Differential Equations for Feynman Integrals

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based on work with M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, L. Tancredi, V. Yundin

arXiv:1401.2979, arXiv:1408.3107

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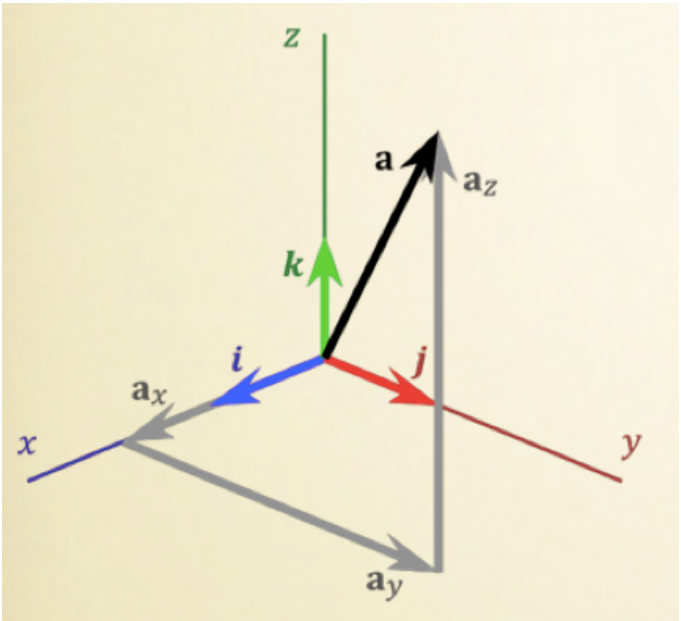
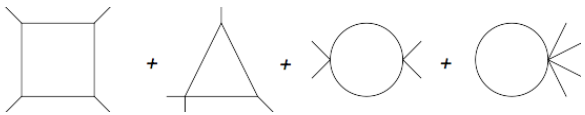
Three Steps of a modern multi-loop and -leg calculation

1) Find integral basis for the process
2) Determine coefficients of the basis elements
3) Calculate elements (integrals) of the basis



Tools

- Integration-by-parts identities (IBP-Ids) Chetyrkin,Tkachov; Laporta
- Integrand Reduction Ossola, Papadopoulos, Pittau; Mastrolia, Ossola; Badger, Frellesvig, Zhang; Zhang; Mastrolia, Mirabella, Ossola, Peraro
- Generalized/Maximal Unitarity, Bern, Dixon, Dunbar, Kosower Kosower, Larsen; Caron-Huot; Carrasco, Johansson; Zhang
- Differential Equations, Kotikov;Remiddi; Gehrmann, Remiddi; Henn;Papadopoulos

	one-loop	two-loop
graphs	only planar	planar and non-planar
integral basis	known 	determined case by case ?
integrals	known	only for certain cases
IR poles	cancellation between one-loop and tree level	cancellation between two- and one-loop and tree level
appearing functions	logs and dilogs	logs, polylogarithms, generalized polylogs, elliptic functions and more?

$\rightarrow$ a lot more exploring has to be done

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Feynman integrals are functions of

- Mandelstam variables
- Internal and external masses
- Spacetime dimensions

Facts

- Not all Feynman integrals are independent
- IBP-ids connect different Feynman integrals
- We can find an integral basis called master integrals

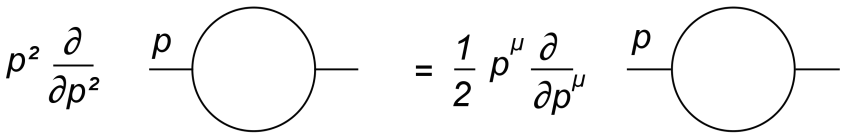
Exploit this by

- Taking derivatives of the master integrals in respect to the kinematic invariants
- Reduce result back to master integrals
- Solve the obtained first order differential equation analytically

