Recent developments on a massive planar pentabox with the SDE approach

Chris Wever (N.C.S.R. Demokritos)

C. Papadopoulos, D. Tommasini, C. Wever *[work in progress]*

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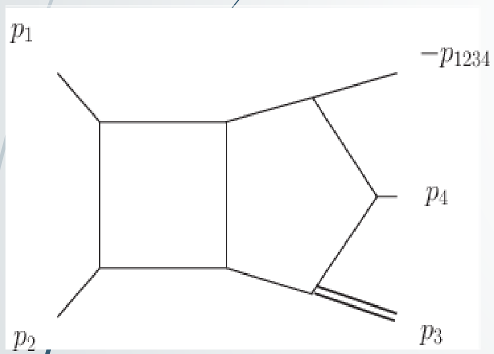
Outline

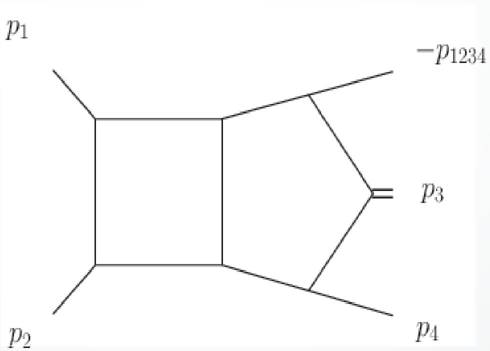
- The SDE approach for a massive planar pentabox
- Boundary conditions
- The massless limit
- Summary and Outlook

A massive planar pentabox

- Interested in two-loop, five-point diagrams with ***one*** external mass
- Massless propagators
- Relevant e.g. for virtual-virtual contribution to $2 \to 3$ LHC processes such as H+2j (Les Houches Wishlist), Z+2j, γ^*+2j production at NNLO QCD
- Three planar topologies:



P1



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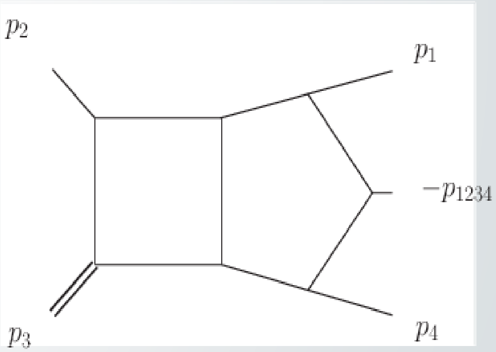

P3

- All other 8 or less propagator (2-loop, 5-point, 1-mass) planar diagrams are reducible to diagrams in the above families

We will use SDE approach *(see talk by C. Papadopoulos)*

Review: SDE approach

[Papadopoulos ’14, Papadopoulos, Tommasini, CW ’14]

- Introduce extra parameter *x* in the denominators of loop integral
- *x*-parameter describes off-shellness of (some) external legs:

Massive external legs: p_1 → x-parametrize → $p_1 + (1-x)q$

Massless external legs: p_2 → x-parametrize → p_2 or xp_2

$x = 1$

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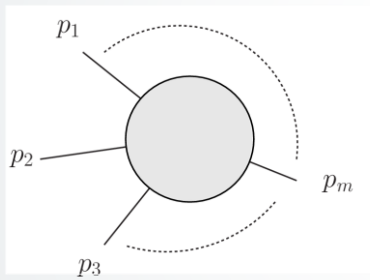
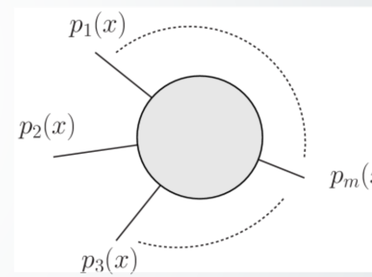
General:



x-parametrize →

← $x = 1$



$$p_i(x) = p_i + (1-x)q_i$$
$$\sum_i q_i = 0$$

$$G_{a_1\cdots a_n}(s,\epsilon) = \int \Big(\prod_i d^d k_i\Big) \frac{1}{D_1^{2a_1}(k,p)\cdots D_n^{2a_n}(k,p)} \longrightarrow G_{a_1\cdots a_n}(x,s,\epsilon) = \int \Big(\prod_i d^d k_i\Big) \frac{1}{D_1^{2a_1}(k,p(x))\cdots D_n^{2a_n}(k,p(x))}$$

$$D_i(k,p) = c_{ij}k_j + d_{ij}p_j, \quad s = \{p_i.p_j\}|_{i,j}$$

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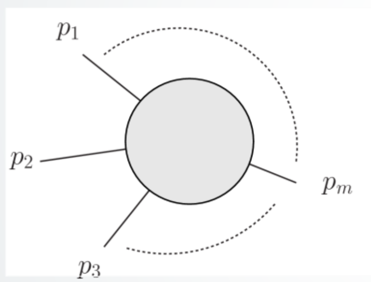

