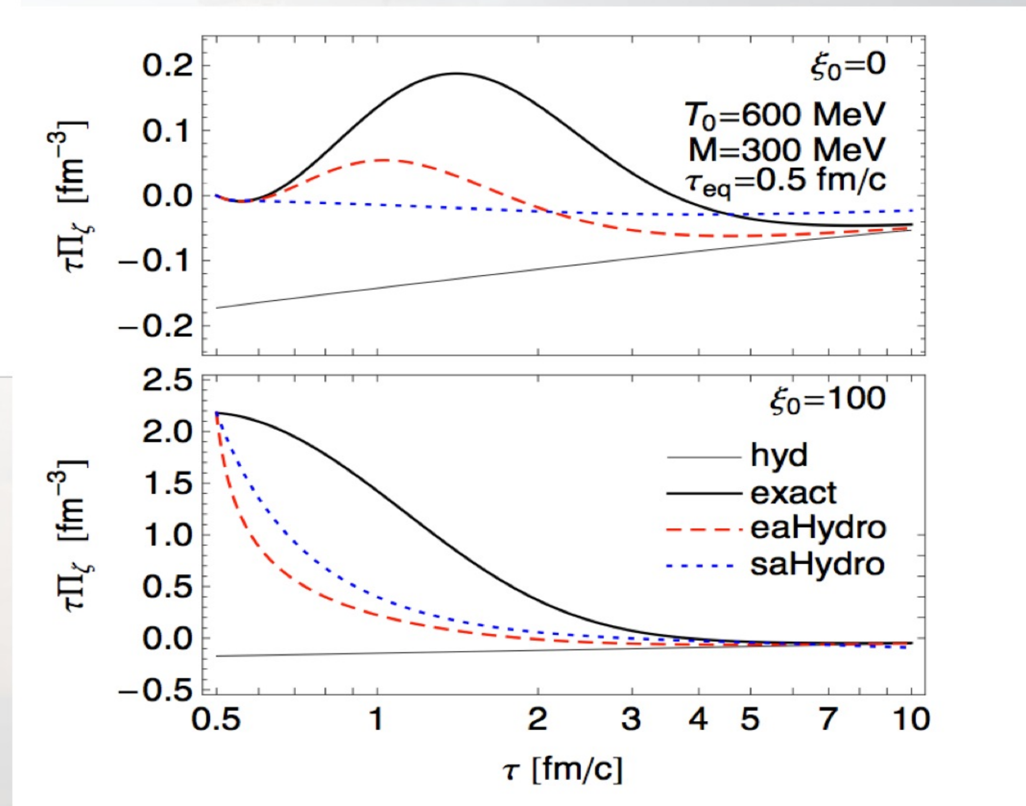
$$\tau_\pi = \tau_{\rm eq.} \quad \eta = \frac{4}{15}\,\varepsilon\tau_{\rm eq.} \Rightarrow \tau_\pi = \frac{5\bar{\eta}}{T}$$

Important even without radial expansion

- W Florkowski, R Ryblewski, M Strickland, L Tinti, Phys. Rev. C 89, 054909 (2014)



Much better agreement with the exact solution



So far so good, but...

Bulk degree of freedom (M Nopoush, R Ryblewski, M Strickland, Phys. Rev. C 90, 014908 (2014))

(3+1)-dimensional framework

No symmetry constraints from boost invariance or cylindrical symmetry

Generalized "Romatschke-Strickland" form

$$f = k \exp\left(-\frac{1}{\lambda}\sqrt{p_\mu \Xi^{\mu\nu} p_\nu}\right)$$

Geometric decomposition

$$\Xi^{\mu\nu} = U^\mu U^\nu + \xi^{\mu\nu} - \phi\Delta^{\mu\nu}$$

Matching with second order viscous hydrodynamics (Israel-Stewart??)

- **<u>L Tinti</u>, arXiv:1411.7268**

(3+1)-dimensional framework

Geometric argument to choose the necessary equations

We need higher moments of the Boltzmann equation

$$\Theta^{\lambda\mu\nu} = \int dP\, p^\lambda p^\mu p^\nu f = \Theta_U U^\lambda U^\mu U^\nu + U^\lambda \Theta^{\mu\nu} + U^\mu \Theta^{\nu\lambda} + U^\nu \Theta^{\lambda\mu}$$

$$\Theta^{\mu\nu} = U_\alpha \Delta^\mu_\beta \Delta^\nu_\gamma\, \Theta^{\alpha\beta\gamma}$$

"Shear equation"

$$D\Theta^{\langle\mu\nu\rangle} + \frac{5}{3}\theta\,\Theta^{\langle\mu\nu\rangle} + 2\Theta^{\langle\mu}_\lambda \sigma^{\nu\rangle\lambda} - 2\Theta^{\langle\mu}_\lambda \omega^{\nu\rangle\lambda} = -\frac{1}{\tau_{\rm eq.}}\Theta^{\langle\mu\nu\rangle}$$

(3+1)-dimensional framework

Close-to-equilibrium limit

$$\Theta^{\langle\mu\nu\rangle} \simeq g(m,T)\pi^{\mu\nu} \qquad \Theta_{\mathsf{tr.}} \simeq \Theta_{\mathsf{tr.}}^{\mathrm{eq.}} + h_{\mathsf{tr.}}(m,T)\,\Pi$$

$$n \simeq n_{\mathrm{eq.}} - h_{\mathsf{0}}(m,T)\,\Pi$$

$$\begin{aligned} D\pi^{\langle\mu\nu\rangle} + \frac{1}{\tau_\pi}\pi^{\mu\nu} &= 2\beta_\pi\,\sigma^{\mu\nu} + 2\,\pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \tau_{\pi\pi}\,\pi_\lambda^{\langle\mu}\sigma^{\nu\rangle\lambda} \\ &\quad -\delta_{\pi\pi}\,\pi^{\mu\nu}\theta + \lambda_{\pi\Pi}\,\Pi\,\pi^{\mu\nu} \end{aligned}$$

$$D\Pi + \frac{1}{\tau_\Pi}\Pi = -\beta_\Pi\,\theta - \delta_{\Pi\Pi}\,\Pi\,\theta + \lambda_{\Pi\pi}\,\pi^{\mu\nu}\sigma_{\mu\nu}$$

Summary & outlook

- Anisotropic hydrodynamics is a reorganization of the hydrodynamic expansion, around an anisotropic distribution.

- Pressure anisotropies already at the leading order. Very good approximation in the exactly solvable 0+1 case

- Generalized ansatz for the leading order. Well defined in the early stages and consistent with second order viscous hydrodynamics close to equilibrium, providing bulk-shear couplings.

- Next: Test with the exact solutions of the Boltzmann equation?