Fix-lines and stability

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AOC-Workshop - CERN

Introduction: close to $3Q_x = N$

Stability Domain

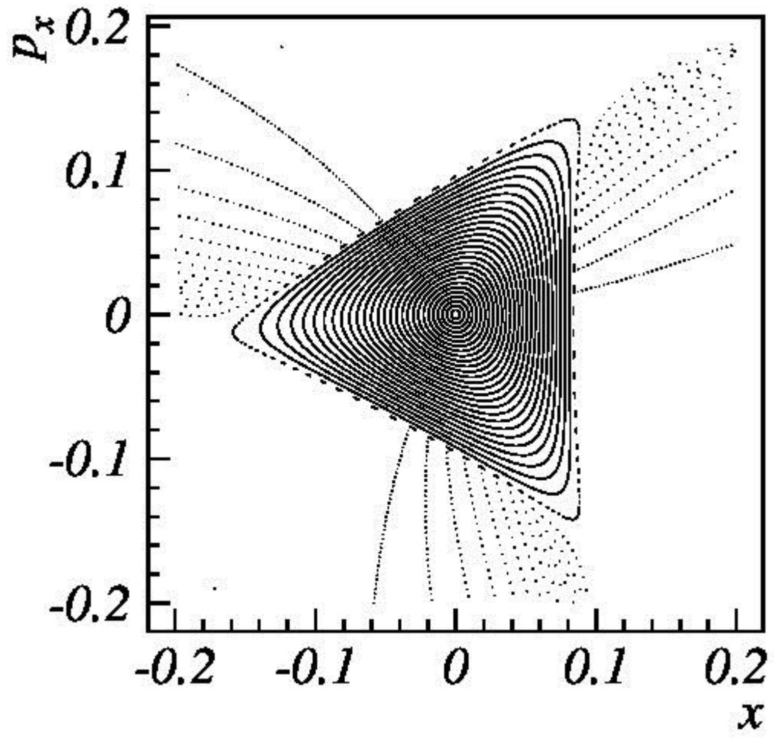
Analytic theory do not exist far from the continuum limit, but possible near resonances

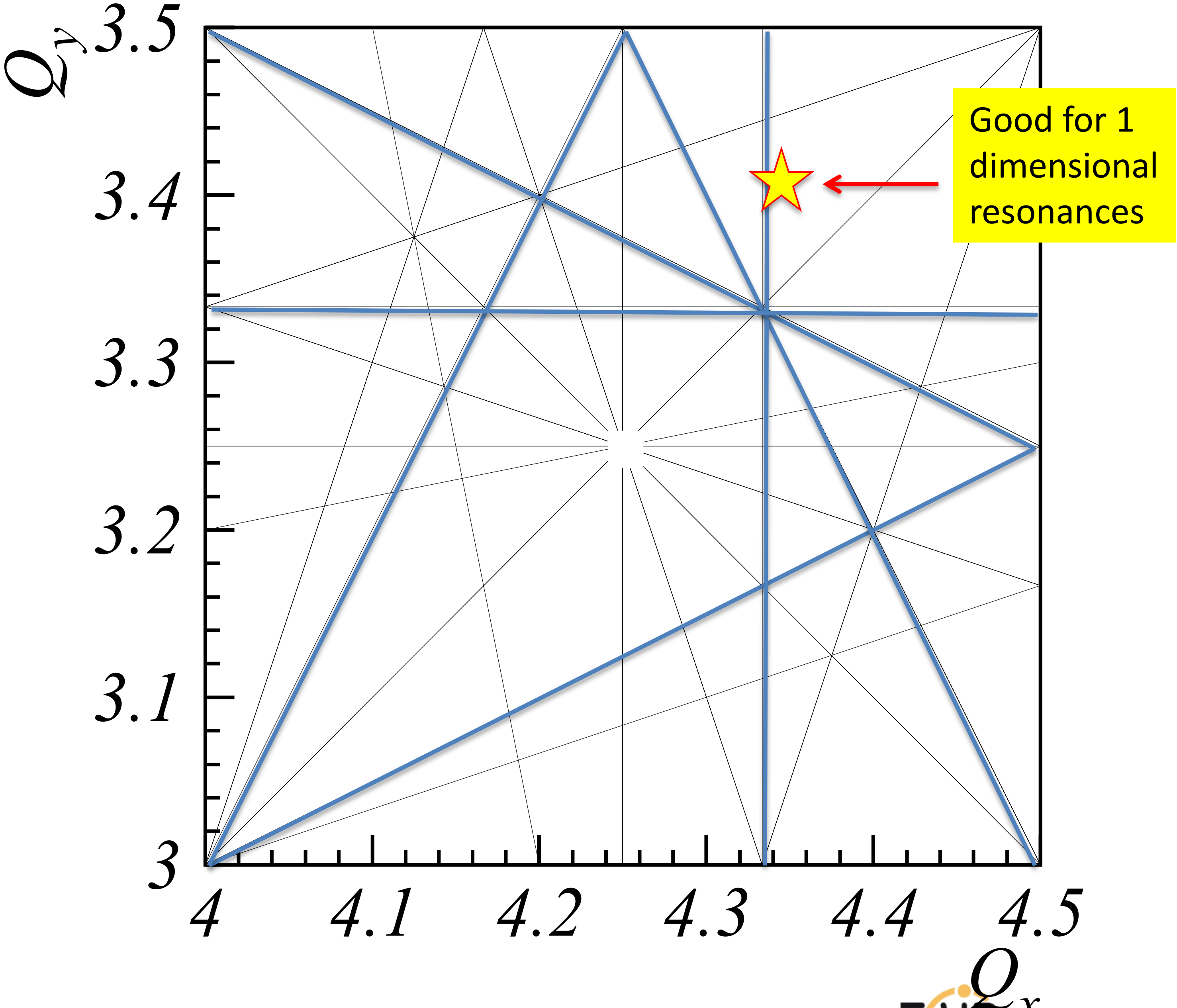
Very useful close to the resonances

Example,
slow extraction near q=1/3

It is possible to give an analytic estimate of the border of stability

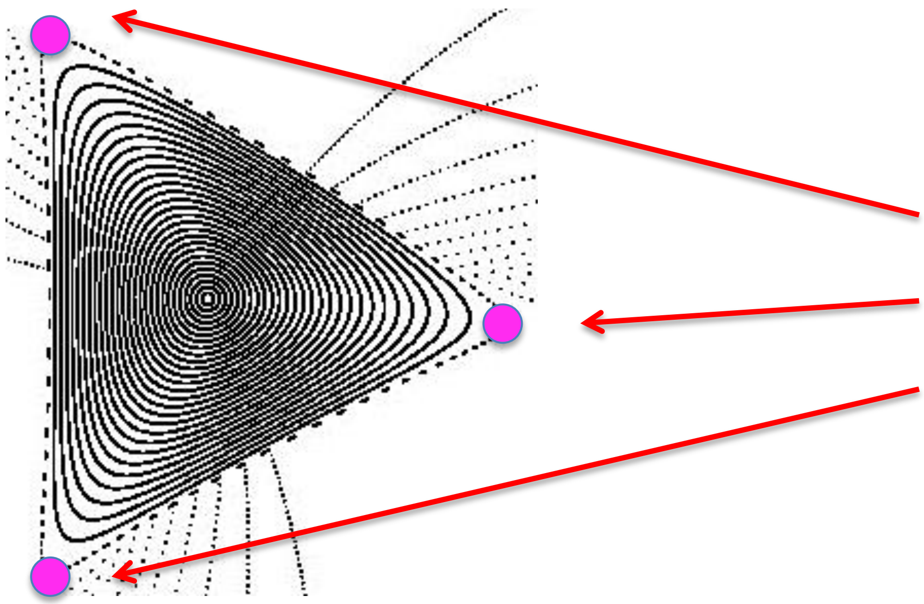
q = 0.327





Q_y
3.5
3.4
3.3
3.2
3.1
3
4
4.1
4.2
4.3
4.4
4.5
Q_x
Good for 1 dimensional resonances