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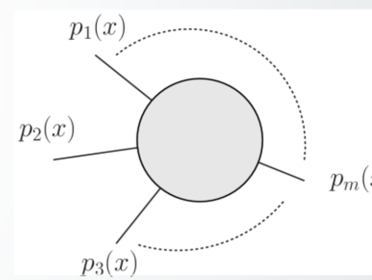

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$$D_i(k,p) = c_{ij}k_j + d_{ij}p_j, \qquad s = \{p_i.p_j\}|_{i,j}$$

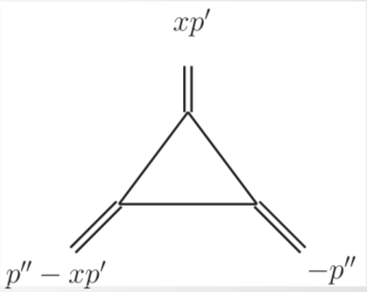
- Take derivative of integral *G* w.r.t. *x*-parameter instead of w.r.t. invariants and reduce r.h.s. by IBP identities:

$$\frac{\partial}{\partial x}\vec{G}^{MI}(x,s,\epsilon) \overset{IBP}{=} \overline{\overline{M}}(x,s,\epsilon).\vec{G}^{MI}(x,s,\epsilon), \qquad s = \{p_i.p_j\}|_{i,j}$$

x-parametrization for pentabox

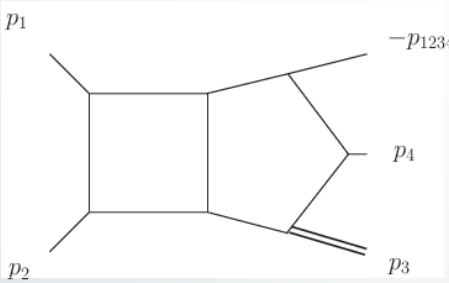
Main criteria for choice of x-parametrization: *require Goncharov Polylog (GP) solution for DE*

- For all MI that we have calculated, the above criteria could be easily met



- Often enough to choose the external legs such that the corresponding massive MI triangles (**found by pinching external legs**) are as follows:

- x-parametrization for **P1** family (74 MI in total):



$$p_1^2 = p_2^2 = p_4^2 = p_{1234}^2 = 0, \quad p_3^2 \neq 0$$

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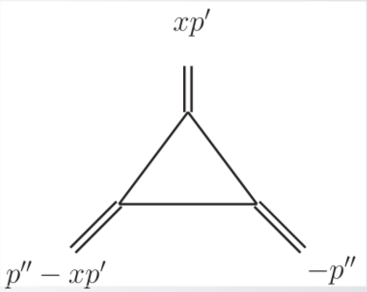

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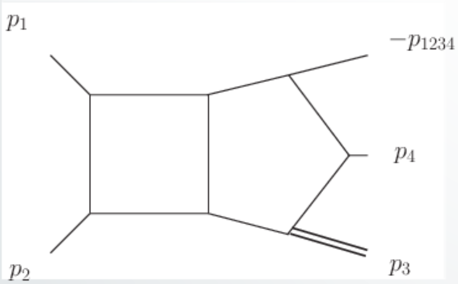
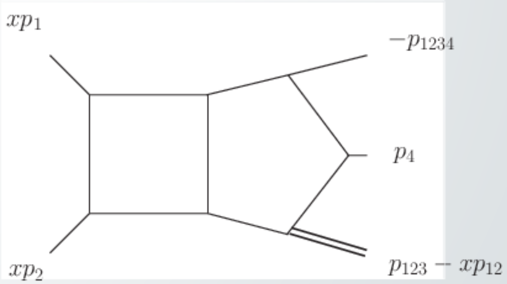


Introduce *x*



$p_1^2 = p_2^2 = p_4^2 = p_{1234}^2 = 0, \quad p_3^2 \neq 0$ → $p_1^2 = p_2^2 = p_3^2 = p_4^2 = p_{1234}^2 = 0$

- DE for **P1** are known and integration underway in terms of GP's
- Reduction for **P2** done (75 MI in total), **P3** underway (bottleneck)

Dealing with boundary conditions

- Integration of a linear DE: $\partial_x G[x,s,\epsilon] = H[x,s,\epsilon] * G[x,s,\epsilon] + \tilde{I}[x,s,\epsilon]$

$$MG[x,s,\epsilon] - MG[x\to 0,s,\epsilon] = \int_0^x dx' I[x',s,\epsilon]$$

$$= \sum_n \int_0^x dx' x'^{-1+n\epsilon} I^{(n)}_{\text{sing}}[s,\epsilon] + \int_0^x dx' \left(I[x',s,\epsilon] - \sum_n x'^{-1+n\epsilon} I^{(n)}_{\text{sing}}[s,\epsilon] \right)$$

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Often correctly reproduces $x \to 0$ behavior of $MG(x,s,\epsilon)$!