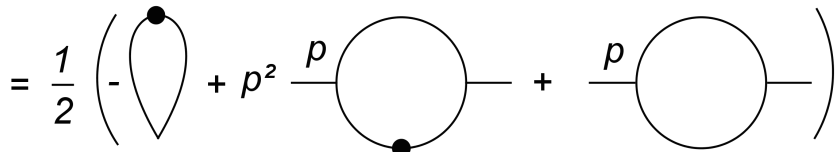
Let's have a look at an easy example

The one-loop massless bubble

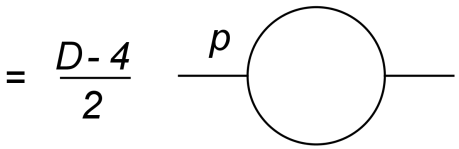
- Construct differential operator



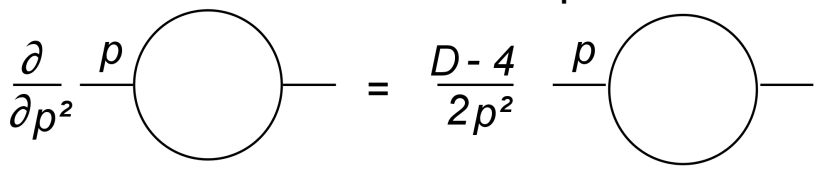
- Apply derivative



- Reduce back to master integrals with IBP-ids



- Gives us the differential equation



First order differential equation

$$\partial_x \vec{f}(x,\epsilon) = A(x,\epsilon)\vec{f}(x,\epsilon)$$

where ϵ is the dimensional regularization parameter $D = 4 - 2\epsilon$.

Properties of $A(x,\epsilon)$

- Block triangular
- Rational in x and ϵ
- Satisfies integrability condition (for at least two invariants x and y)

$$\partial_y A(x,\epsilon) - \partial_x A(y,\epsilon) + [A(x,\epsilon), A(y,\epsilon)] = 0$$

Bottom-up Approach

- Solve each line in $A(x,\epsilon)$ bottom up
- Previously solved integrals will appear as inhomogenous parts of the next DEQ

Matrix Approach Henn

- Conjecture: We can find a basis such that

$$\partial_x \vec{g}(x,\epsilon) = \epsilon \tilde{A}(x)\vec{g}(x,\epsilon)$$

- Makes integration simple
- But finding the basis is difficult

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Change of Basis

$$\begin{aligned} \vec{g}(x,\epsilon) &= B(x,\epsilon)\vec{f}(x,\epsilon) \\ \hat{A}(x,\epsilon) &= B^{-1}(x,\epsilon)A(x,\epsilon)B(x,\epsilon) - B^{-1}(x,\epsilon)\partial_x B(x,\epsilon) \end{aligned}$$

Reformulate the Problem

Find a change of basis which brings us to the canonical form
$\Rightarrow$ Solving a DEQ for $B(x,\epsilon)$

$$\partial_x B(x,\epsilon) = A(x,\epsilon)B(x,\epsilon) - \epsilon B(x,\epsilon)\hat{A}(x)$$

Can be as hard as initial problem

Assumption — Ageri, Di Vita, Mastrolia, Mirabella, Schlenk, Tancredi, **U.S.**

Linear DEQ

$$A(x,\epsilon) = A_0(x) + \epsilon A_1(x)$$

We have to solve

$$\partial_x B(x) = A_0(x)B(x)$$

Can be done with Magnus Theorem

Magnus theorem

Starting from a first order differential equation

$$\partial_x \vec{f}(x) = A(x)\vec{f}(x)$$

The solution is given by the Magnus exponential

$$\vec{f}(x) = e^{\Omega[A](x,x_0)}\vec{f}(x_0) \equiv e^{\Omega[A](x)}\vec{f}(x_0) \qquad \Omega[A](x) = \sum_{n=1}^{\infty} \Omega_n[A](x)$$

$$\Omega_1[A](x) = \int_{x_0}^{x} d\tau_1 A(\tau_1)$$

$$\Omega_2[A](x) = \frac{1}{2}\int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 [A(\tau_1), A(\tau_2)] \ , \ldots$$