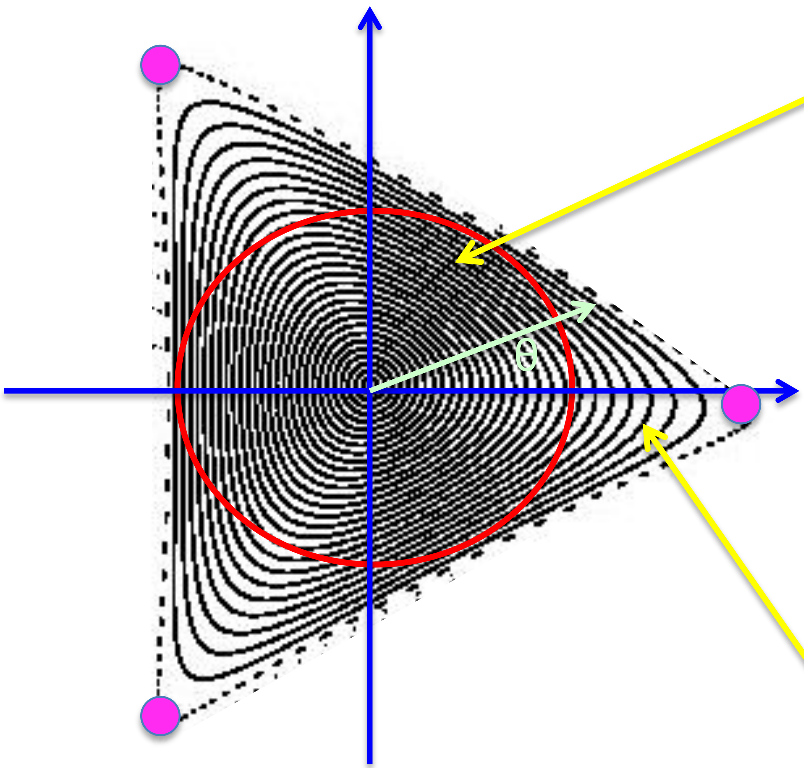
In proximity of $3Q_x = N$



Unstable fix points

stable orbits are “bounded” by the unstable fix points

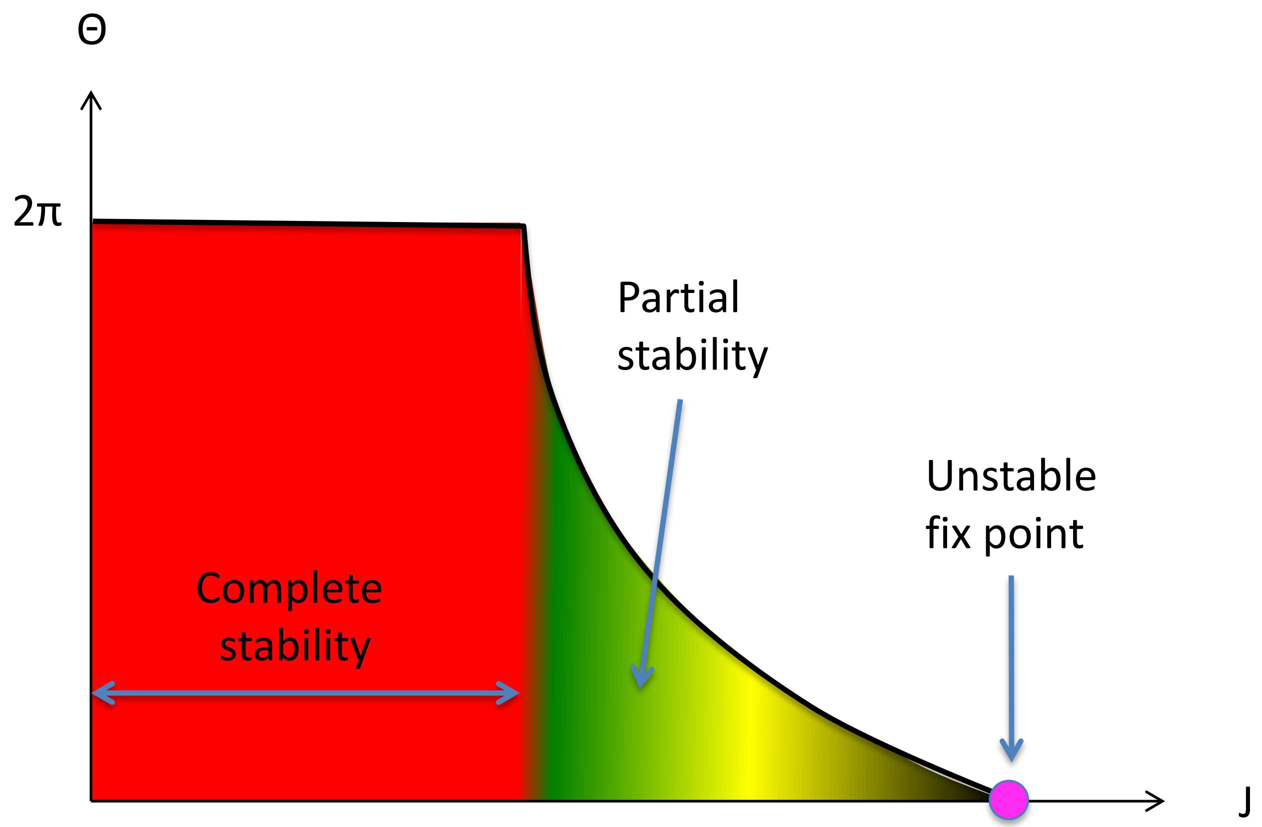
Properties of the stability domain



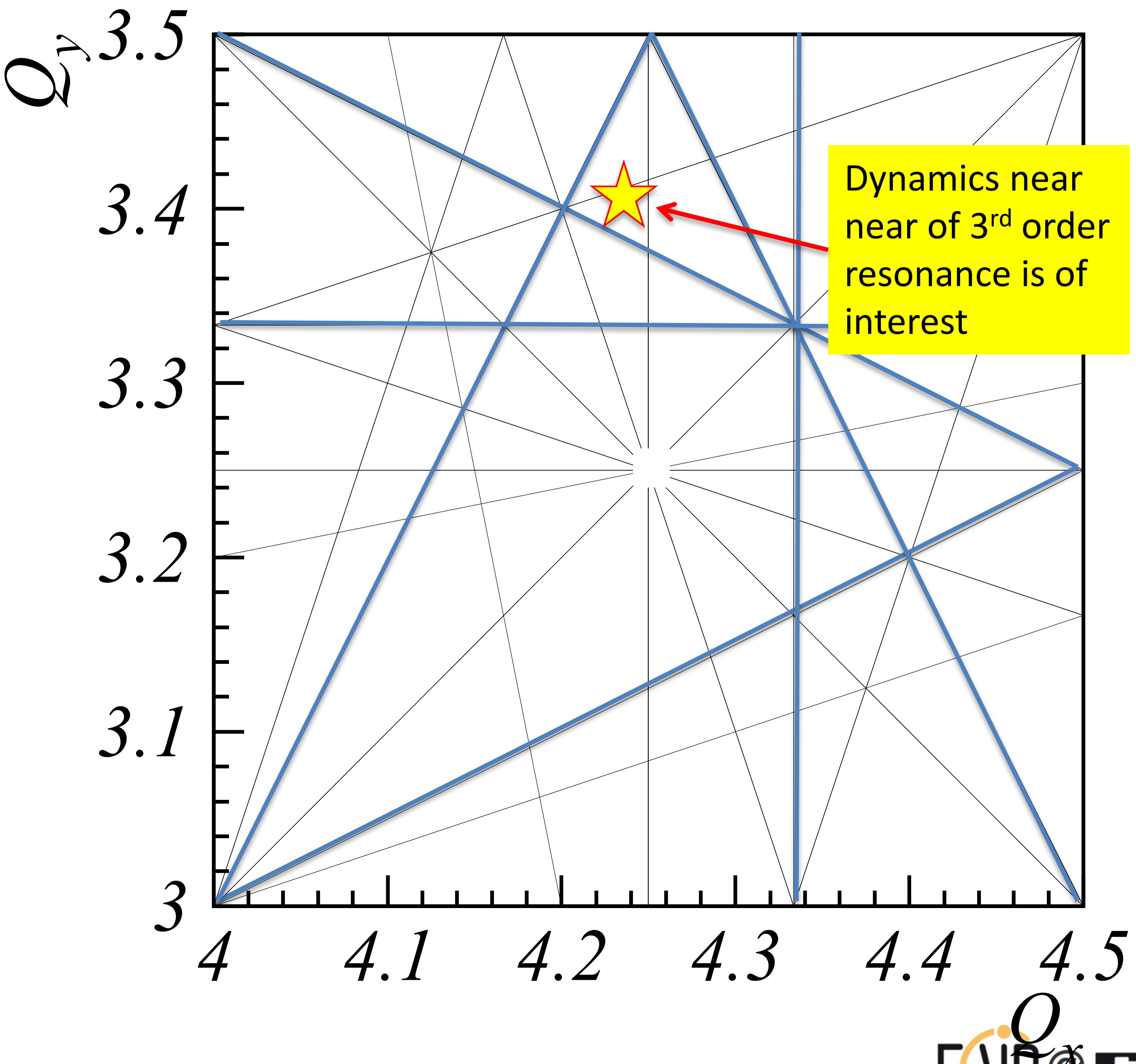
every particle inside the circle is “stable”

Only at special angles particles out of the circle are “stable”

Representation of stability

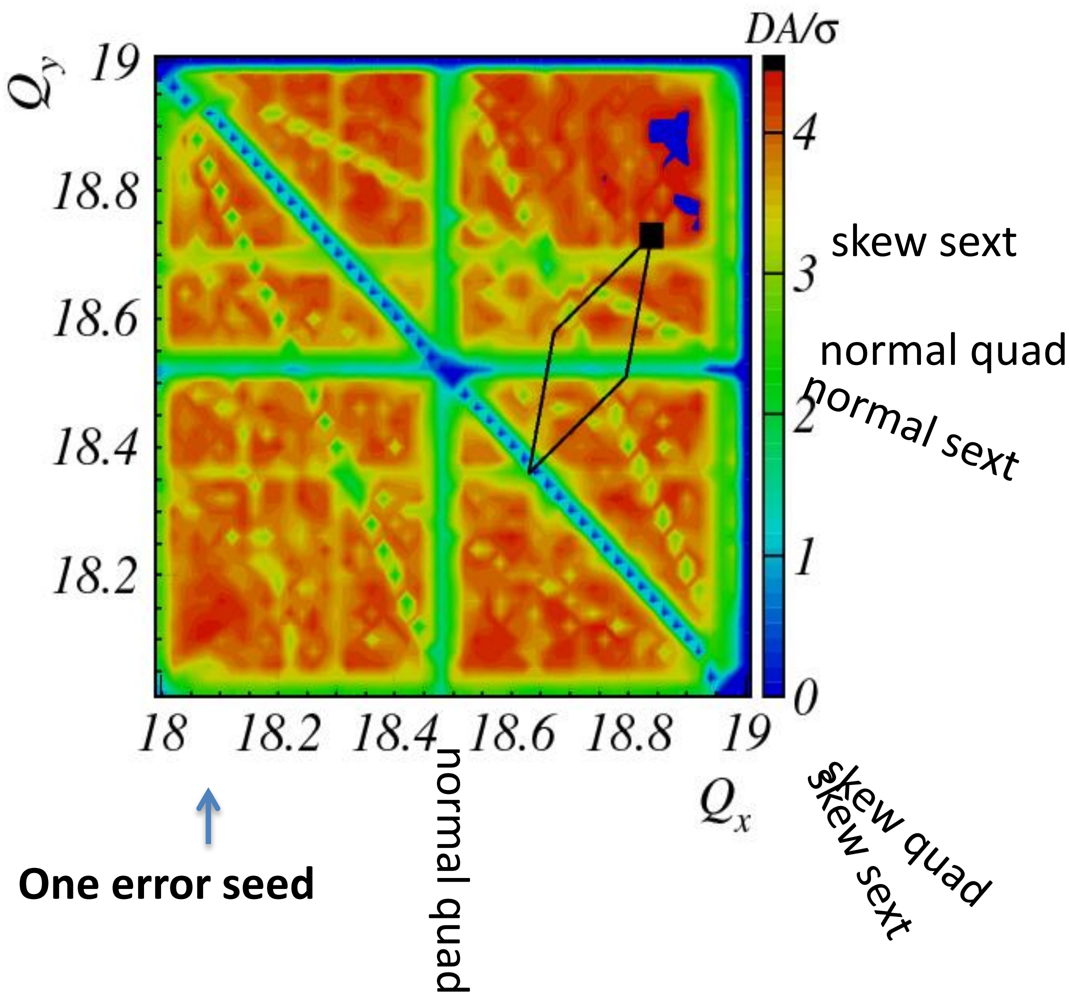


Close to $Q_x + 2Q_y = N$



Q_y
3.5
3.4
3.3
3.2
3.1
3
4
4.1
4.2
4.3
4.4
4.5
Q_x
Dynamics near near of 3rd order resonance is of interest

Relevance

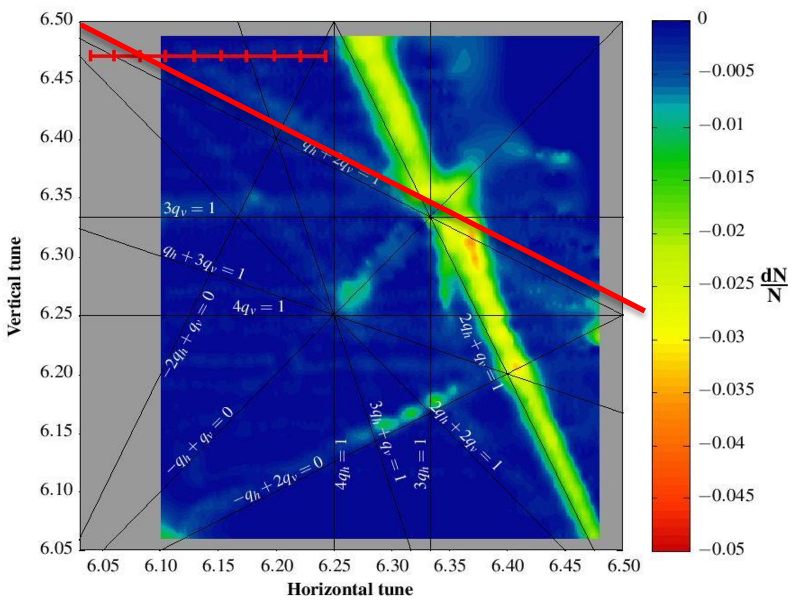


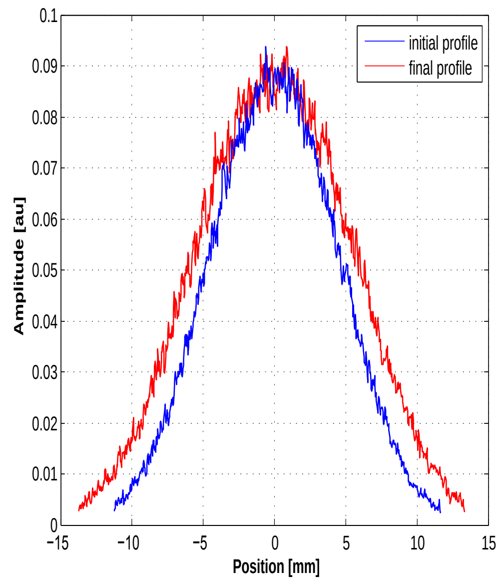
Periodic resonance crossing of a coupled 3rd order resonance induced by space charge

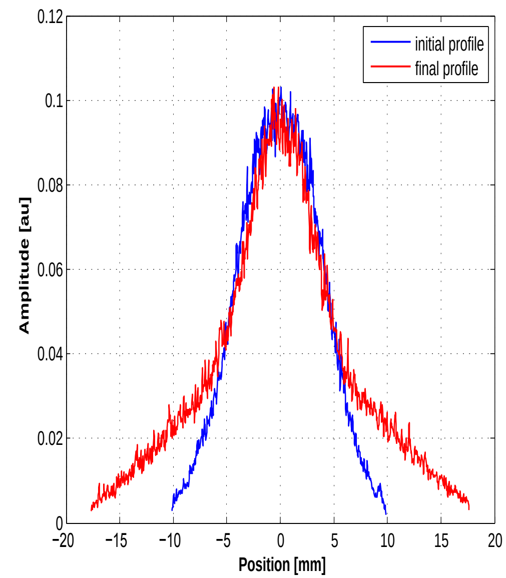
Issues of beam halo prediction and of resonance compensation

Experimental signature of something strange

Resonance: $q_x + 2\, q_y = 19$







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Unfortunately coupled resonances require a 4D treatment of the dynamics

$$nQ_x + mQ_y = N$$

structure of resonances is controlled by the equations

$$\frac{d^2x}{ds^2} - k_x x = \quad Re\left[\sum_{n=2}^{M}(k_n(s) + ij_n(s))\frac{(x+iy)^n}{n!}\right]$$

$$\frac{d^2y}{ds^2} - k_y y = -Im\left[\sum_{n=2}^{M}(k_n(s) + ij_n(s))\frac{(x+iy)^n}{n!}\right]$$

Do we find fix points in proximity of $2Q_y + Q_x = N$?

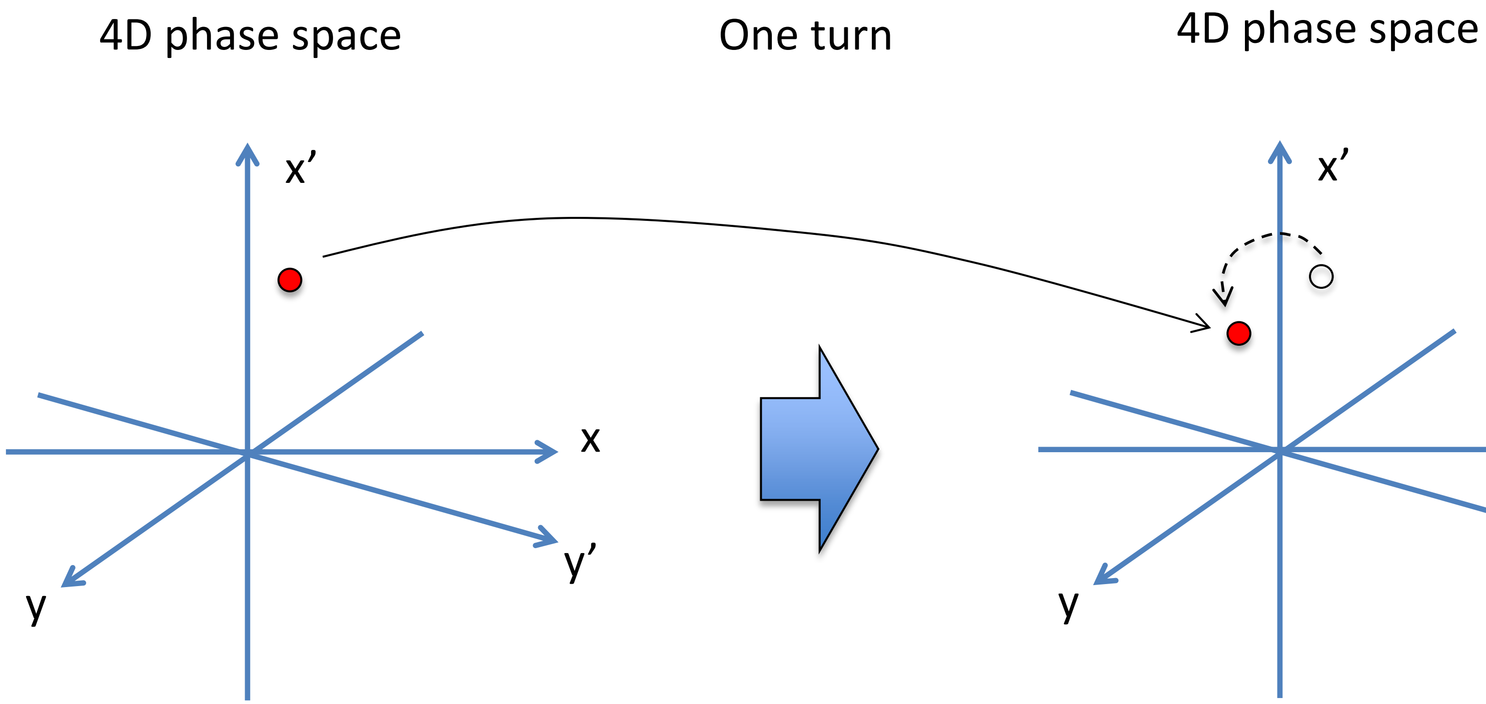
Wrong concept!