- Integrand $I[x]$ contains *branch points* or *poles* at $x = \{x_1, x_2, \dots, \infty\}$ of form $(x - x_i)^{m+n\epsilon}$
- Also possible to integrate from either $x_1, x_2, \dots, \infty$ instead from integration boundary $x = 0$

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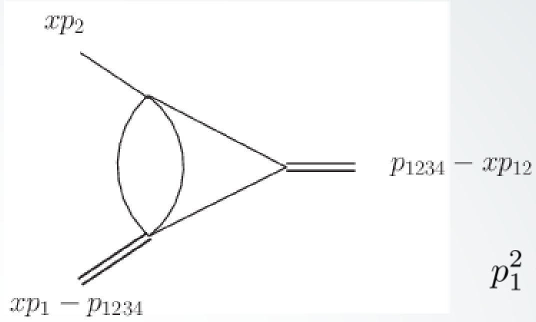
Observation: Boundary always captured by integration from $x = 0$ or appropriate x_i

→ Not well understood yet why this is so and if will persist in future!

- Alternative: use analytical/regularity constraints or asymptotic expansion in $x \to x_i$

Example of boundary calculation

- Two-loop triangle: $G[x] =$



$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = p_{1234}^2 = 0$$

- DE: $\partial_x \left(MG\right)[x, s, \epsilon] = C[s, \epsilon] \left(1 - \frac{x s_{12}}{s_{12} - s_{34}}\right)^{-1+\epsilon} \left(1 - \frac{x s_{12}}{s_{12} - s_{34} + s_{51}}\right)^{-1+\epsilon}$

- A naïve integration from lower boundary $x = 0$ misses a boundary term (collinear region)

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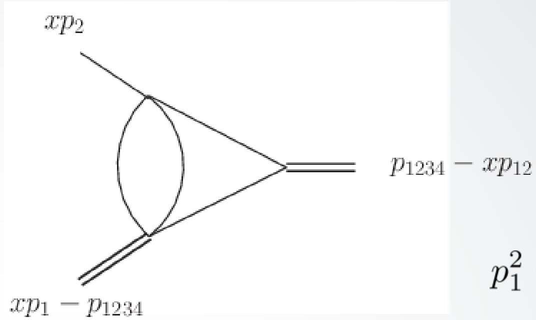

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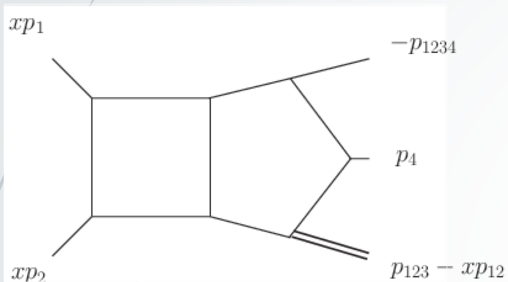
- A naïve integration from lower boundary $x = 0$ misses a boundary term (collinear region)
- Try instead to integrate from poles $x_1 = (s_{12} - s_{34})/s_{12}$ or $x_2 = (s_{12} - s_{34} + s_{51})/s_{12}$
- Integrating from $x_1 = (s_{12} - s_{34})/s_{12}$ (one-mass case) misses boundary term (collinear)
- From $x_2 = (s_{12} - s_{34} + s_{51})/s_{12}$ (equal-mass case) we capture boundary term (hard):

$$MG[x,s,\epsilon] = C[s,\epsilon] \left(- \left(1 - \frac{x_2 s_{12}}{s_{12} - s_{34}}\right)^{-1+\epsilon} \left(1 - \frac{x s_{12}}{s_{12} - s_{34} + s_{51}}\right)^{\epsilon} \left(\frac{s_{12} - s_{34} + s_{51}}{s_{12}\epsilon}\right) + \int_{x_2}^{x} dx' I_{\text{integrable}}[x'] \right)$$

The massless pentabox case

- Massless pentabox = $x \to 1$ *under integral sign* = $x \to 1$ *of hard region contribution* →
- For massless limit of G: resum logs of $(1-x)$ into $(1-x)^{n\epsilon} \to 0$ and *afterwards* $x \to 1$



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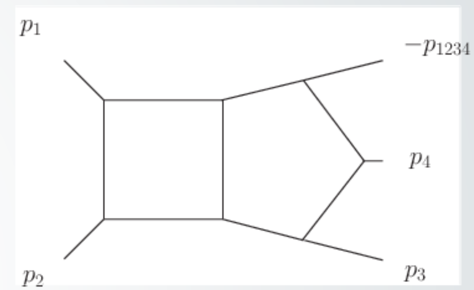

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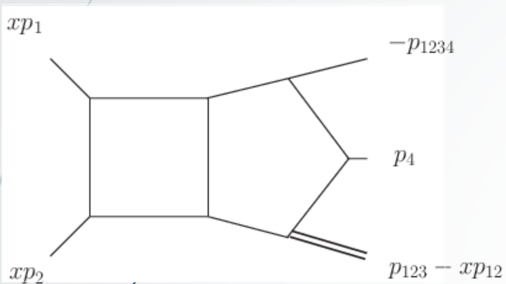