Connected to the Dyson Series

$$\vec{f}(x) = \left(1 + \sum_{n=1}^{\infty} P_n(x)\right)\vec{f}(x_0) \ , \ P_n(x) = \int_{x_0}^{x} d\tau_1 \ldots \int_{x_0}^{\tau_{n-1}} d\tau_n \ A(\tau_1)A(\tau_2)\ldots A(\tau_n)$$

$$P_1(x) = \Omega_1(x)$$

$$P_2(x) = \Omega_2(x) + \frac{1}{2}\Omega_1^2(x) \ , \ldots .$$

Finding the canonical form with Magnus

From a linear DEQ

$$A(x,\epsilon) = A_0(x) + \epsilon A_1(x)$$

Magnus Theorem provides a basis change to the canonical form

$$B(x) = e^{\Omega[A_0](x)} \qquad \vec{g}(x,\epsilon) = B(x)\vec{f}(x,\epsilon)$$

Reducing DEQ

Lee (2014)

DEQ is rational in x
Mosers algorithm reduces all poles to simple poles (if possible)

$$A(x,\epsilon) = \sum_{n=1}^{k} \frac{\mathcal{S}_n(\epsilon)}{x - x_n} \qquad \text{with } x_n \text{ being constant}$$

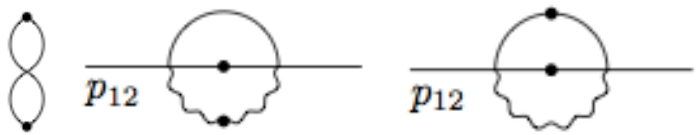
Referred to as Fuchsian form

If Eigenvalues of $\mathcal{S}_n$ are linear in ϵ with integer coefficients
We can shift them to multiples of ϵ

A similarity transformation $\epsilon\tilde{\mathcal{S}} = T^{-1}(\epsilon)\mathcal{S}(\epsilon)T(\epsilon)$ gives us the canonical form

Example: massive QED Sunrise

- Three master integrals



- Dimensionless variable

$$-\frac{s}{m^2} = \frac{(1-x)^2}{x}$$

- Linear and Fuchsian DEQ

$$\partial_x \vec{f}(x) = A(x,\epsilon)\vec{f}(x)$$

$$A(x,\epsilon) = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\epsilon}{2(1-x)} - \frac{\epsilon}{2(1+x)} & -\frac{1+6\epsilon}{1+x} + \frac{1+3\epsilon}{x} + \frac{1}{1-x} & -\frac{1-2\epsilon}{2(1-x)} - \frac{\epsilon}{x} - \frac{1-6\epsilon}{2(1+x)} \\ 0 & \frac{2\epsilon}{x} + \frac{4\epsilon}{1-x} & \frac{1}{x} + \frac{2}{1-x} \end{pmatrix}$$

Let's apply Magnus and Lees algorithm

QED Sunrise with Magnus

Split ϵ^0-part into diagonal and off-diagonal part and transform with diagonal part first

$$A_0(x) = D_0(x) + N_0(x), \qquad B_1 = e^{\int dx\, D_0(x)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{x}{1-x^2} & 0 \\ 0 & 0 & \frac{x}{(1-x)^2} \end{pmatrix}$$

Transformed DEQ

$$\hat{A}(x) = B_1^{-1} A(x) B_1 - B_1^{-1} \partial_x B_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\epsilon}{x} & \frac{3\epsilon}{x} - \frac{6\epsilon}{1+x} & -\frac{1-2\epsilon}{(x-1)^2} - \frac{\epsilon}{x} \\ 0 & \frac{2\epsilon}{x} & 0 \end{pmatrix}$$

QED Sunrise with Magnus

Transform with off-diagonal part

$$\hat{A}_0(x) = \hat{N}_0(x), \qquad B_2 = e^{\int dx\, \hat{N}_0(x)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{1-x} \\ 0 & 0 & 1 \end{pmatrix}$$

Transformed DEQ II

$$\tilde{A}(x) = B_2^{-1}\hat{A}(x)B_2 - B_2^{-1}\partial_x B_2$$

$$\tilde{A}(x) = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{\epsilon}{x} & \frac{2\epsilon}{1-x} + \frac{5\epsilon}{x} - \frac{6\epsilon}{1+x} & -\frac{2\epsilon}{1-x} - \frac{6\epsilon}{x} + \frac{3\epsilon}{1+x} \\ 0 & \frac{2\epsilon}{x} & -\frac{2\epsilon}{1-x} - \frac{2\epsilon}{x} \end{pmatrix}$$