For $2Q_y + Q_x = N$ there are **fix-lines**

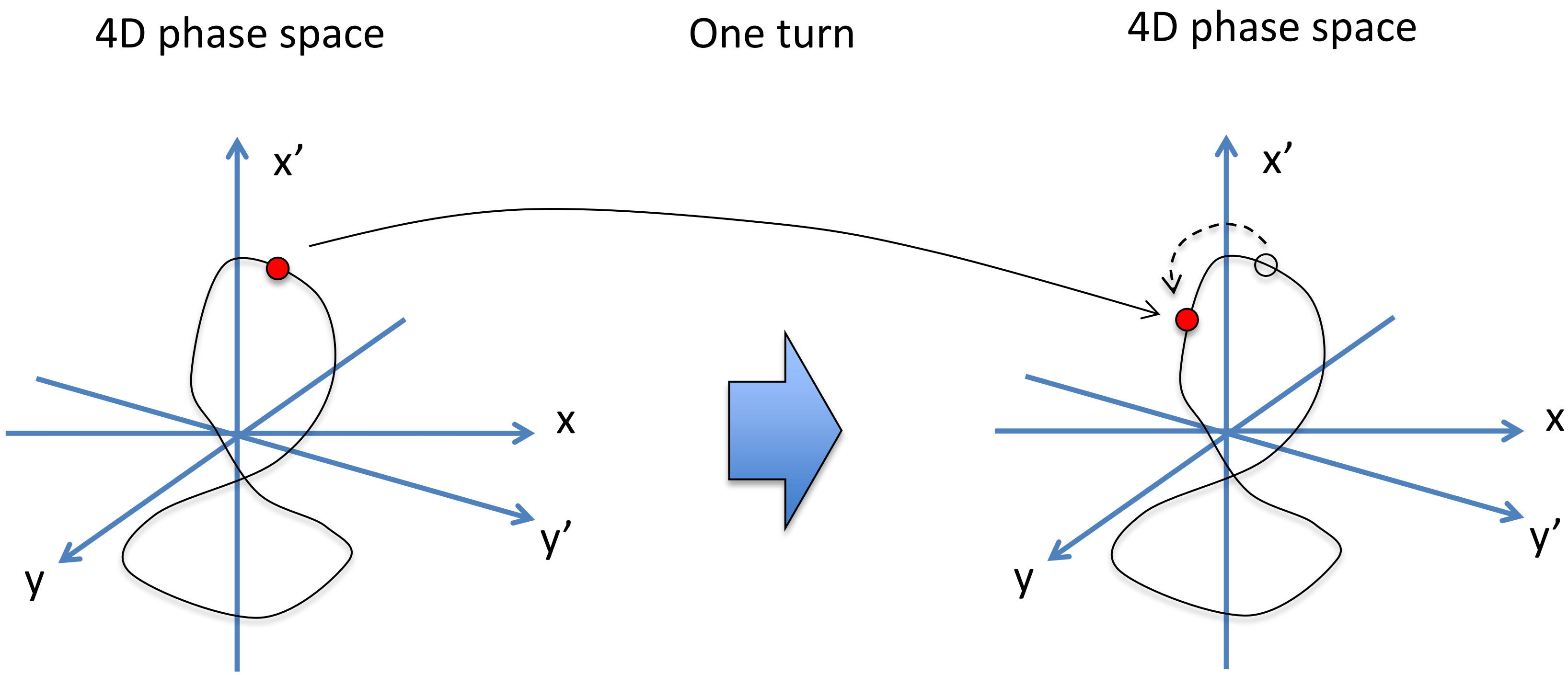
F. Schmidt
PhD

These are closed lines in the 4D phase space of which we can only see the projections

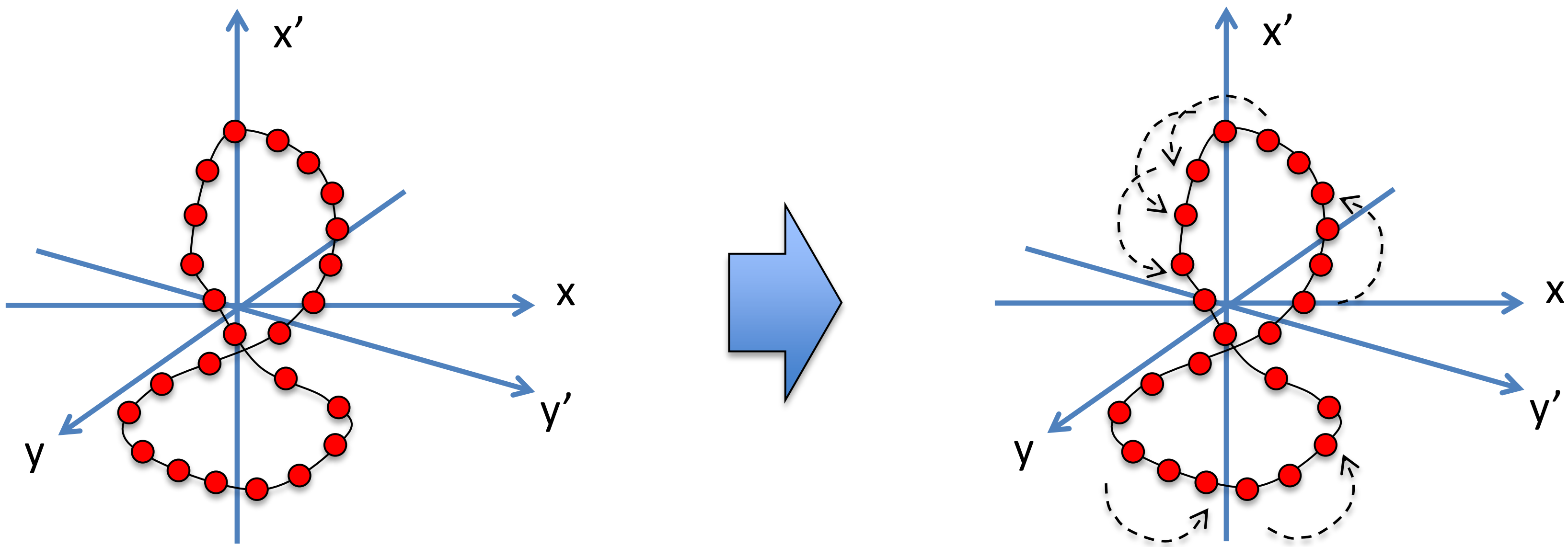
Close to the resonance



Close to the resonance

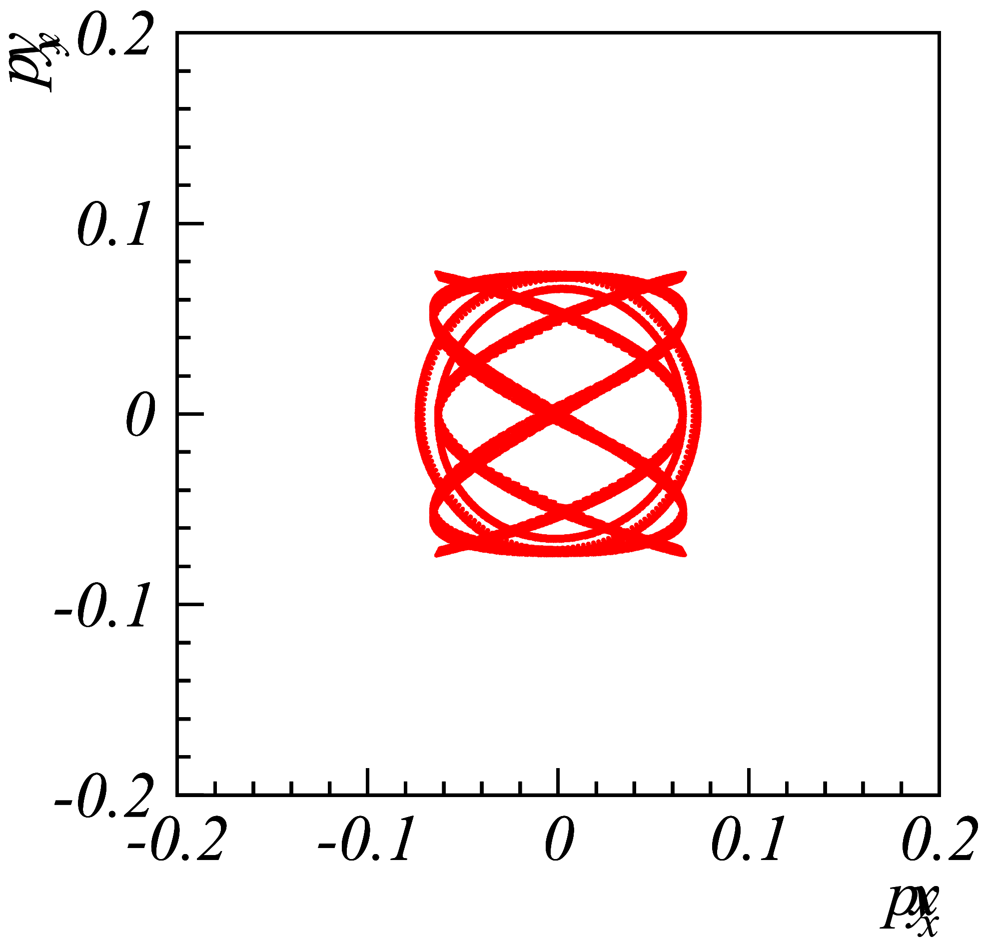


4D phase space One turn 4D phase space



After one turn each point on the fix-line is mapped into the fix-line

Fix-line projections



Frank
Schmidt
PhD

Fundamental Questions

(Where Do We Come From? What Are We? Where Are We Going?)

1) Can we characterize these objects with a mathematic expression?
2) Can we predict their extension as functions of lattice nonlinear errors?
3) How many fix-lines do we have ?
4) Can a fix-line be stable or unstable? What does it mean ?
5) Do we have "secondary" tunes? What does it mean ?
6) Do we have a concept of "island"?
7) How are fix-lines related to the stability domain ?

1) Analytic form

projection of a fix-line close to the 3rd order coupled resonance

$$x_t = \sqrt{\beta_x a_x} \cos[-2(Q_y - t_y)t + \pi M]$$
$$y_t = \sqrt{\beta_y a_y} \cos[(Q_y - t_y)t]$$

a_x, a_y are the invariant of the fix line, they depends on the “distance” from the resonance, and they are **CONSTANT**

M is an integer that depends on the sign of $\Delta_r = -N + Q_x + 2Q_y$ the “distance” of the resonance

t_y parameter that specify the canonical transformation in which makes a_x, a_y time independent.

t parameter that parametrize the fix-line

The shape of the line is set by the order of the resonance.

The amplitude of the fix-line is determined by a_x, a_y

Therefore ignoring the position of a particle in a fix-line, the fix-line can be identified by a proper pair of a_x, a_y

2) Fix-line extension function of lattice

Fix-line is given by

$$0 = \Delta_r \Lambda (-1)^M \left[\frac{1}{2\sqrt{\tilde{a}_x}} \tilde{a}_y + 2\sqrt{\tilde{a}_x} \right] + \frac{\Delta_r^2}{2}$$

with $\Lambda = \sqrt{\Lambda_c^2 + \Lambda_s^2}$ $\cos\theta = \Lambda_c / \Lambda$ $\sin\theta = \Lambda_s / \Lambda$

$$\Lambda_c = -\sum_j \frac{1}{8L} K_{2j} \sqrt{\beta_{xj}} \beta_{yj} \cos\left[2\pi \frac{s_j}{L} N + \mathscr{D}_x(s_j) + 2\mathscr{D}_y(s_j) \right]$$

$$\Lambda_s = -\sum_j \frac{1}{8L} K_{2j} \sqrt{\beta_{xj}} \beta_{yj} \sin\left[2\pi \frac{s_j}{L} N + \mathscr{D}_x(s_j) + 2\mathscr{D}_y(s_j) \right]$$

K_{2j} are the normal integrated sextupolar strength of all errors and correctors

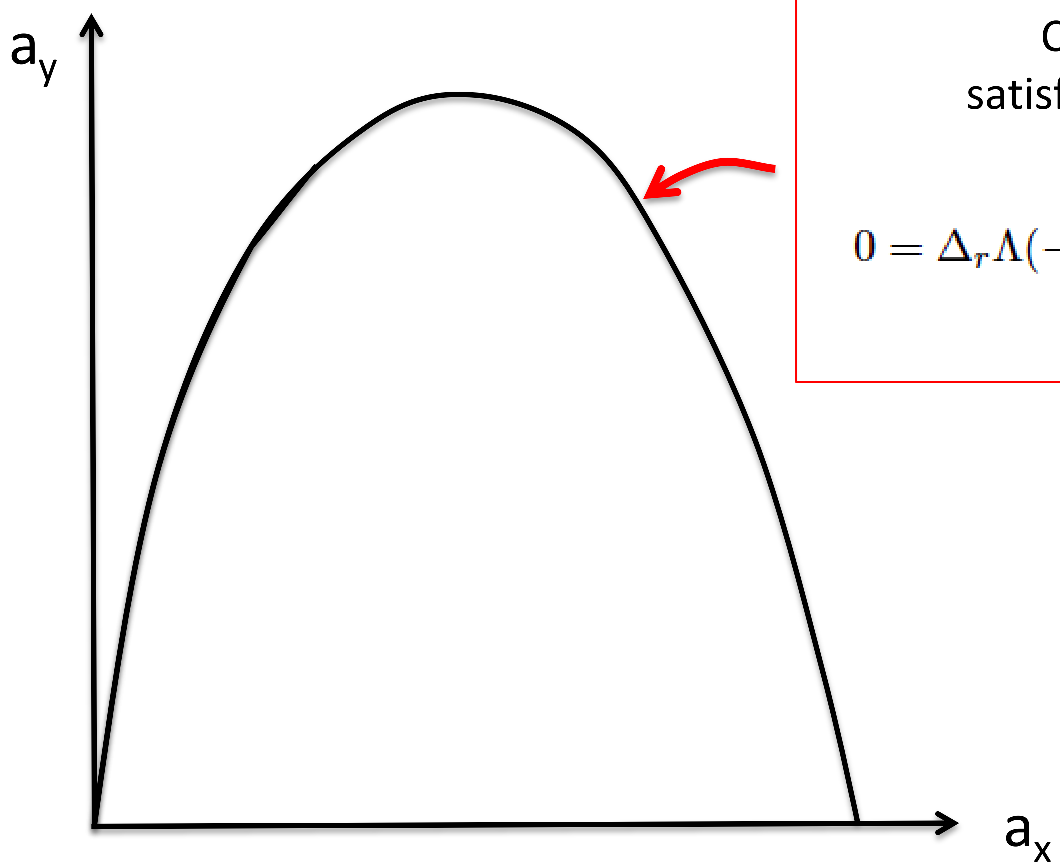
3) How many fix-lines ?