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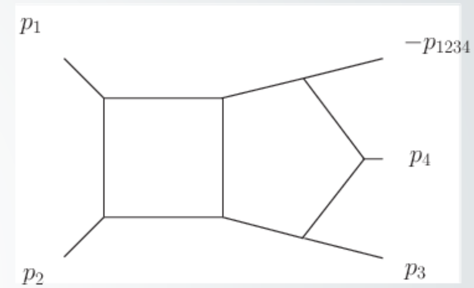

- Limits $x \to 1$ and $\epsilon \to 0$ <u>may not commute</u>:

$$MG[x,s,\epsilon] - MG[x \to 0, s, \epsilon] = \int_0^x dx' I[x',s,\epsilon]$$

May contain $(1-x')^{-1+n\epsilon}$ singularities!

$$= \sum_n \int_0^x dx' x'^{-1+n\epsilon} I_{\text{sing}}^{(n)}[s,\epsilon] + \int_0^x dx' \left(I[x',s,\epsilon] - \sum_n x'^{-1+n\epsilon} I_{\text{sing}}^{(n)}[s,\epsilon] \right)$$

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May contain divergent $\log(1-x)$ terms!

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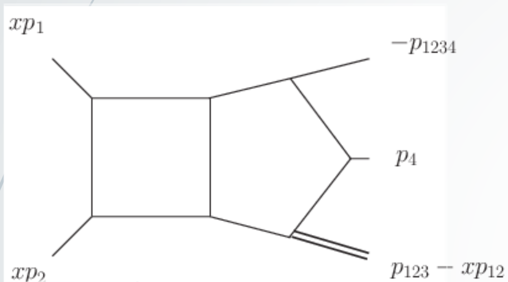

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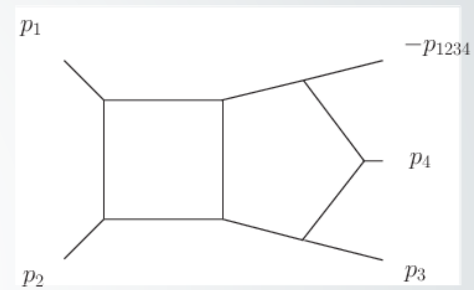

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May contain divergent $\log(1-x)$ terms!

- Possibility: integrate from pole $x = 1$ instead of $x = 0$, but then might miss boundary term
- Even if boundary behavior captured, would have to integrate twice: 1) $x \neq 1$ and 2) $x = 1$
- How to perform the resummation in algorithmic and efficient manner?

Resumming logs of $(1-x)$

[J. Henn, A.V. Smirnov, V.A. Smirnov '13]

- The $(1-x)^{n\epsilon}$ behavior is captured by the DE itself!
- Corresponds to singularities in $(1-x)$ in the DE
- Exponents n are the residues of these singularities
- For coupled systems one has:

$$\partial_x(MG)[x] = \sum_n (1-x)^{-1+n\epsilon} H_{-1}^{(n)}.(MG)[x] + \sum_n (1-x)^{-1+n\epsilon} \tilde{I}_{-1}^{(n)} + \mathcal{O}((1-x)^0)$$

- Solution (generally might contain powers of $\log(1-x)$):

$$(MG)[x \sim 1] = \sum_n c_n (1-x)^{n\epsilon}$$

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$$(MG)[x\sim 1] = \sum_n c_n (1-x)^{n\epsilon}$$

- Exponents n are determined by the DE itself
- The coefficients c_n found by *matching* logs of $(1-x)$ to solution of $x \neq 1$ case:

$$(MG)[x] = \sum_n c_n \left(1 + n\epsilon \log(1-x) + \frac{n^2\epsilon^2}{2}\log(1-x)^2 + \cdots\right)$$

- Massless pentabox: $G[x=1] = \left.\frac{\sum_n c_n (1-x)^{n\epsilon}}{M}\right|_{(1-x)^{n\epsilon}\to 0,\ x\to 1}$

Summary and Outlook

- In progress: two-loop pentaboxes with one massive external leg
- SDE method captures boundary terms by choosing the boundary at an appropriate branch point or pole
- Massless limit captured by resumming logs of $(1 - x)$
- Can be done by algorithmic matching

Summary and Outlook

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Thank you very much!

Backup slides

Comparison of DE methods

Traditional DE method:

- Choose $\tilde{s} = \{f(p_i.p_j)\}$ and use chain rule to relate differentials of (independent) momenta and invariants:

$$p_i.\frac{\partial}{\partial p_j}\mathrm{F}(\tilde{s}) = \sum_k p_i.\frac{\partial \tilde{s}_k}{\partial p_j}\frac{\partial}{\partial \tilde{s}_k}\mathrm{F}(\tilde{s})$$

- Solve above linear equations:

$$\frac{\partial}{\partial \tilde{s}_k} = g_k(\{p_i.\frac{\partial}{\partial p_j}\})$$

- Differentiate w.r.t. invariant(s) $\tilde{s}_k$:

$$\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = g_k(\{p_i.\frac{\partial}{\partial p_j}\})\vec{G}^{MI}(\tilde{s},\epsilon) \stackrel{IBP}{=} \overline{\overline{M}}_k(\tilde{s},\epsilon).\vec{G}^{MI}(\tilde{s},\epsilon)$$