Masters for canonical basis are given by

$$\vec{g} = B_2^{-1}B_1^{-1}\vec{f} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{(1-x)(x+1)}{x} & \frac{1-x}{x} \\ 0 & 0 & \frac{(x-1)^2}{x} \end{pmatrix} \vec{f}$$

QED Sunrise with Lee

Focus only on 2x2 system of the Sunrise, since the rest of the system is already ϵ-factorized
System is Fuchsian
$\Rightarrow$ Shift all Eigenvalues to multiples of ϵ

$$x \to 1 : \{-1-2\epsilon, -2+2\epsilon\} \quad x \to 0 : \{1+\epsilon, 1+2\epsilon\}$$
$$x \to -1 : \{0, -1-6\epsilon\} \quad x \to \infty : \{1+\epsilon, 1+2\epsilon\}$$

A balance transformation will shift one eigenvalue by $+1$ the other one by -1

$$\mathcal{B}(\mathbb{P}, x_1, x_2|x) = \bar{\mathbb{P}} + c\frac{x-x_2}{x-x_1}\mathbb{P}, \quad \mathbb{P} = u(\lambda_1)v^\dagger(\lambda_2), \quad \bar{\mathbb{P}} = 1-\mathbb{P}$$

After four transformation we brought all Eigenvalues in the right form

$$x \to 1 : \{-2\epsilon, 2\epsilon\} \quad x \to 0 : \{\epsilon, 2\epsilon\}$$
$$x \to -1 : \{0, -6\epsilon\} \quad x \to \infty : \{\epsilon, 2\epsilon\}$$

QED Sunrise with Lee

The last step is to find a similarity transformation which brings us to the canonical form
For each pole matrix we solve

$$\frac{A(x,\epsilon)}{\epsilon} T(\epsilon,\mu) = T(\epsilon,\mu) \frac{A(x,\mu)}{\mu}$$

Finally we arrive at

$$\partial_x \vec{g} = \epsilon \begin{pmatrix} \frac{5}{x} - \frac{6}{1+x} + \frac{2}{1-x} & \frac{6}{x} - \frac{3}{1+x} + \frac{2}{1-x} \\ -\frac{2}{x} & -\frac{2}{1-x} - \frac{2}{x} \end{pmatrix} \vec{g}$$

Combining all transformation we find for our masters

$$\vec{g} = \begin{pmatrix} \frac{(1-x)(1+x)}{x} & \frac{1-x}{x} \\ 0 & -\frac{(x-1)^2}{x} \end{pmatrix} \vec{f} \quad \text{Reminder: } B_{\text{Magnus}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{(1-x)(x+1)}{x} & \frac{1-x}{x} \\ 0 & 0 & \frac{(x-1)^2}{x} \end{pmatrix}$$

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Solving DEQ in the matrix approach

The canonical form

$$\partial_x g(x,\epsilon) = \epsilon \tilde{A}(x) g(x,\epsilon)$$

implies

$$\vec{g}(x,\epsilon) = \left(1 + \sum_{n=1}^{\infty} \epsilon^n P_n(x)\right) \vec{g}(x_0,\epsilon) \quad P_n(x) = \int_{x_0}^{x} d\tau_1 \ldots \int_{x_0}^{\tau_{n-1}} d\tau_n \, \tilde{A}(\tau_1)\tilde{A}(\tau_2)\ldots\tilde{A}(\tau_n)$$

- $\tilde{A}(x)$ determines the types of functions which will appear
- If $\tilde{A}(x)$ is Fuchsian

$$\tilde{A}(x) = \sum_{n=1}^{k} \frac{\mathcal{S}_n}{x - x_n} \qquad \text{with } x_n \text{ being constant}$$

then $g(x,\epsilon)$ can be written in terms of Goncharov Polylogarithms

Goncharov Polylogarithm

Goncharov

Basic definition

$$
\begin{aligned}
G(a_1;\, x) &= \int_0^x \frac{dt}{t - a_1} \\
G(a_1, ..., a_n;\, x) &= \int_0^x \frac{dt}{t - a_1} G(a_2, ..., a_n\,; t) \\
G(\vec{0}_n;\, x) &= \frac{1}{n!}\log^n(x)
\end{aligned}
$$

n is the weight of the Polylog

Connections to other functions:

- Logarithm: $G(a_1, ..., a_1;\, x) = \frac{1}{n!}\log^n(1 - \frac{x}{a_1})$
- classical Polylog $G(\vec{0}_n, 1;\, x) = -\mathrm{Li}_n(x)$
- Harmonic Polylogs (HPLs) Remiddi, Vermaseren
  $G(\vec{a};\, x) = (-1)^p H(\vec{a}, x)$ with $a_i \in \{-1, 0, 1\}$
- Two-dimensional harmonic polylogarithms Gehrmann, Remiddi
  $G(\vec{a};\, x) = (-1)^p H(\vec{a}, x)$ with $a_i \in \{0, 1, -y, 1 - y\}$

Properties

Invariant under rescaling

for $k \in \mathbb{C}$

$$G(k\vec{a};\, k\,x) = G(\vec{a};\, x)$$

Shuffle relations

$$G(\vec{a};\, x)G(\vec{b};\, x) = \sum_{\vec{c} \in \vec{a} \sqcup\!\sqcup \vec{b}} G(\vec{c};\, x)$$

where $\vec{a} \sqcup\!\sqcup \vec{b}$ means taking every permutation of the elements in $\vec{a}$ and $\vec{b}$ such that the individual ordering of $\vec{a}$ and $\vec{b}$ is kept

e.g. $G(a_1;\, x)G(b_1, b_2;\, x) = G(a_1, b_1, b_2;\, x) + G(b_1, a_1, b_2;\, x) + G(b_1, b_2, a_1;\, x)$

Many more known/unknown relations
$\Rightarrow$ Led to development of Symbol and Coproduct

Goncharov, Spradlin, Vergu, Volovich; Duhr

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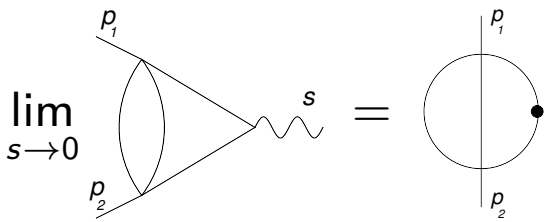
Recap: Solution of the canonical DEQ

$$\vec{g}(x,\epsilon) = \left(1 + \sum_{n=1}^{\infty} \epsilon^n P_n(x)\right) \vec{g}(x_0,\epsilon)$$

We can obtain $\vec{g}(x_0,\epsilon)$ by

Known limits

- Taking the limit $x \to x_0$ to a known function
- Fix the boundary constant by matching the solution to the known function



Pseudo-thresholds

- Solution has physical and unphysical divergences
- Physical divergences correspond to particle thresholds
- Unphysical divergences are absent in the final result
- Demanding their absence gives relations between the boundary constants of the master integrals
- Leftover constants must be provided (usually elementary integrals)