Near $3Q_x = N$ there are 3 unstable fix points

Near $2Q_y + Q_x = N$ there are **infinite** fix-lines !!

What is the meaning of this ?

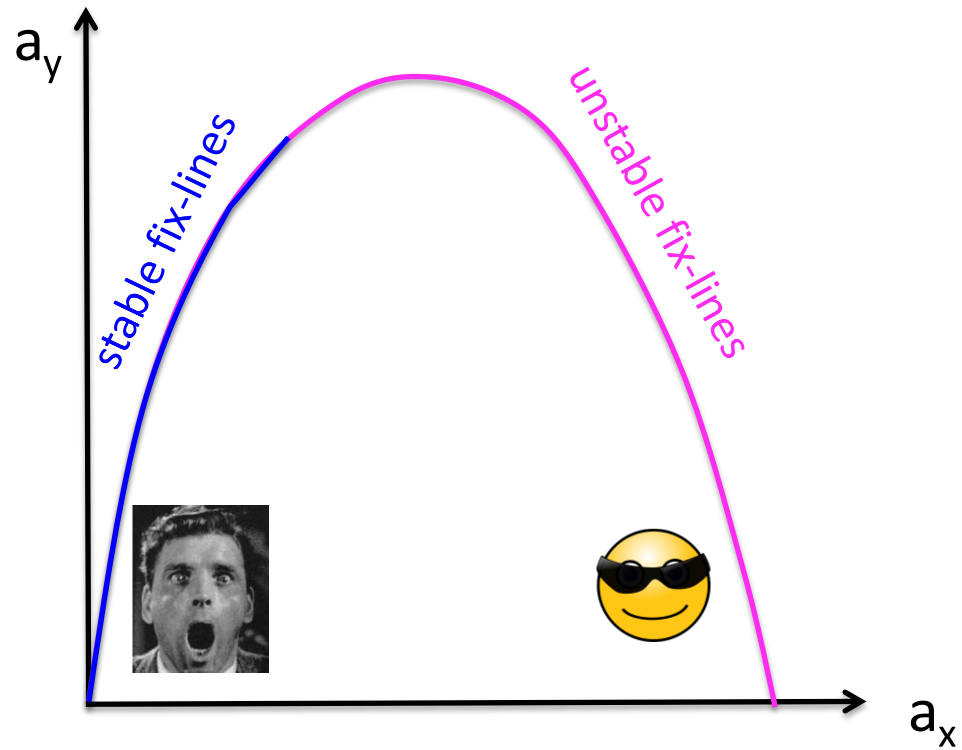
All the infinite fix-lines



Collection of all (a_x, a_y)
satisfying the fix-line equation

$$0 = \Delta_r \Lambda (-1)^M \left[\frac{1}{2\sqrt{\tilde{a}_x}} \tilde{a}_y + 2\sqrt{\tilde{a}_x} \right] + \frac{\Delta_r^2}{2}$$

4) Fix-lines are stable or unstable?



$$0 = \Delta_r \Lambda (-1)^M \left[\frac{1}{2\sqrt{\tilde{a}_x}} \tilde{a}_y + 2\sqrt{\tilde{a}_x} \right] + \frac{\Delta_r^2}{2}$$

5) The secondary tune: the dynamics off the fix-line

$$x = \sqrt{\beta_x a_x} \cos(\phi_x)$$
$$y = \sqrt{\beta_y a_y} \cos(\phi_y)$$

Particle coordinates parameterization

a_x, a_y are constant

$$\phi_{x/y} = \varphi_{x/y} + \int_0^s \frac{1}{\beta_{x/y}} ds$$

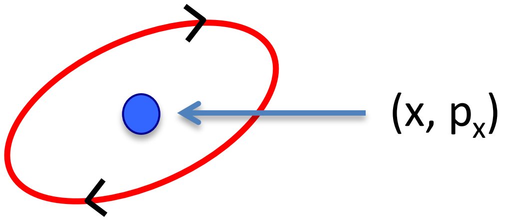
In a linear lattice

Near the resonance $a_x, \varphi_x, a_y, \varphi_y$ becomes time dependent

Only on **a fix-line** $a_x, \varphi_x, a_y, \varphi_y$ are constant

The problem of the secondary tune

In a 1D resonance
stable points means that

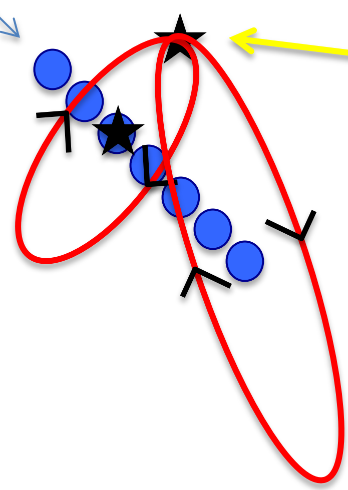


The angular velocity → secondary tunes

The problem of the secondary tune

What does it means if we have infinite fix-lines ?

fix-lines in (a_x, a_y)



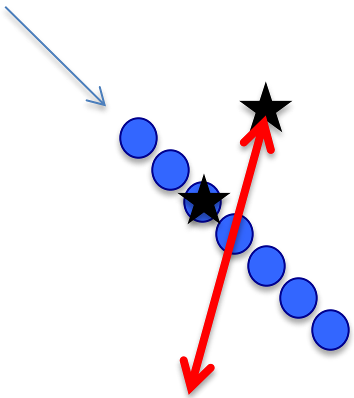
around which fix-line is this particle oscillating

?

The problem of the secondary tune

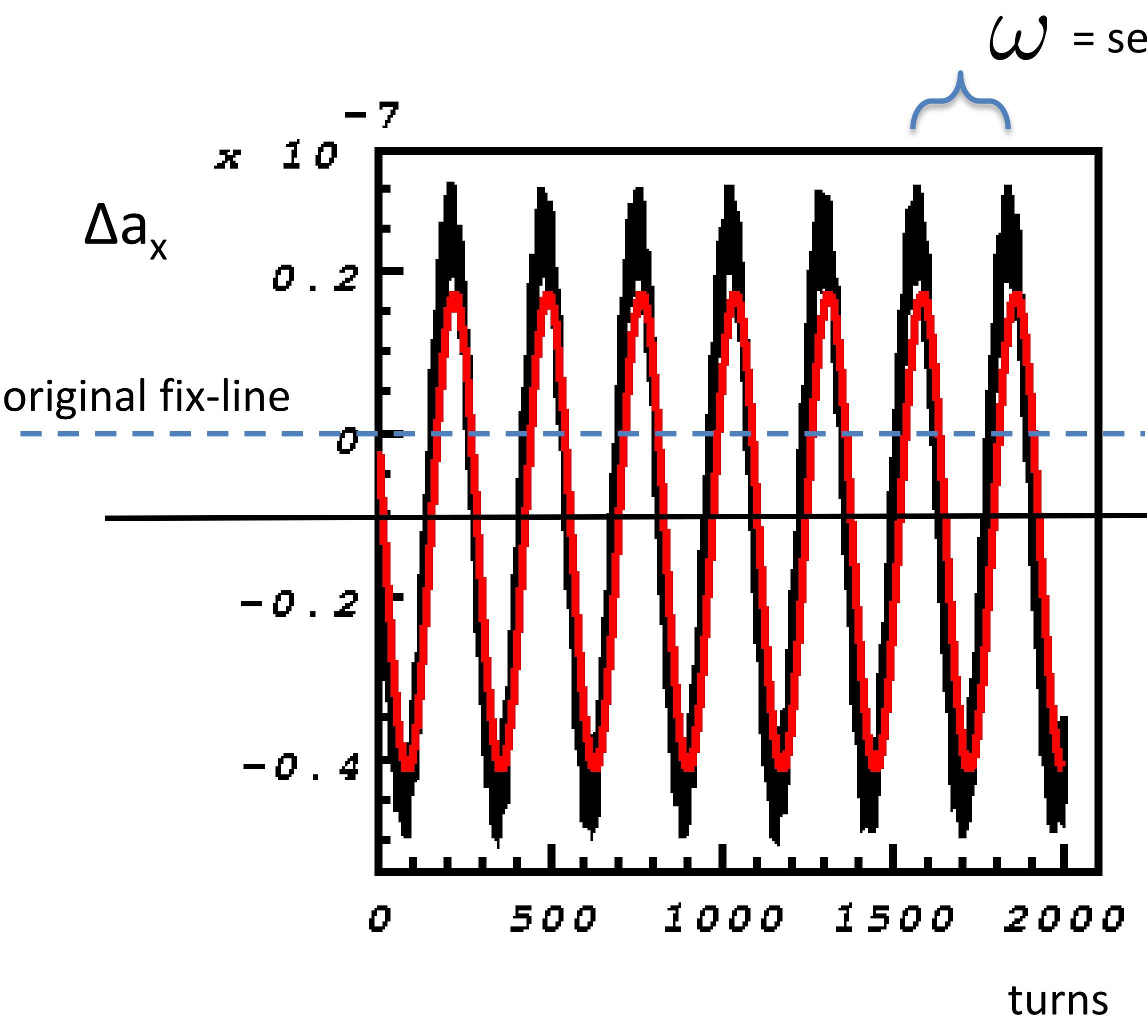
What does it means if we have infinite fix-lines ?

fix-lines in (a_x, a_y)



There is only one special direction of oscillation,
which identify a unique fix-line