- Make rotation $\vec{G}^{MI} \to \overline{\overline{A}}.\vec{G}^{MI}$ such that:

$$\frac{\partial}{\partial \tilde{s}_k}\vec{G}^{MI}(\tilde{s},\epsilon) = \epsilon\overline{\overline{M}}_k(\tilde{s}).\vec{G}^{MI}(\tilde{s},\epsilon)$$ [Henn '13]

- Solve perturbatively in ϵ to get GP's if $\tilde{s} = \{f(p_i.p_j)\}$ chosen properly
- Solve DE of different $\tilde{s}_k$, to capture boundary condition

Simplified DE method:

- Introduce external parameter x to capture off-shellness of external momenta:

$$G_{a_1\cdots a_n}(s,\epsilon) = \int\Big(\prod_i d^dk_i\Big)\frac{1}{D_1^{2a_1}(k,p(x))\cdots D_n^{2a_n}(k,p(x))}$$

$$p_i(x) = p_i + (1-x)q_i,\quad \sum_i q_i = 0,\quad \boxed{s = \{p_i.p_j\}|_{i,j}}$$

- Parametrization: pinched massive triangles should have legs (not fully constraining):

$$q_1(x) = xp', q_2(x) = p'' - xp',\ \ p'^2 = m_1, p''^2 = m_3$$

- Differentiate w.r.t. parameter x:

$$\frac{\partial}{\partial x}\vec{G}^{MI}(x,s,\epsilon) \stackrel{IBP}{=} \overline{\overline{M}}(x,s,\epsilon).\vec{G}^{MI}(x,s,\epsilon)$$

- Check if *constant term* $(\epsilon = 0)$ *of residues of homogeneous term* for *every* DE is an *<u>integer</u>*: 1) if yes, solve DE by "bottom-up" approach to express in GP's; 2) if no, change parametrization and check DE again
- Boundary term almost always captured, if not: try $x \to 1/x$ or asymptotic expnansion

Bottom-up approach

- Notation: upper index "(m)" in integrals $G^{(m)}_{\{a_1 \dots a_n\}}$ denotes amount of positive indices, i.e. amount of denominators/propagators

$$G^{(m)}_{a_1 \cdots a_n} = \int \left(\prod_i d^d k_i \right) \underbrace{\frac{1}{D_1^{2a_1}(k,p) \cdots D_n^{2a_n}(k,p)}}_{m \text{ propagators, (positive indices) } a_i}$$

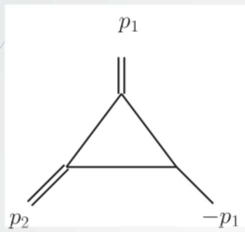
- In practice individual DE's of MI are of the form:

$$\frac{\partial}{\partial x} G^{(m)}_{a_1 \cdots a_n}(x,s,\epsilon) = \sum_{m'=m_0}^{m} \sum_{b_1,\cdots b_n} \text{Rational}^{b_1,\cdots b_n}_{a_1 \cdots a_n}(x,s,\epsilon) G^{(m')}_{b_1 \cdots b_n}(x,s,\epsilon)$$

Bottom-up:

- Solve first for all MI with least amount of denominators m_0 (these are often already known to all orders in ϵ or often calculable with other methods)
- After solving all MI with m denominators ($m \geq m_0$), solve all MI with $m+1$ denominators
- Often: $$G^{(m_0)}_{a_1 \cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon} \left(\sum \text{Rational}(x) GP(\cdots; x) \right)$$

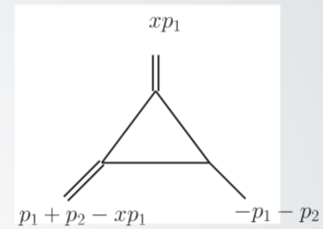
Example: one-loop triangle



$$G_{111}(m_1, m_2) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^2(k+p_1)^2(k+p_1+p_2)^2}$$

$$p_1^2 = m_1, p_2^2 = m_2, (p_1+p_2)^2 = 0$$

Parametrize p_2 off-shellness with *x*



$$G_{111}(m_1, x) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^2(k+xp_1)^2(k+p_1+p_2)^2}$$

$$p_1^2 = m_1, p_2^2 = 0, (p_1+p_2-xp_1)^2 \neq 0, (p_1+p_2)^2 = 0$$

- Differentiate to *x* and use IBP to reduce:

$$\frac{\partial}{\partial x} G_{111}(x) = \frac{-x^{-2-\epsilon}}{\epsilon^2 m_1}((-m_1 - i.0)^{-\epsilon}(1+2\epsilon)x^{-\epsilon} - (m_1 - i.0)^{-\epsilon}(1-x)^{-1-\epsilon}(1+\epsilon - x(1+2\epsilon)))$$