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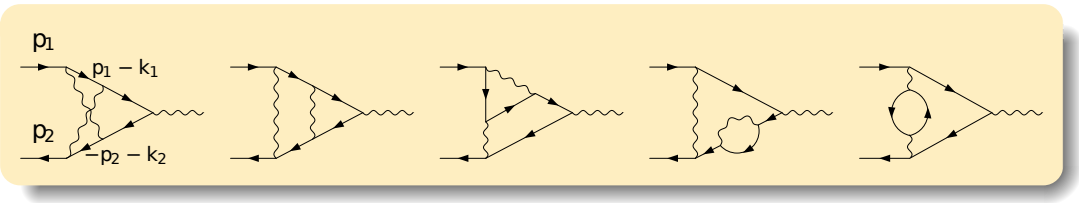
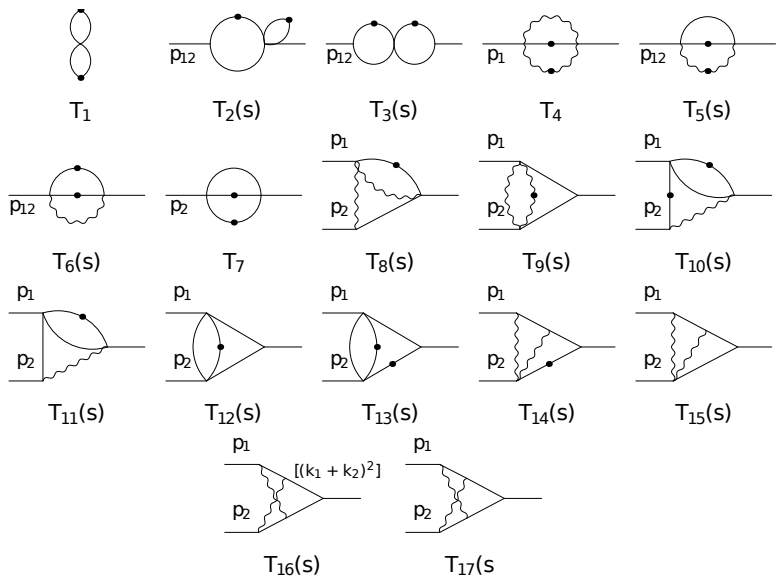
massive QED vertex at two-loop

Bonciani, Mastrolia, Remiddi

17 MI's for all relevant topologies



$$-\frac{s}{m^2} = \frac{(1-x)^2}{x}$$



The f's obey an ϵ-linear DE [ADVMMSST '14]

$$f_1 = \epsilon^2 T_1 \quad f_2 = \epsilon^2 T_2 \quad f_3 = \epsilon^2 T_3 \quad f_4 = \epsilon^2 T_4 \quad f_5 = \epsilon^2 T_5$$
$$f_6 = \epsilon^2 T_6 \quad f_7 = \epsilon^2 T_7 \quad f_8 = \epsilon^3 T_8 \quad f_9 = \epsilon^3 T_9 \quad f_{10} = \epsilon^2 T_{10}$$
$$f_{11} = \epsilon^3 T_{11} \quad f_{12} = \epsilon^3 T_{12} \quad f_{13} = \epsilon^2 T_{13} \quad f_{14} = \epsilon^3 T_{14} \quad f_{15} = \epsilon^4 T_{15}$$
$$f_{16} = \epsilon^4 T_{16} \quad f_{17} = \epsilon^4 T_{17}$$

After getting rid of A_0, the g's obey a canonical DE

$$\partial_x g(\epsilon, x) = \epsilon\, \hat{A}_1(x)\, g(\epsilon, x) \qquad \hat{A}_1(x) = \frac{M_1}{x} + \frac{M_2}{1+x} + \frac{M_3}{1-x}$$

$M_1 =$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	0	0	0	5	−6	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	2	−2	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	0	0	−4	0	−2	0	−2	0	0	0	0	0	0	0	0	0
0	0	0	−2	0	0	0	0	2	0	0	0	0	0	0	0	0
$-\frac{1}{2}$	0	0	0	1	−2	−3	0	0	3	3	0	0	0	0	0	0
0	0	0	0	1	−1	2	0	0	−2	−2	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	−1	−1	0	0	0	0
0	−1	0	0	0	0	−3	0	0	0	0	3	3	0	0	0	0
0	−1	0	0	1	$-\frac{1}{2}$	0	2	2	0	0	0	0	2	2	0	0
0	0	0	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	0	0	0	−1	−1	0	0
$-\frac{1}{2}$	0	0	−2	−1	0	−2	1	0	2	0	−2	0	0	−2	−2	2
0	0	0	0	−1	$\frac{1}{2}$	0	3	−2	0	−6	−2	0	0	−4	−4	4

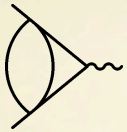
$M_2 =$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	−4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	−6	3	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	−4	0	0	0	0	0	0	0
0	0	0	0	−1	$\frac{1}{2}$	0	0	0	−4	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0
0	0	0	0	0	0	0	−6	0	0	0	0	−2	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	−4	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−4