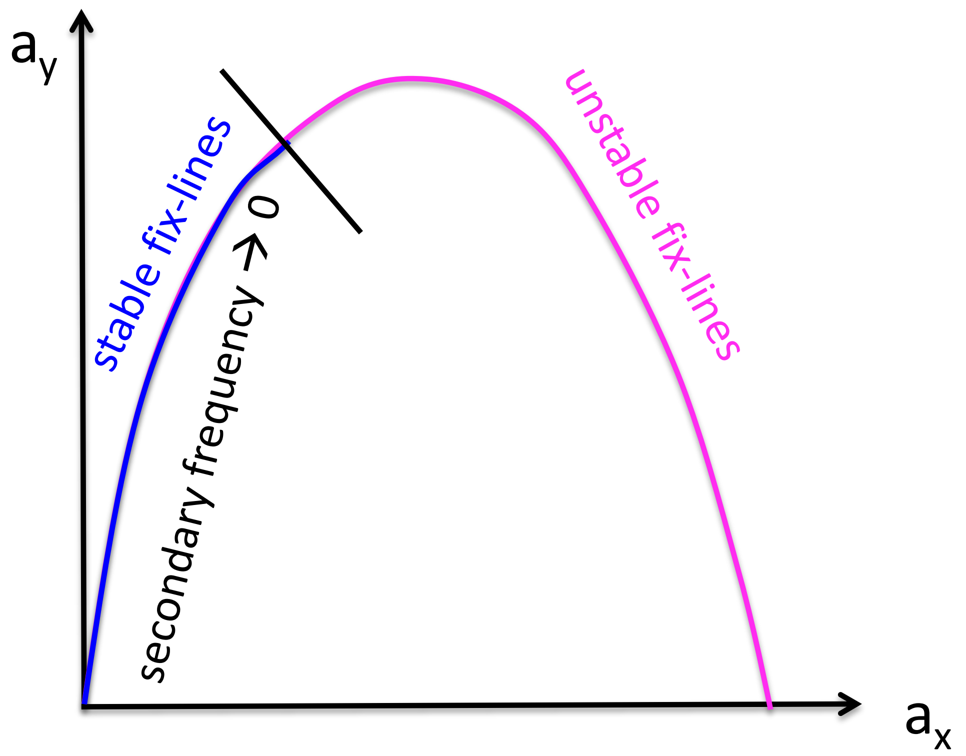
A very strange stability…



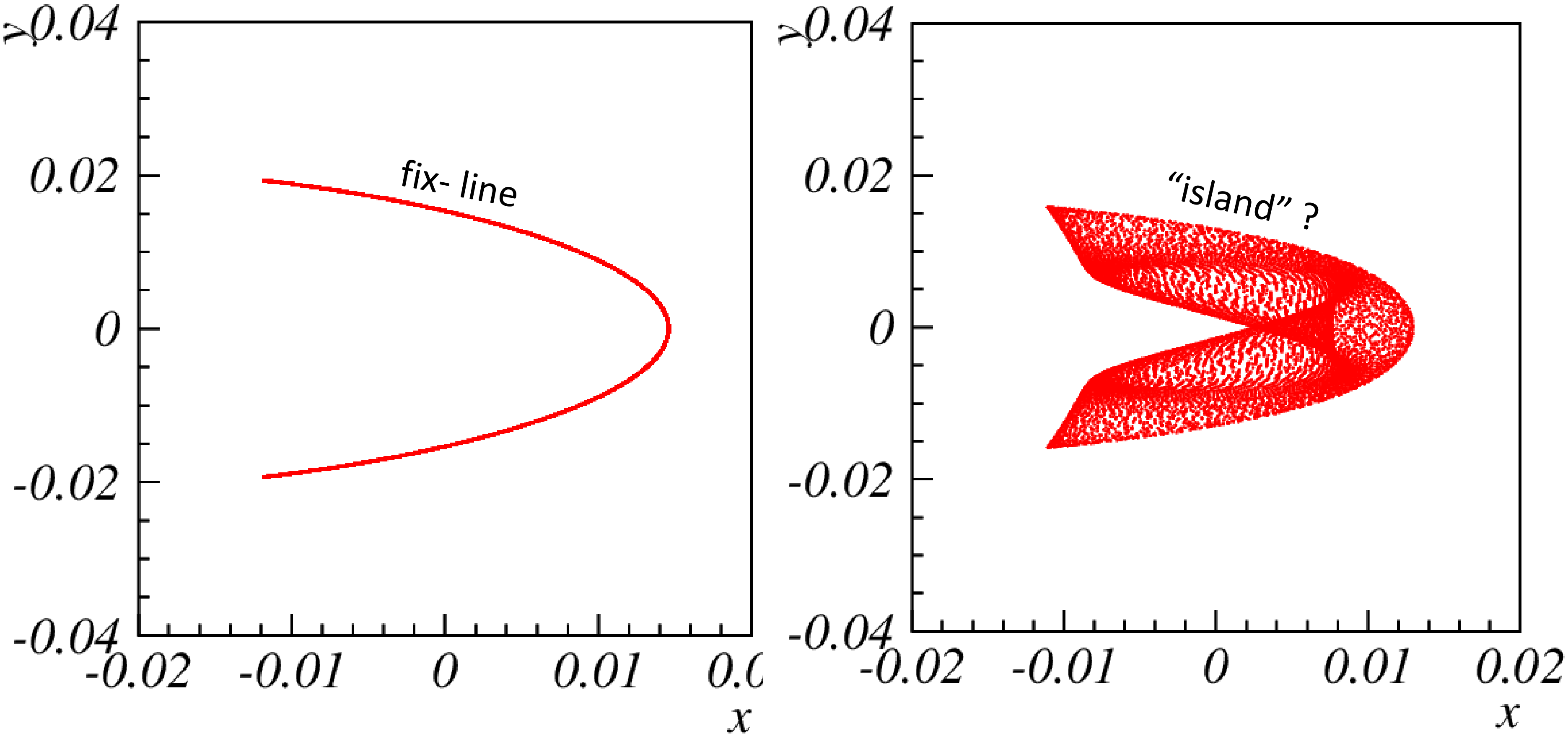
it oscillates around another fix-line

- simulations
- theory

All the fix-lines



6) Islands or not Islands… that is the problem



7) Stability & fix-lines

$$x = \sqrt{\beta_x \tilde{a}_x} \cos(\tilde{\varphi}_x)$$

$$y = \sqrt{\beta_y \tilde{a}_y} \cos(\tilde{\varphi}_y)$$

particle coordinates parameterization

$$\Omega = \tilde{\varphi}_x + 2\tilde{\varphi}_y$$

← An interesting, and strange coordinate

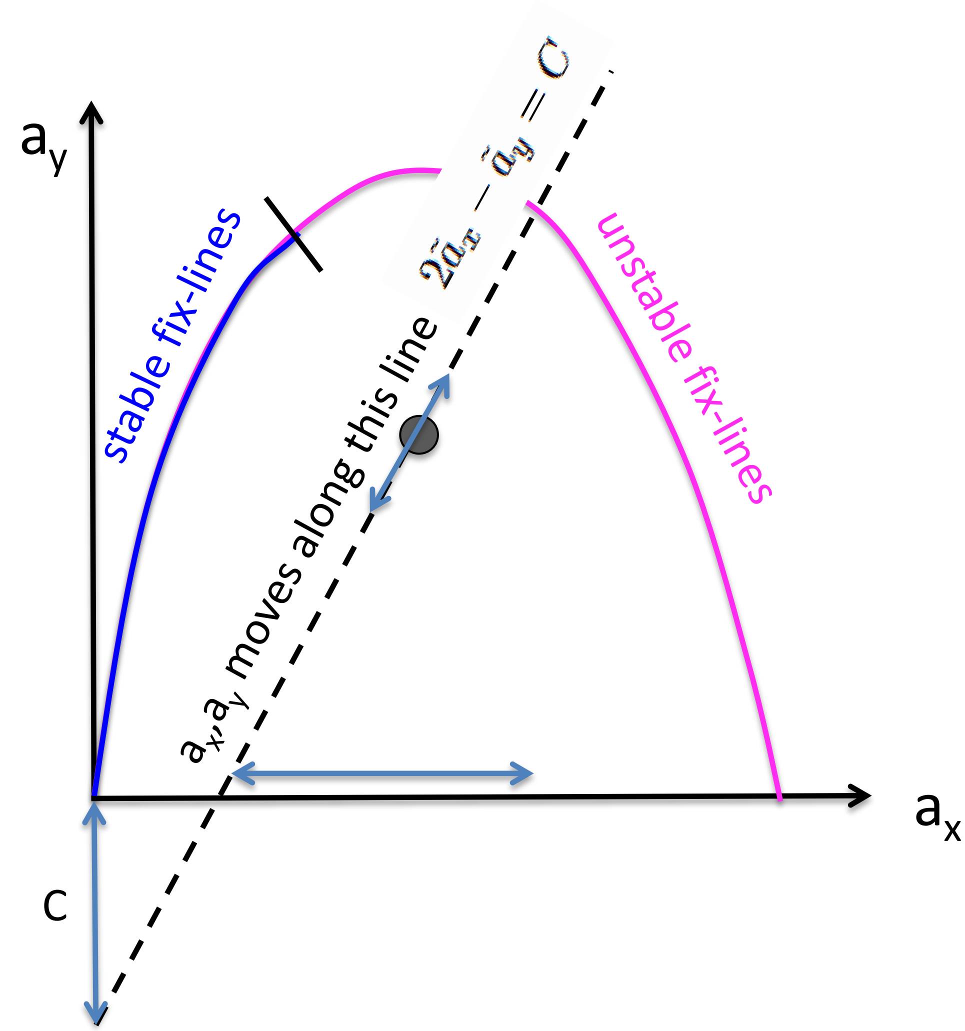
Two invariants of motion

$$2\tilde{a}_x - \tilde{a}_y = C$$

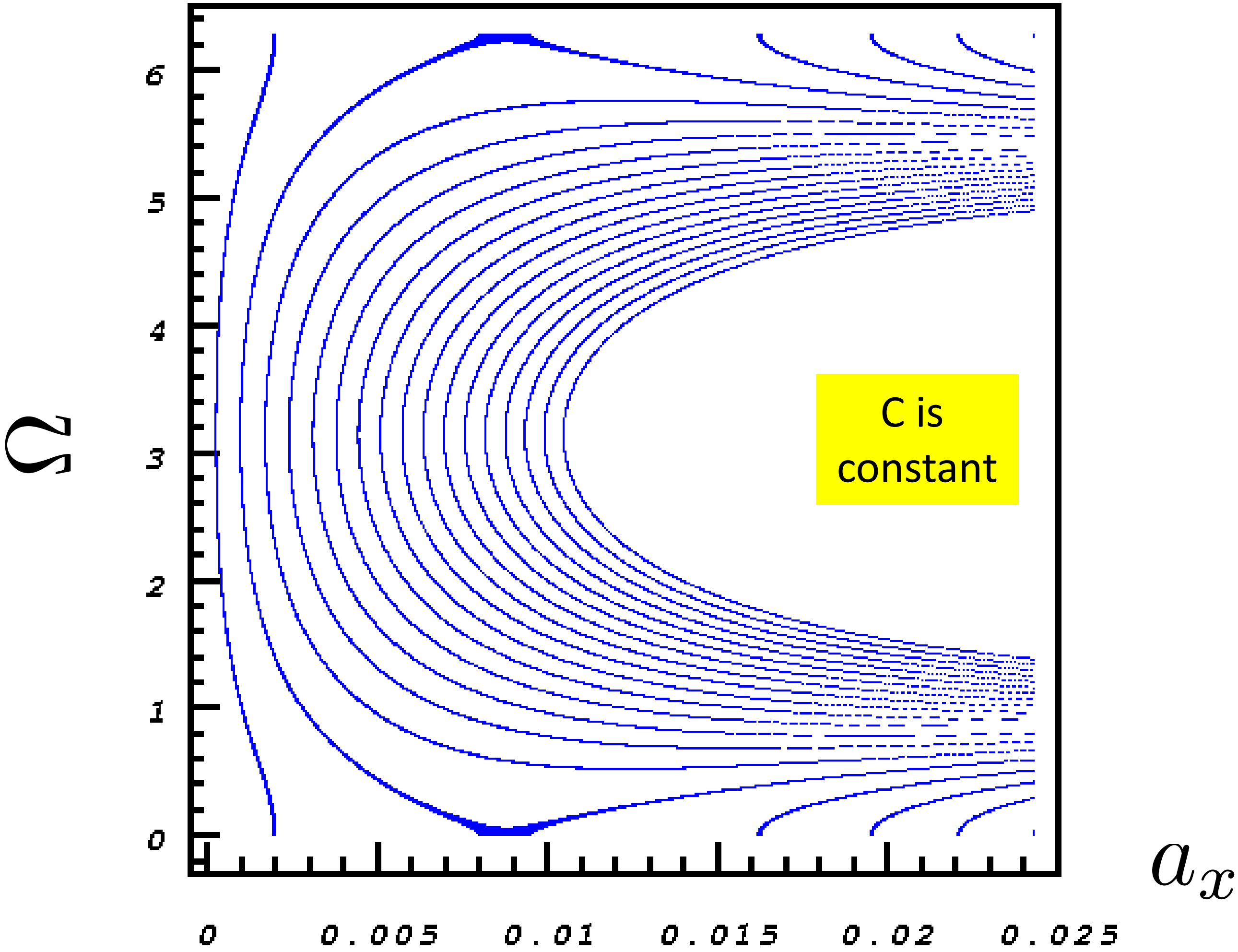
$$I(\tilde{a}_x, \Omega) = 2\Lambda \sqrt{\tilde{a}_x}(2\tilde{a}_x - C)\cos[\Omega] + \tilde{a}_x \Delta_r$$