- Subtracting the singularities and expanding the finite part leads to:

$$\begin{aligned} G_{111}(x) &= G_{111}(0) + \int_0^x dx' \frac{-x'^{-2-\epsilon}}{\epsilon^2 m_1}((-m_1 - i.0)^{-\epsilon}(1+2\epsilon)x'^{-\epsilon} - (m_1 - i.0)^{-\epsilon}(1-x')^{-1-\epsilon}(1+\epsilon - x'(1+2\epsilon))) \\ &= \underbrace{G_{111}(0)}_{=0} + \frac{-(m_1 - i.0)^{-\epsilon}x^{-\epsilon} + (-m_1 - i.0)^{-\epsilon}x^{-2\epsilon}}{m_1 x \epsilon^2} + \frac{(m_1 - i.0)^{-\epsilon}(-x^{-\epsilon} + (x + GP(1;x)))}{m_1 x \epsilon} + \mathcal{O}(\epsilon^0) \end{aligned}$$

- Agrees with expansion of exact solution: $G_{111}(m_1 * x^2, m_2 = (-m_1)x(1-x)) = \frac{c_\Gamma(\epsilon)}{\epsilon^2} \frac{(-m_1 x^2)^{-\epsilon} - (-(-m_1)x(1-x))^{-\epsilon}}{m_1 x^2 - (-m_1)x(1-x)}$

GP-structure of solution

- Assume for $m' < m$ denominators: $G^{(m')}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \mathrm{Rational}(x) GP(\cdots;x)\Big), \quad m' < m$

- For simplicity we assume here a non-coupled DE for a MI with m denominators:

$$\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) = H(x,s,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) + \sum_{m'=1}^{m-1} \sum_{b_1,\cdots b_n} \mathrm{Rational}^{(b_1,\cdots b_n)}(x,s,\epsilon) G^{(m')}_{b_1\cdots b_n}(x,s,\epsilon)$$

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dependence on invariants s suppressed

$$\begin{aligned}
\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) &= H(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\left(\sum \mathrm{Rational}(x) GP(\cdots;x)\right) \\
&= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\left(\sum \mathrm{Rational}(x) GP(\cdots;x)\right) \longrightarrow \\
\frac{\partial}{\partial x}\left(M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon)\right) &= M(x,\epsilon) \sum_{n,l} x^{-n+l\epsilon}\left(\sum \mathrm{Rational}(x) GP(\cdots;x)\right), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}
\end{aligned}$$

GP-structure of solution

- Assume for $m' < m$ denominators: $G^{(m')}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\left(\sum \text{Rational}(x) GP(\cdots;x)\right), \quad m' < m$

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dependence on invariants s suppressed

$$\begin{aligned}
\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) &= H(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\left(\sum \text{Rational}(x) GP(\cdots;x)\right) \\
&= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\left(\sum \text{Rational}(x) GP(\cdots;x)\right) \longrightarrow \\
\frac{\partial}{\partial x}\left(M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon)\right) &= M(x,\epsilon) \sum_{n,l} x^{-n+l\epsilon}\left(\sum \text{Rational}(x) GP(\cdots;x)\right), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}
\end{aligned}$$

- Formal solution:

$$\begin{aligned}
M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) &= (M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} + \sum_{n,l}\prod_{\text{poles } x^{(0)}} \int_0^x dx' \left(x'^{-n+l\epsilon}(x' - x^{(0)})^{-\epsilon c_{x^{(0)}}}\right)\left(\sum (x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}(x') GP(\cdots;x')\right) \\
&= (M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} + \sum_{\tilde{n},l}\int_0^x dx' x'^{-\tilde{n}+l\epsilon} I_{\tilde{n},l}(\epsilon) + \sum_k \epsilon^k \prod_{\text{poles } x^{(0)}} \sum \int_0^x dx' \underbrace{(x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}_k(x')}_{\text{Rational}_k(x') \text{ if } r_{x^{(0)}} \in \mathbb{Z}} GP(\cdots;x')
\end{aligned}$$

GP-structure of solution

- Assume for $m' < m$ denominators: $$G^{(m')}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big), \quad m' < m$$

- For simplicity we assume here a non-coupled DE for a MI with m denominators:

$$\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) = H(x,s,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) + \sum_{m'=1}^{m-1}\sum_{b_1,\cdots b_n} \text{Rational}^{(b_1,\cdots b_n)}(x,s,\epsilon) G^{(m')}_{b_1\cdots b_n}(x,s,\epsilon)$$

dependence on invariants *s* suppressed

$$\begin{aligned}
\frac{\partial}{\partial x} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) &= H(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big) \\
&= \sum_{\text{poles } x^{(0)}} \frac{r_{x^{(0)}} + \epsilon c_{x^{(0)}}(\epsilon)}{(x - x^{(0)})} G^{(m)}_{a_1\cdots a_n}(x,\epsilon) + \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big) \longrightarrow \\
\frac{\partial}{\partial x}\Big(M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,\epsilon)\Big) &= M(x,\epsilon)\sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big), \quad M(x,\epsilon) = \prod_{\text{poles } x^{(0)}} (x - x^{(0)})^{-r_{x^{(0)}} - \epsilon c_{x^{(0)}}(\epsilon)}
\end{aligned}$$