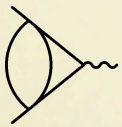
$M_3 =$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	2	−2	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	−2	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	−2	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	−6	0	0	2	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	−2	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	−12	0	0	0	0	0	4	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	2	0	0	0	−4	0	0	−4	−2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4

Bonciani, Remiddi, **P.M.** (2013)



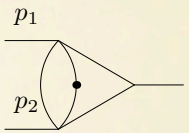
$$
\begin{aligned}
M_{-2} &= \frac{1}{2}\,,\\
M_{-1} &= \frac{5}{2} - \left[1 - \frac{2}{(1-x)}\right] H(0,x)\,,\\
M_{0} &= \frac{19}{2} + \zeta(2) + \left[1 - \frac{2}{(1-x)}\right][\zeta(2) - 5H(0,x) + 2H(-1,0,x)]\\
&\quad + \frac{2}{(1-x)} H(0,0,x) + \left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right][\zeta(2) H(0,x)\\
&\quad + H(0,0,0,x)]\,.
\end{aligned}
$$



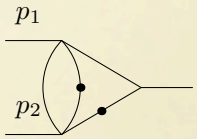
$(p_2 \cdot k_1)$

$$
\begin{aligned}
\frac{N_{-2}}{a} &= \frac{1}{8} + \frac{1}{16}\left[x + \frac{1}{x}\right]\,,\\
\frac{N_{-1}}{a} &= \frac{9}{32}\left[2 + x + \frac{1}{x}\right] - \frac{1}{8}\left[4 + x - \frac{1}{x}\right] H(0,x) + \frac{1}{(1-x)} H(0,x)\,,\\
\frac{N_{0}}{a} &= \frac{63}{32} + \frac{\zeta(2)}{2} + \frac{63}{64}\left[\left(1 + \frac{16}{63}\zeta(2)\right) x + \frac{1}{x}\right] - \frac{\zeta(2)}{(1-x)} - \frac{1}{16}\left[32 + 9x\right.\\
&\quad \left. - \frac{9}{x}\right] H(0,x) + \frac{(16 + \zeta(2))}{4(1-x)} H(0,x) - \frac{\zeta(2)}{4(1+x)} H(0,x) - \frac{1}{4}\left[2 - \frac{1}{x}\right.\\
&\quad \left. - \frac{4}{(1-x)}\right] H(0,0,x) + \frac{1}{4}\left[4 + x - \frac{1}{x} - \frac{8}{(1-x)}\right] H(-1,0,x)\\
&\quad + \frac{1}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] H(0,0,0,x)\,.
\end{aligned}
$$

AdVMMSST (2014)



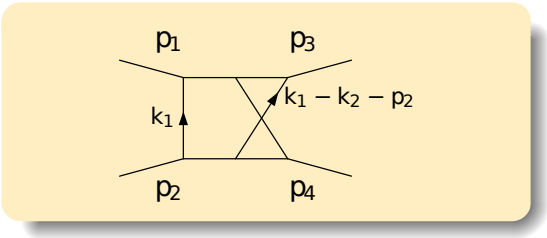
$$
\begin{aligned}
g_{12}^{(0)} &= 0\,,\\
g_{12}^{(1)} &= 0\,,\\
g_{12}^{(2)} &= 0\,,\\
g_{12}^{(3)} &= -\,\mathrm{H}(0,0,0;x) - \zeta_2 \mathrm{H}(0;x)\,,\\
g_{12}^{(4)} &= -\,2\,\mathrm{H}(-1,0,0,0;x) + 2\,\mathrm{H}(0,-1,0,0;x) + 2\,\mathrm{H}(0,0,-1,0;x)\\
&\quad - 3\,\mathrm{H}(0,0,0,0;x) - 4\,\mathrm{H}(0,1,0,0;x) + \zeta_2(-2\,\mathrm{H}(-1,0;x)\\
&\quad + 6\,\mathrm{H}(0,-1;x) - \mathrm{H}(0,0;x)) + 2\,\zeta_3\,\mathrm{H}(0;x) + \frac{\zeta_4}{4}\,,
\end{aligned}
$$