a strange invariant

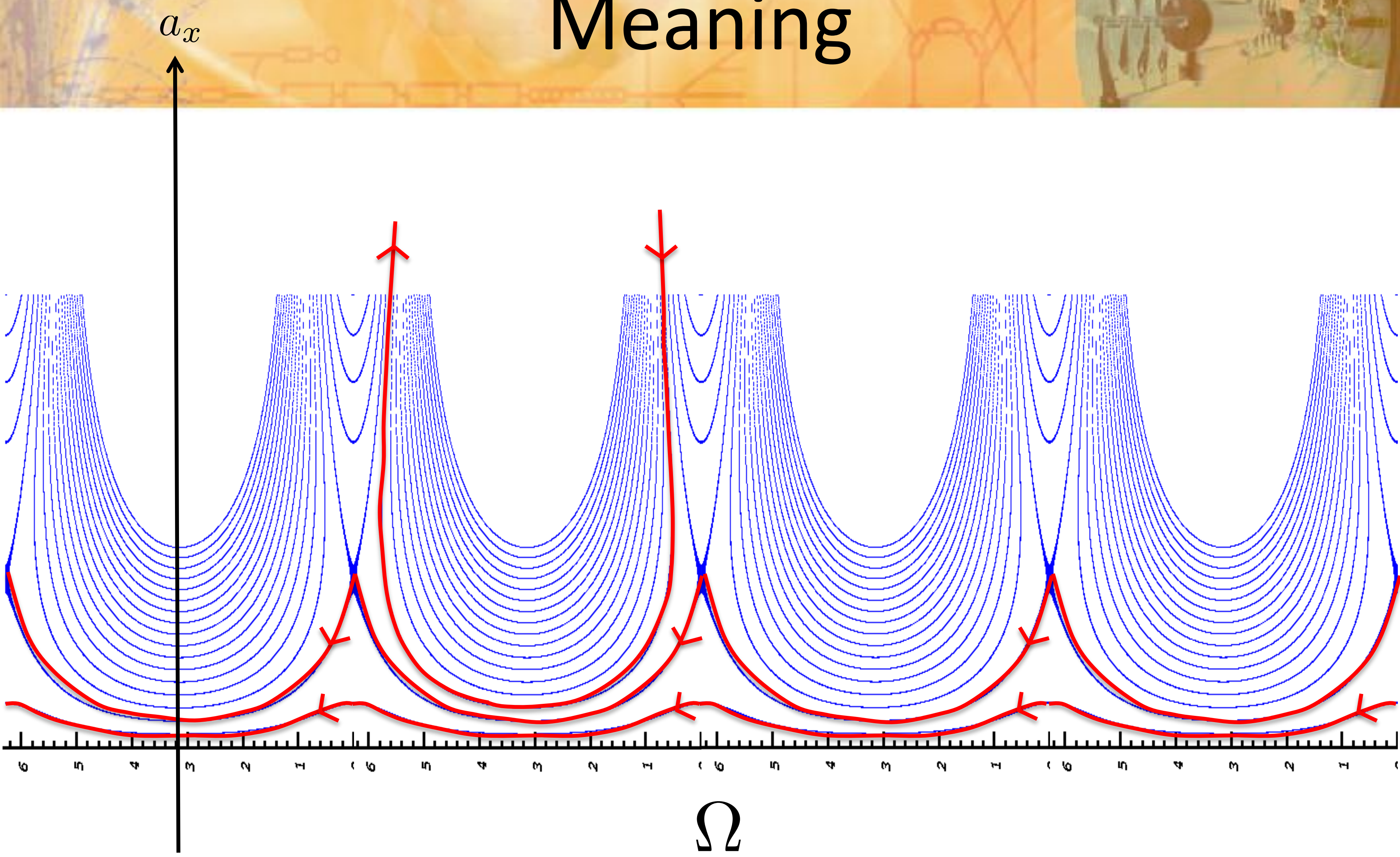
Consequences of the first invariant



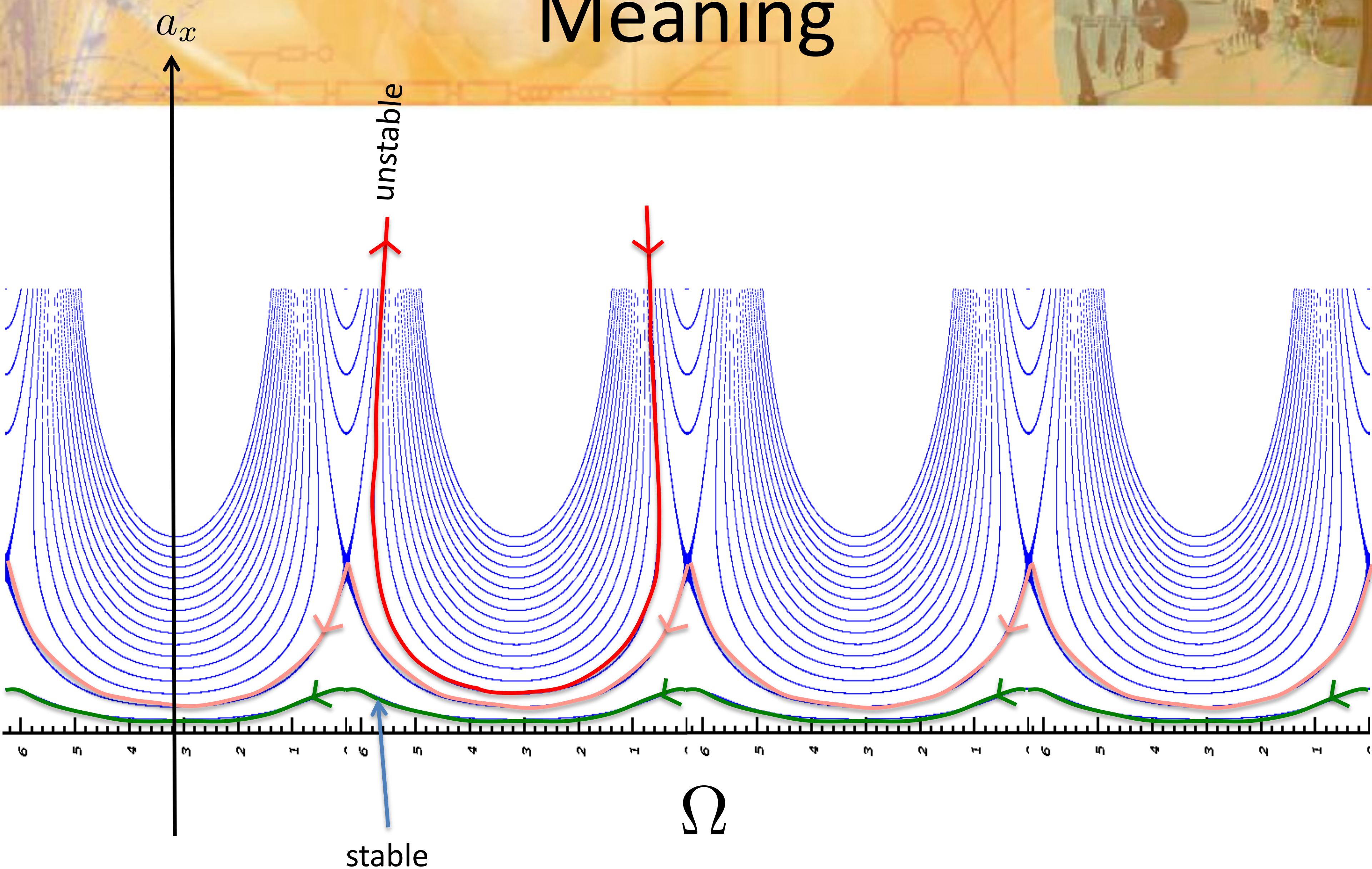
The second invariant: level lines

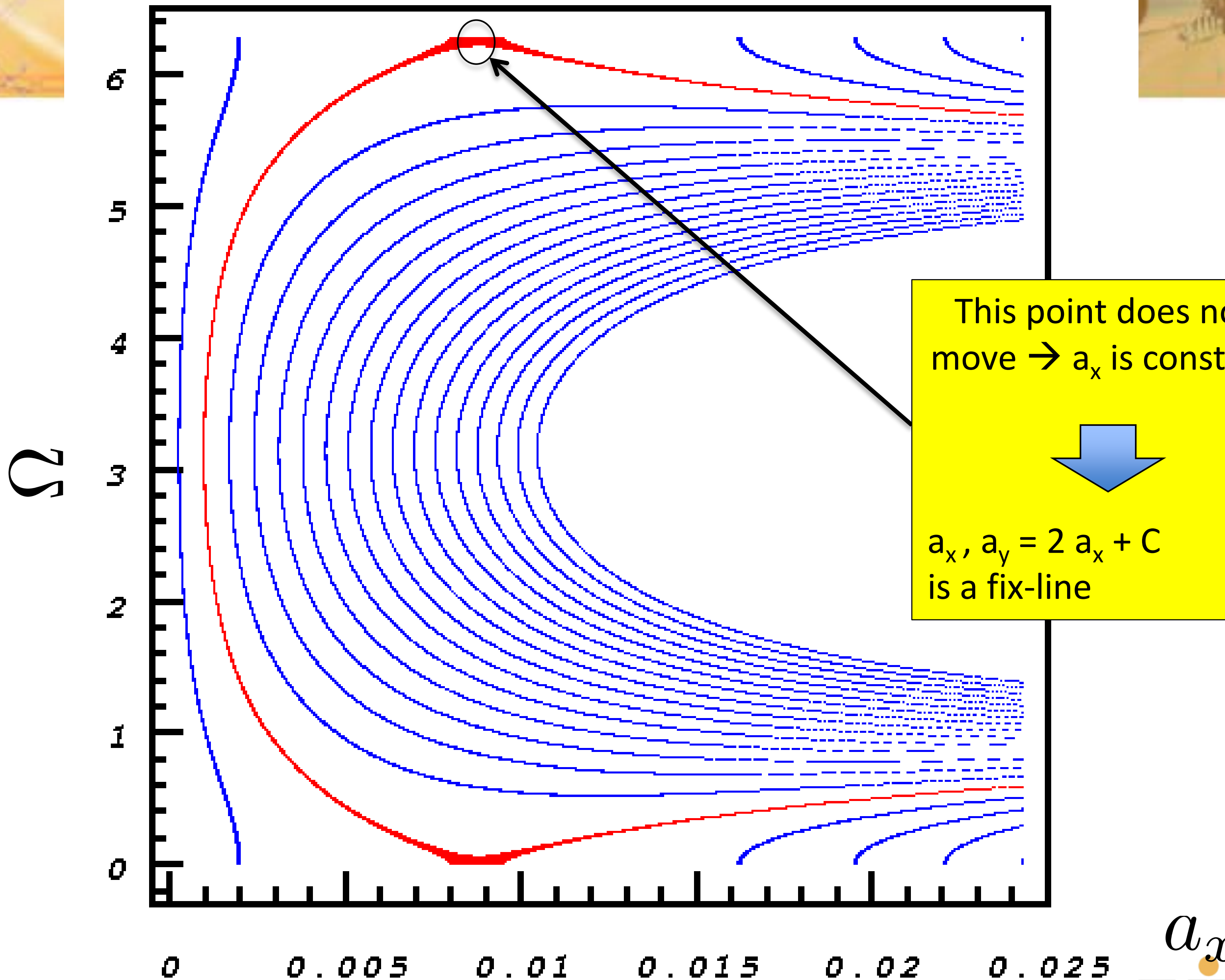


Meaning



Meaning

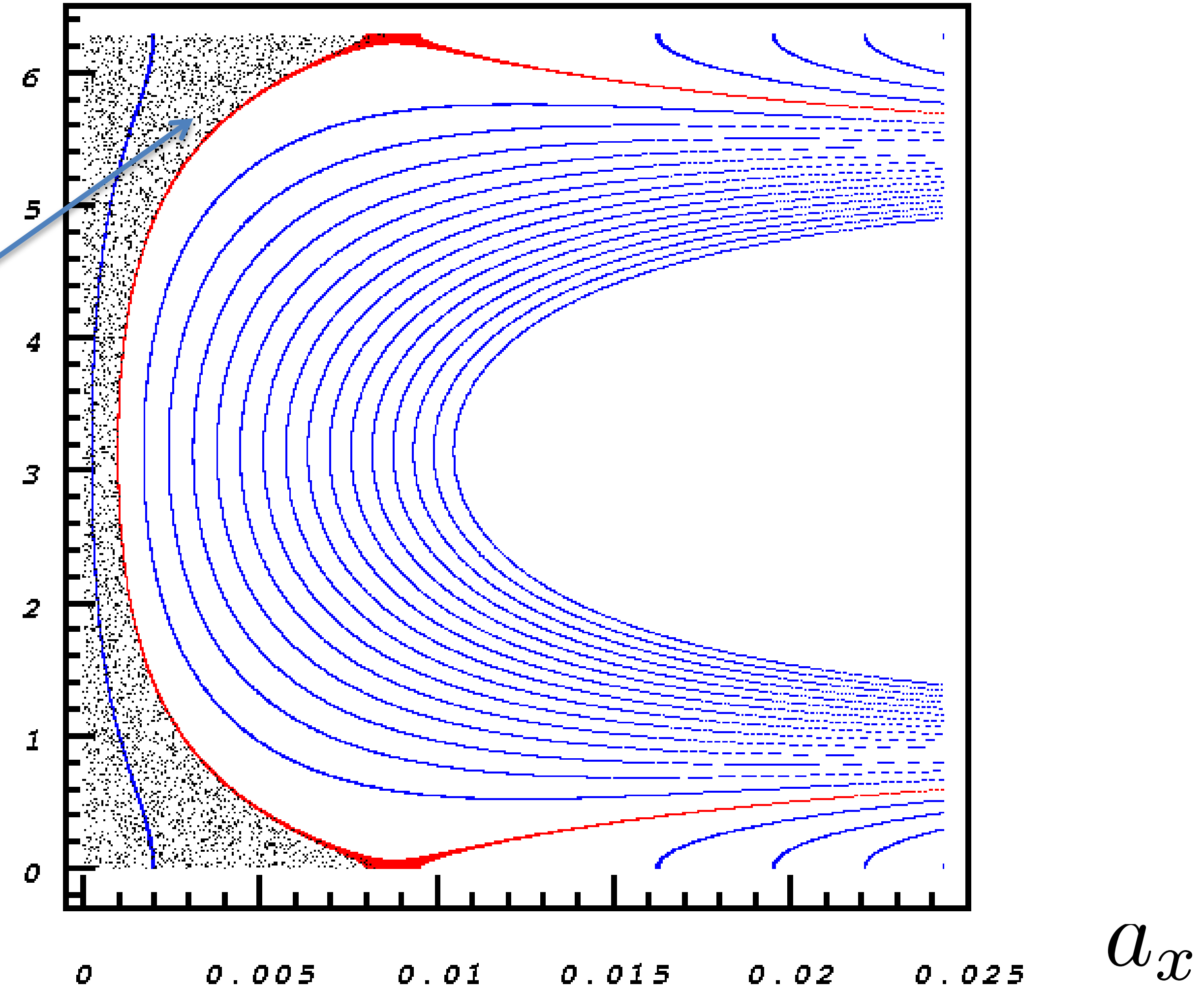




This point does not move → a_x is constant

a_x, $a_y = 2\,a_x + C$
is a fix-line

Comparison with simulations

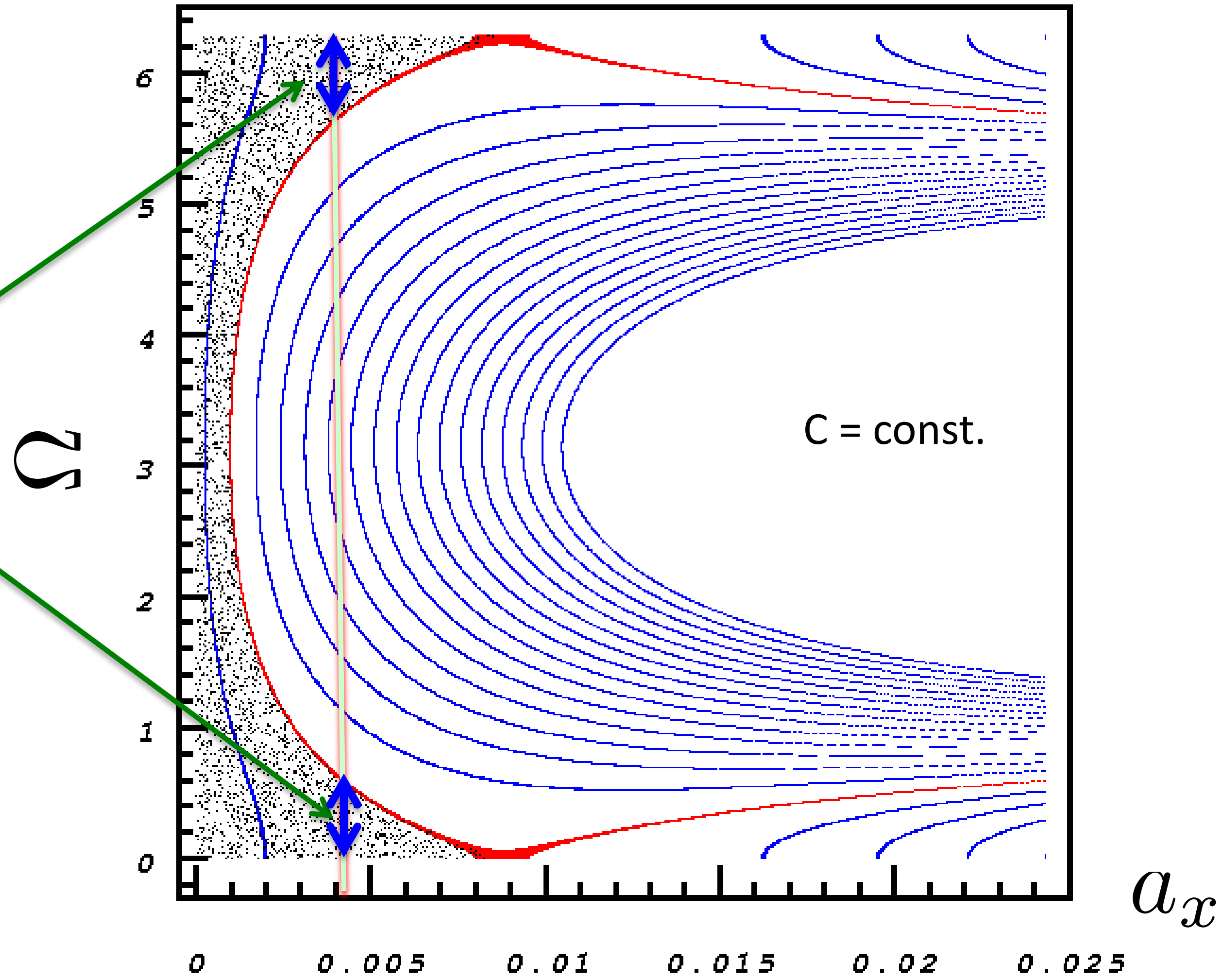


Initial condition of particles that are stable

very nice !