- Formal solution:

$$\begin{aligned}
M(x,\epsilon) G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) &= (M * G^{(m)}_{a_1\cdots a_n})_{x\to 0} + \sum_{n,l}\prod_{\text{poles } x^{(0)}} \int_0^x dx' \Big(x'^{-n+l\epsilon}(x' - x^{(0)})^{-\epsilon c_{x^{(0)}}}\Big)\Big(\sum (x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}(x') GP(\cdots;x')\Big) \\
&= \underbrace{(M * G^{(m)}_{a_1\cdots a_n})_{x\to 0}}_{\text{boundary condition}} + \sum_{\tilde{n},l} \underbrace{\int_0^x dx' x'^{-\tilde{n}+l\epsilon} I_{\tilde{n},l}(\epsilon)}_{x^{-\tilde{n}+l\epsilon+1}\tilde{I}_{\tilde{n},l}(\epsilon)} + \sum_k \epsilon^k \prod_{\text{poles } x^{(0)}} \underbrace{\sum \int_0^x dx' \underbrace{(x' - x^{(0)})^{-r_{x^{(0)}}} \text{Rational}_k(x')}_{\text{Rational}_k(x') \text{ if } r_{x^{(0)}} \in \mathbb{Z}} GP(\cdots;x')}_{\sum \text{Rational}_k(x) GP(\cdots;x) \text{ if } r_{x^{(0)}} \in \mathbb{Z}}
\end{aligned}$$

→ MI expressible in GP's: $$G^{(m)}_{a_1\cdots a_n}(x,s,\epsilon) = \sum_{n,l} x^{-n+l\epsilon}\Big(\sum \text{Rational}(x) GP(\cdots;x)\Big)$$

Fine print for coupled DE's: if the non-diagonal piece of $\epsilon = 0$ term of *matrix H* is nilpotent (e.g. triangular) and if diagonal elements of *matrices* $r_{x^{(0)}}$ are integers, then above "GP-argument" is still valid

Uniform weight solution of DE

- In general matrix in DE is dependent on ϵ:

$$\frac{\partial}{\partial \tilde{s}_k} \vec{G}^{MI}(\tilde{s},\epsilon) = \overline{\overline{M}}_k(\tilde{s},\epsilon).\vec{G}^{MI}(\tilde{s},\epsilon)$$

- **Conjecture:** possible to make a rotation $\vec{G}^{MI} \to \overline{\overline{A}}.\vec{G}^{MI}$ such that:

$$\boxed{\frac{\partial}{\partial \tilde{s}_k} \vec{G}^{MI}(\tilde{s},\epsilon) = \epsilon\overline{\overline{M}}_k(\tilde{s}).\vec{G}^{MI}(\tilde{s},\epsilon)}$$

[Henn '13]

- Explicitly shown to be true for many examples [Henn '13, Henn, Smirnov et al '13-'14]
- **If** set of invariants $\tilde{s} = \{f(p_i.p_j)\}$ chosen correctly: $\overline{\overline{M}}_k(\tilde{s}) = \sum_{\text{poles } \tilde{s}_k^{(0)}} \frac{\overline{\overline{M}}_k^{\tilde{s}_k^{(0)}}}{(\tilde{s}_k - \tilde{s}_k^{(0)})}$
- Solution is *uniform* in *weight* of GP's:

$$\vec{G}^{MI}(\tilde{s},\epsilon) = Pe^{\epsilon\int_{C[0,\tilde{s}]}\overline{\overline{M}}_k(\tilde{s}'_k)}\vec{G}^{MI}(0,\epsilon) = (\mathbf{1} + \epsilon\int_0^{\tilde{s}_k}\overline{\overline{M}}_k(\tilde{s}'_k) + \cdots)\underbrace{\vec{G}^{MI}(0,\epsilon)}_{\vec{G}_0^{MI}+\epsilon\vec{G}_1^{MI}+\cdots}$$

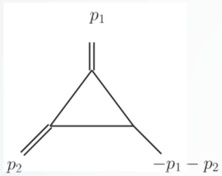
$$= \underbrace{\vec{G}_0^{MI}}_{\text{weight i}} + \underbrace{\epsilon(\underbrace{\vec{G}_1^{MI}}_{\text{weight i+1}} + \sum_{\text{poles } \tilde{s}_k^{(0)}} \overbrace{\left(\int_0^{\tilde{s}_k}\frac{d\tilde{s}'_k}{(\tilde{s}'_k - \tilde{s}_k^{(0)})}\right)}^{GP(\tilde{s}_k^{(0)};\tilde{s}_k)}\overline{\overline{M}}_k^{\tilde{s}_k^{(0)}}.\underbrace{\vec{G}_0^{MI}}_{\text{weight i}})}_{\text{weight i+1}} + \cdots$$

Example of tradition DE method: one-loop triangle (1/2)

- Consider again one-loop triangles with 2 massive legs and massless propagators:

$$G_{a_1a_2a_3}(\tilde{s}) \quad = \quad \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^{2a_1}(k+p_1)^{2a_2}(k+p_1+p_2)^{2a_3}}, \quad p_1^2 = m_1, p_2^2 = m_2, (p_1+p_2)^2 = m_3 = 0$$

$$G_{111} =$$



- General function:

$$p_i \cdot \frac{\partial}{\partial p_j} \mathrm{F}(m_1,m_2,m_3) = \sum_{k=1}^{3} p_i \cdot \frac{\partial \tilde{s}_k}{\partial p_j} \frac{\partial}{\partial \tilde{s}_k} \mathrm{F}(m_1,m_2,m_3), \quad i,j \in \{1,2\}$$

$$\tilde{s}_1 = p_1^2 = m_1, \tilde{s}_2 = p_2^2 = m_2, \tilde{s}_3 = (p_1+p_2)^2 = m_3$$

- Four linear equations, of which three independent because of invariance under Lorentz transformation [Remiddi & Gehrmann '00], in three unknowns: $\{\frac{\partial}{\partial m_1}, \frac{\partial}{\partial m_2}, \frac{\partial}{\partial m_3}\}$

- Solve linear equations: $\frac{\partial}{\partial m_k} = g_k(p_1 \cdot \frac{\partial}{\partial p_1}, p_2 \cdot \frac{\partial}{\partial p_2}, p_2 \cdot \frac{\partial}{\partial p_1}), \quad k = 1,2,3$

$$\frac{\partial}{\partial m_1} G_{111} = \frac{1-2\epsilon}{\epsilon(m_1-m_2)^2}(G_{011} - (1+\epsilon(1-\frac{m_2}{m_1}))G_{110}), \quad \frac{\partial}{\partial m_2} G_{111} = \frac{\partial}{\partial m_1} G_{111} \; (m_1 \leftrightarrow m_2, G_{011} \leftrightarrow G_{110})$$

Example of tradition DE method: one-loop triangle (2/2)

$$\frac{\partial}{\partial m_1}G_{111} = \frac{1}{\epsilon^2(m_1-m_2)^2}((-m_2)^{-\epsilon}+(-m_1)^{-\epsilon}(1+\epsilon)-\epsilon m_2(-m_1)^{-1-\epsilon}) =: F[m_1,m_2], \quad \frac{\partial}{\partial m_2}G_{111} = F[m_2,m_1]$$

- Solve by usual subtraction procedure: $F_{\text{sing}}[m_1,m_2] = \frac{-1}{\epsilon m_2}(-m_1)^{-1-\epsilon}$

$$\begin{aligned} G_{111}(m_1,m_2) &= G_{111}(0,m_2) + \int_0^{m_1} F_{\text{sing}}[m_1',m_2] + \int_0^{m_1}(F[m_1',m_2]-F_{\text{sing}}[m_1',m_2]) \\ &= G_{111}(0,m_2) - \frac{(-m_1)^{-\epsilon}}{\epsilon^2 m_2} + \int_0^{m_1}\Big(\frac{(1-(-m_2)^{-\epsilon})GP(;-m_1')}{\epsilon^2(m_2-m_1')^2} - \frac{(m_2-m_1')GP(;-m_1')+m_2GP(0;-m_1')}{\epsilon m_2(m_2-m_1')^2} + \mathcal{O}(\epsilon^0)\Big) \\ &= G_{111}(0,m_2) - \frac{(-m_1)^{-\epsilon}}{\epsilon^2 m_2} + \Big(\frac{m_1(1-(-m_2)^{-\epsilon})}{\epsilon^2 m_2(m_1-m_2)} + \frac{m_1 GP(0;-m_1)}{\epsilon m_2(m_2-m_1)}\Big) + \mathcal{O}(\epsilon^0) \end{aligned}$$

- Boundary condition follows by plugging in above solution in $\frac{\partial}{\partial m_2}G_{111} = F[m_2,m_1]$

$$\frac{\partial}{\partial m_2}G_{111}(0,m_2) = \frac{(1+\epsilon)}{\epsilon^2}(-m_2)^{-2-\epsilon} \to G_{111}(0,m_2) = \frac{-(-m_2)^{-1-\epsilon}}{\epsilon^2} + \underbrace{G_{111}(0,0)}_{\text{scaleless}=0} = \frac{-(-m_2)^{-1-\epsilon}}{\epsilon^2}$$

- Agrees with exact solution: $G_{111} = \frac{c_\Gamma(\epsilon)}{\epsilon^2}\frac{(-m_1)^{-\epsilon}-(-m_2)^{-\epsilon}}{m_1-m_2} = \frac{c_\Gamma(\epsilon)}{m_1-m_2}\Big(-\frac{1}{\epsilon}\log(\frac{-m_1}{-m_2}) + \mathcal{O}(\epsilon^0)\Big)$

Open questions

- Is there a way to pre-empt the choice of x-parametrization without having to calculate the DE?
- Why are the boundary conditions naturally taken into account by the DE?
- How do the DE in the x-parametrization method relate exactly to those in the traditional DE method?