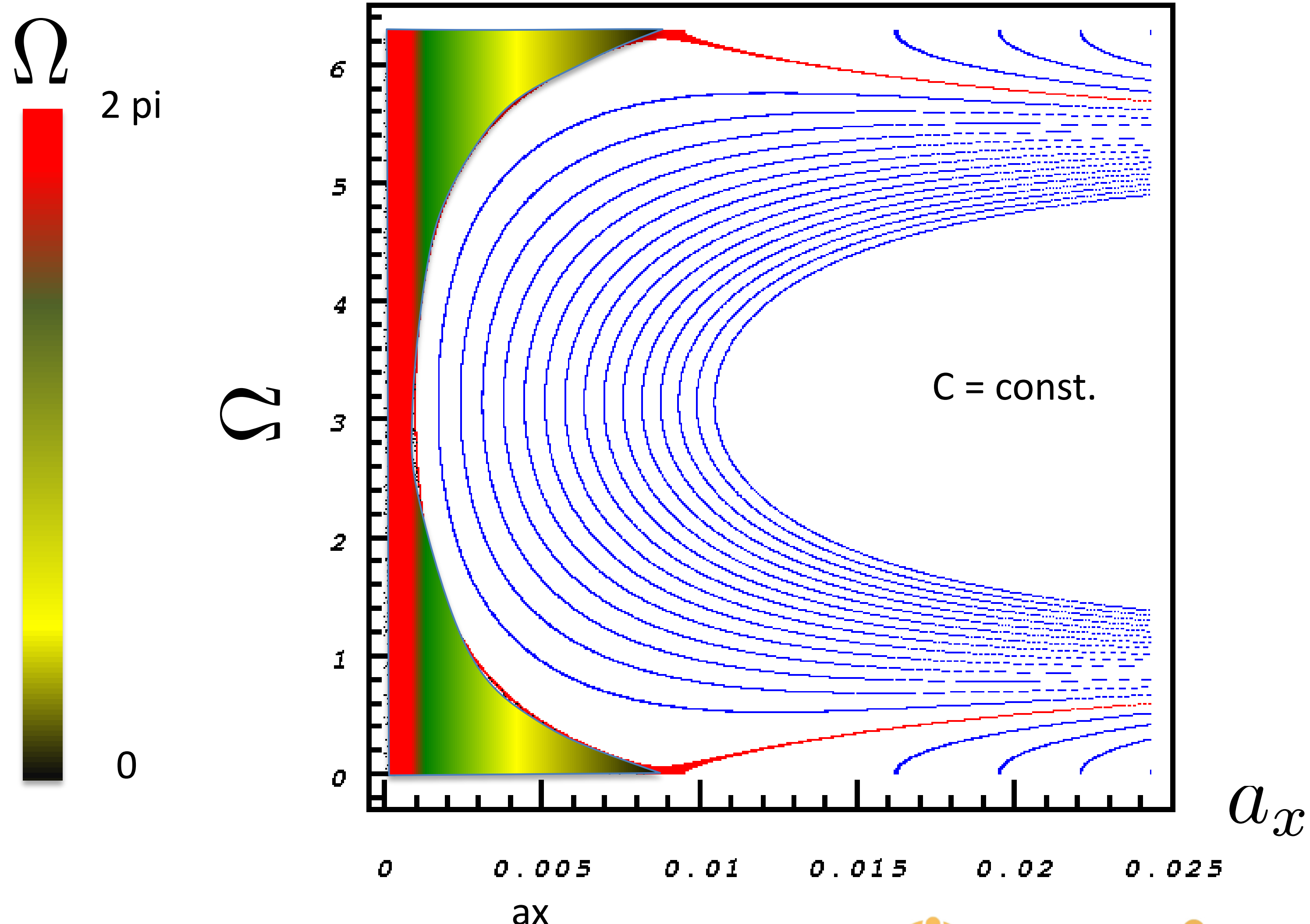
Measuring the stable area

Particle with initial a_x, a_y fix C, then the allowed $\Delta\Omega$ is this



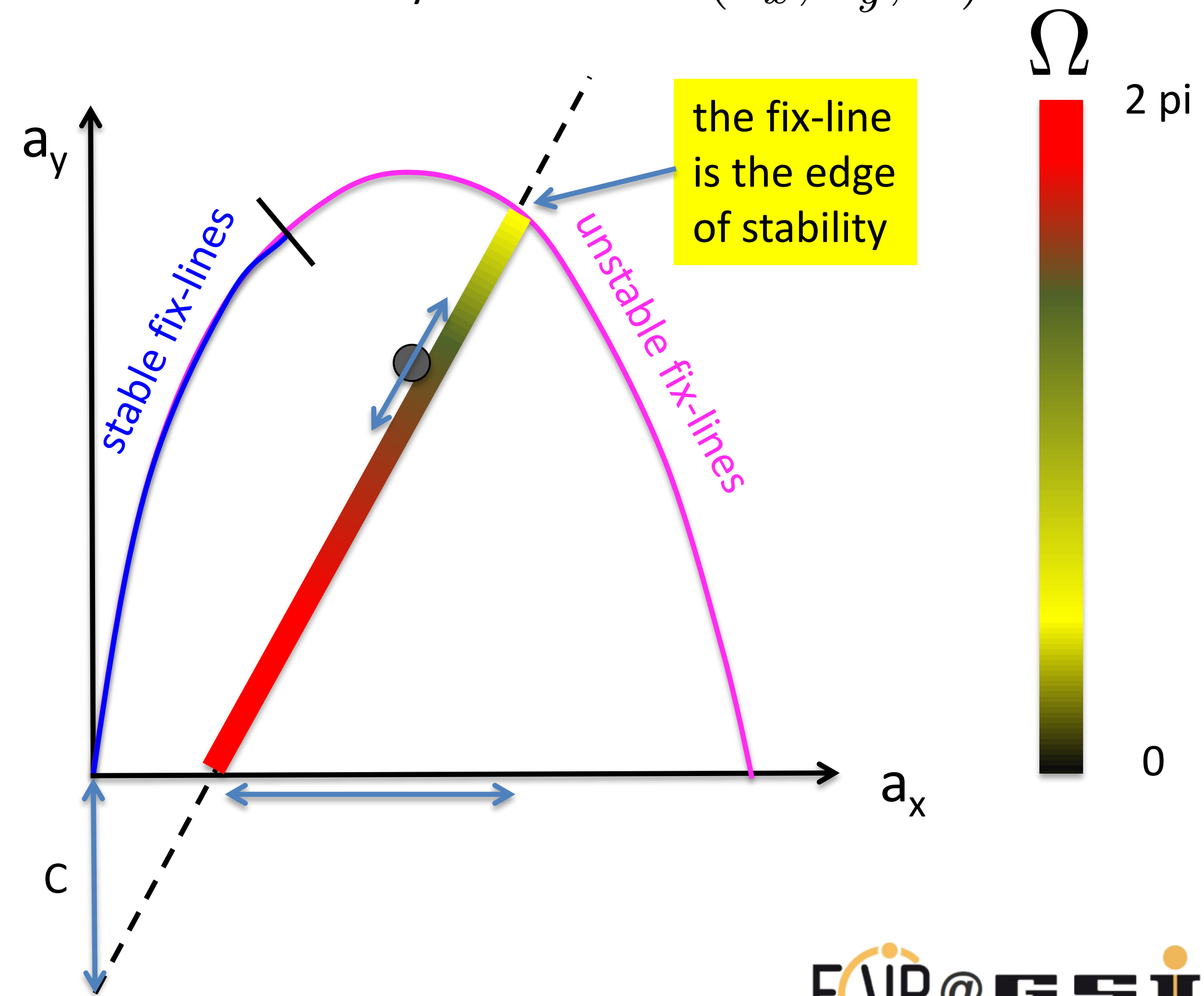
a_x

Measuring the stable area

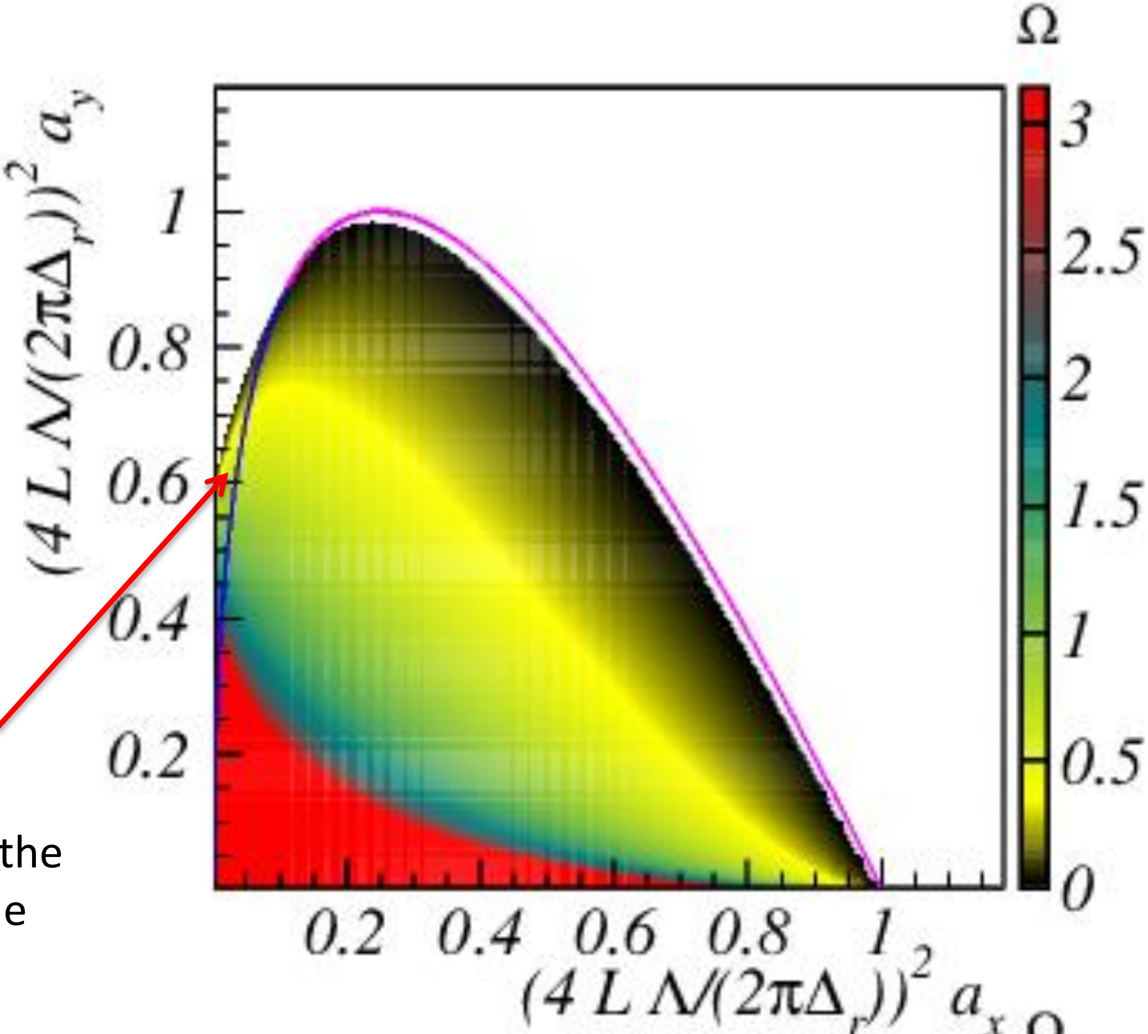


Stability domain

We can characterize the stability domain with (a_x, a_y, Ω)



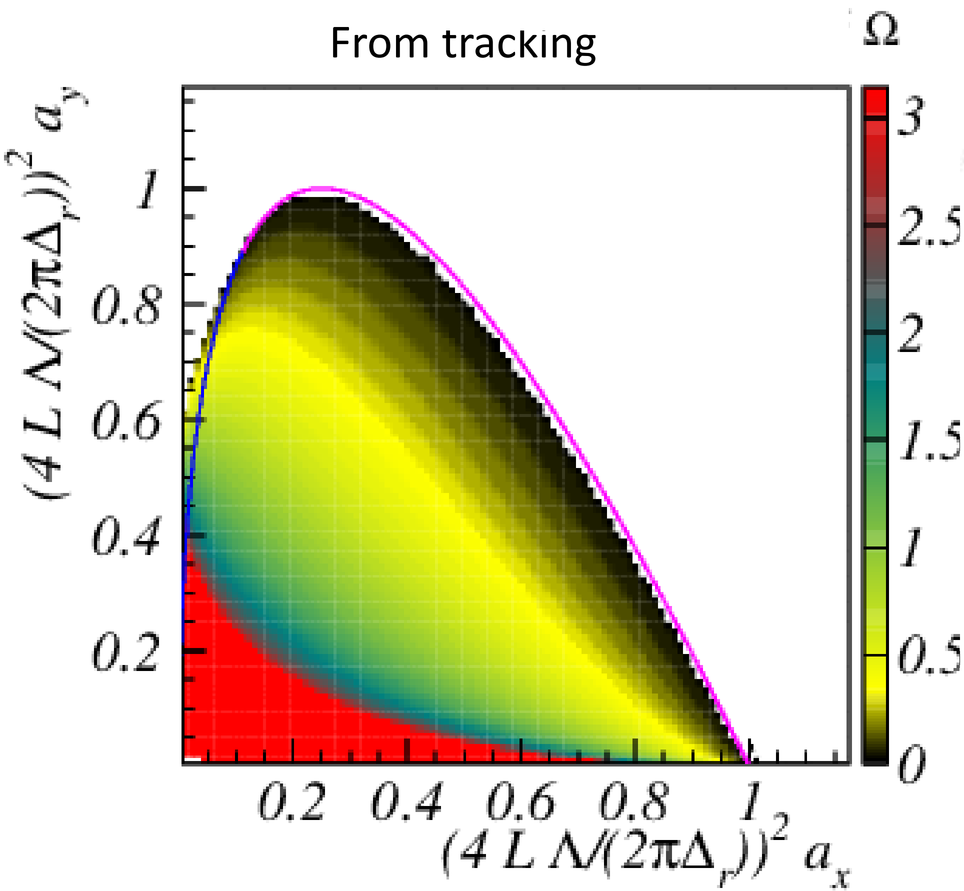
Stability domain



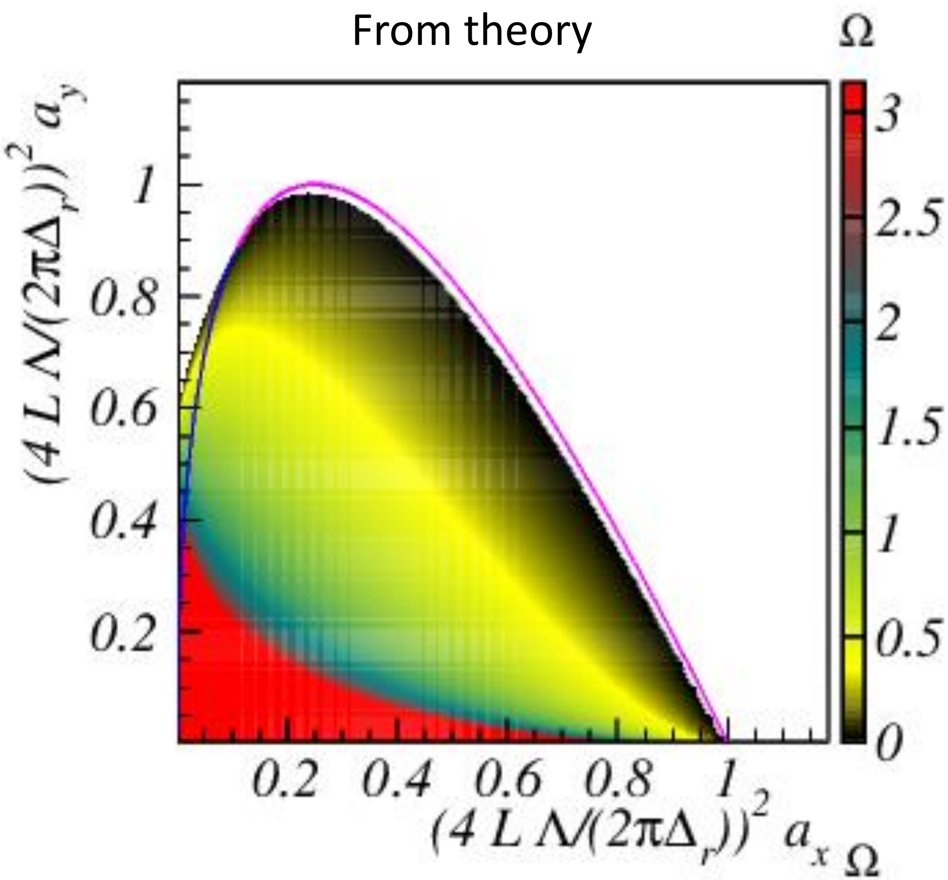
Here the coordinates are beyond the stable fix-line

Comparison with simulations

From tracking



From theory



Really good! also in situation off of the “single harmonics limit”

Advantages

fine exploration of the stability domain with tracking

100 processors
2 hours
30x30x100 initial conditions
tracking 1000 turns

CPU time = $7x10^5$ sec.

fine exploration of the stability domain with analytic theory

stability during infinite turns !

CPU time = 3 seconds

gain > $2x10^5$

Conclusion/Outlook

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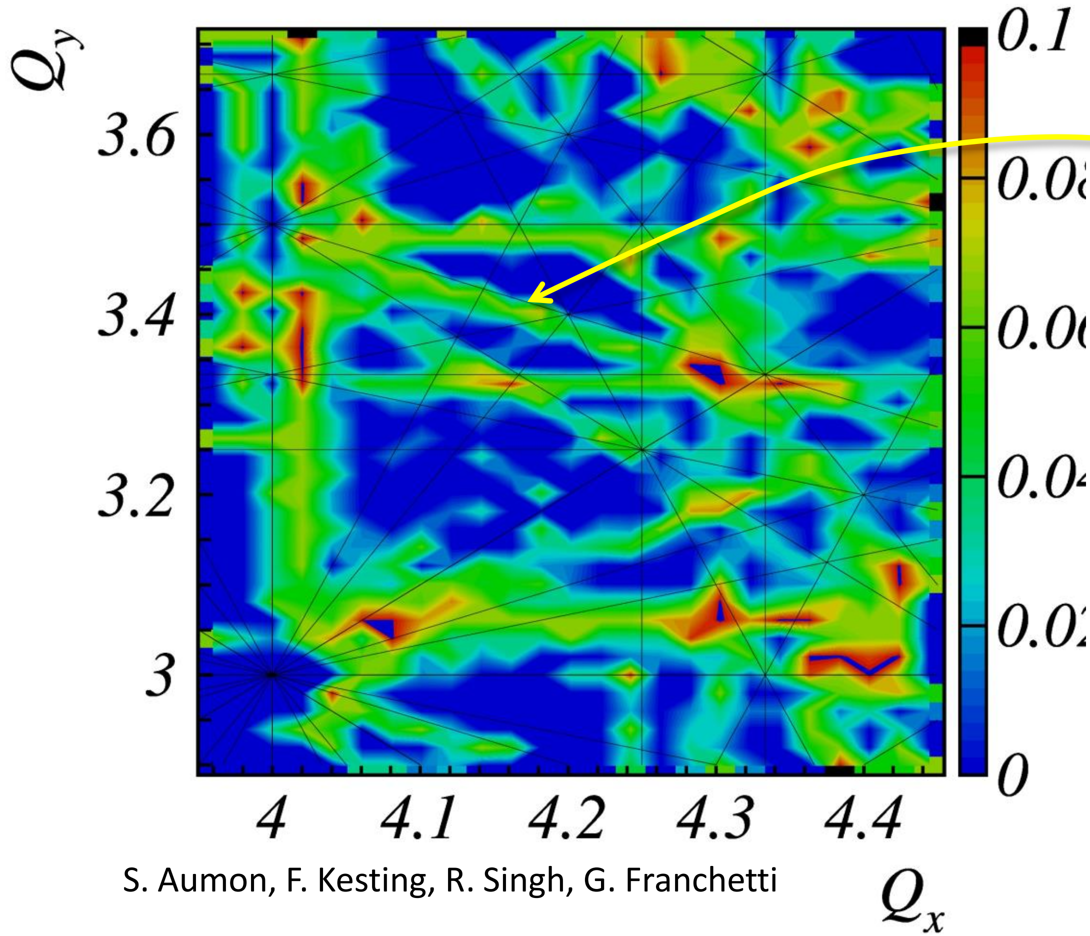

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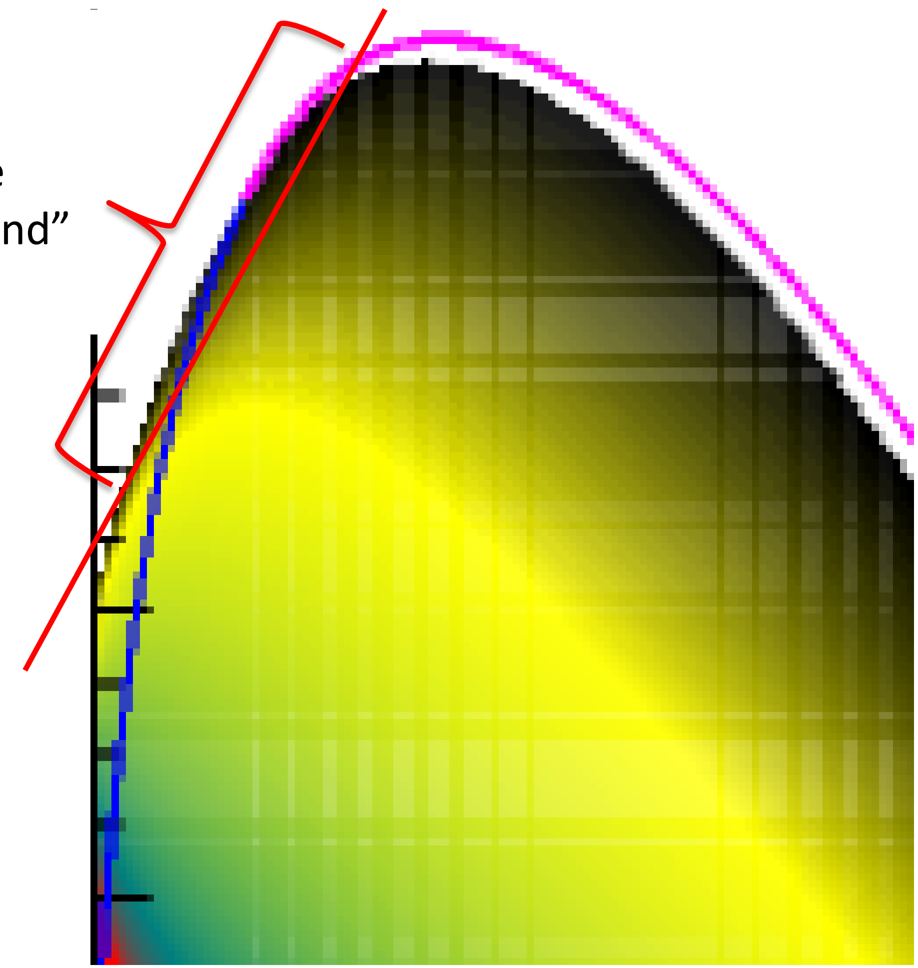
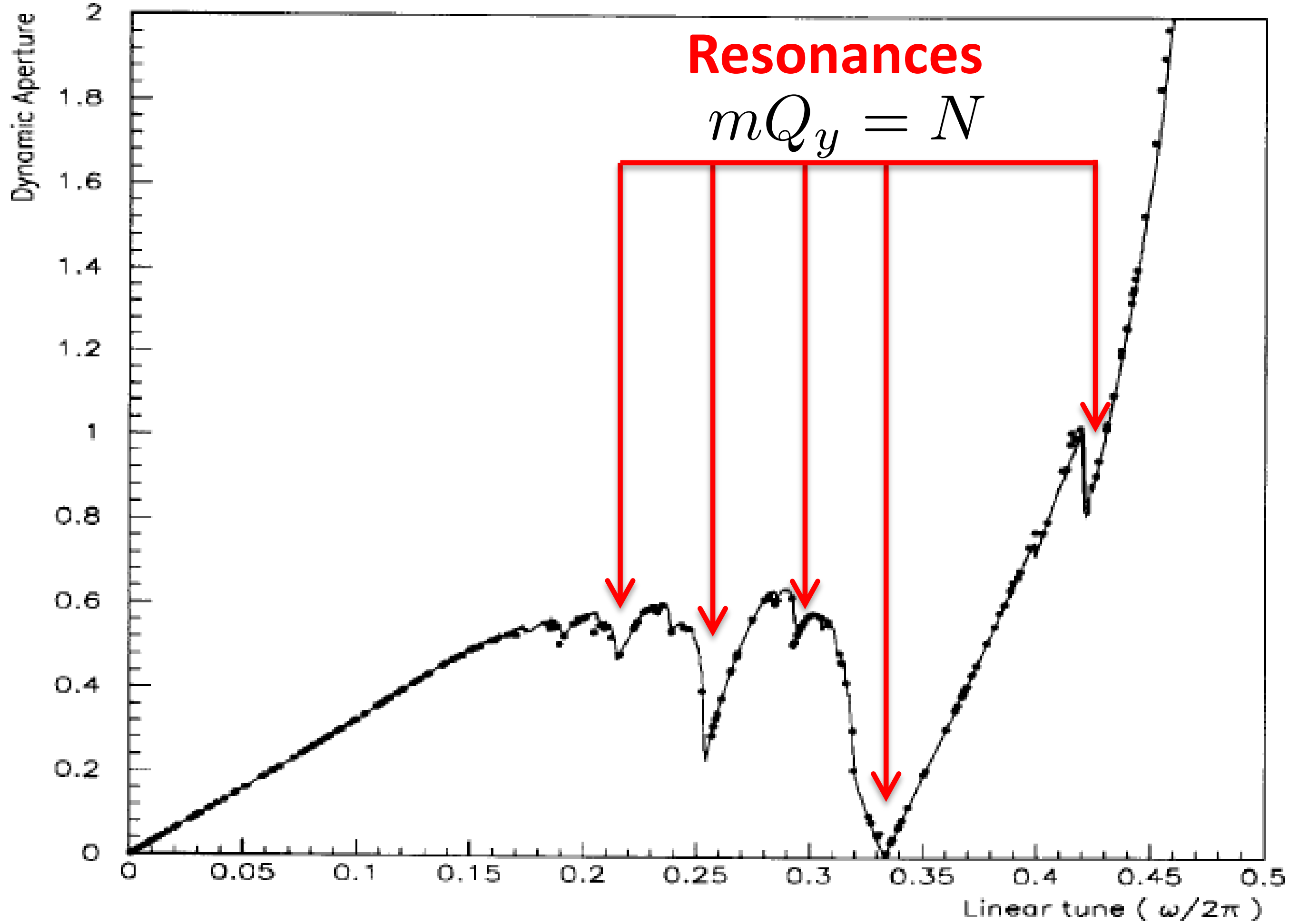
Now we know what is the dynamics here

Fundamental ingredient to create the asymmetric halo (hence to predict halo extension and maybe density), for high intensity bunched beams

This is the amplitude of an “island”



Stability in a 1D system



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