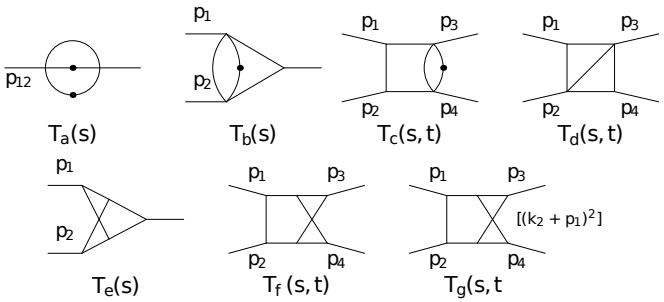
$$
\begin{aligned}
g_{13}^{(0)} &= 0\,,\\
g_{13}^{(1)} &= 0\,,\\
g_{13}^{(2)} &= \mathrm{H}(0,0;x) + \frac{3\,\zeta_2}{2}\,,\\
g_{13}^{(3)} &= -\,2\,\mathrm{H}(-1,0,0;x) - 2\,\mathrm{H}(0,-1,0;x) + 4\,\mathrm{H}(0,0,0;x) + 4\,\mathrm{H}(1,0,0;x)\\
&\quad + \zeta_2(-6\,\mathrm{H}(-1;x) + 2\,\mathrm{H}(0;x) - 3\log 2) - \frac{\zeta_3}{4}\,,\\
g_{13}^{(4)} &= 4\,\mathrm{H}(-1,-1,0,0;x) + 4\,\mathrm{H}(-1,0,-1,0;x) - 8\,\mathrm{H}(-1,0,0,0;x)\\
&\quad - 8\,\mathrm{H}(-1,1,0,0;x) + 4\,\mathrm{H}(0,-1,-1,0;x) - 8\,\mathrm{H}(0,-1,0,0;x)\\
&\quad - 8\,\mathrm{H}(0,0,-1,0;x) + 10\,\mathrm{H}(0,0,0,0;x) + 12\,\mathrm{H}(0,1,0,0;x)\\
&\quad - 8\,\mathrm{H}(1,-1,0,0;x) - 8\,\mathrm{H}(1,0,-1,0;x) + 16\,\mathrm{H}(1,0,0,0;x)\\
&\quad + 16\,\mathrm{H}(1,1,0,0;x) + 12\,\mathrm{Li}_4\frac{1}{2} + \frac{\log^4 2}{2} + 2\,\zeta_2\,(12\log 2\,\mathrm{H}(-1;x)\\
&\quad + 12\log 2\,\mathrm{H}(1;x) + 6\,\mathrm{H}(-1,-1;x) - 2\,\mathrm{H}(-1,0;x) - 8\,\mathrm{H}(0,-1;x)\\
&\quad + \mathrm{H}(0,0;x) - 12\,\mathrm{H}(1,-1;x) + 4\,\mathrm{H}(1,0;x) + 3\log^2 2)\\
&\quad - 2\,\zeta_3(5\,\mathrm{H}(-1;x) + 4\,\mathrm{H}(0;x) + 11\,\mathrm{H}(1;x)) - \frac{47\,\zeta_4}{4}\,,
\end{aligned}
$$

Non-planar massless box Tausk (1999); Anastasiou, Gehrmann, Oleari, Remiddi, Tausk

12 MI's for the crossed topology



$$x = -\frac{t}{s}, \quad s > 0\,, t < 0\,, |s| > |t|$$



The f's obey an ϵ-linear DE [ADVMMSST '14]

$$f_1 = \epsilon^2\, s\, T_a(s)\,, \quad f_2 = \epsilon^2\, t\, T_a(t)\,, \quad f_3 = \epsilon^2\, u\, T_a(u)\,,$$
$$f_4 = \epsilon^3\, s\, T_b(s)\,, \quad f_5 = \epsilon^3\, s\, t\, T_c(s,t)\,, \quad f_6 = \epsilon^3\, s\, u\, T_c(s,u)\,,$$
$$f_7 = \epsilon^4\, u\, T_d(s,t)\,, \quad f_8 = \epsilon^4\, s\, T_d(t,u)\,, \quad f_9 = \epsilon^4\, t\, T_d(u,s)\,,$$
$$f_{10} = \epsilon^4\, s^2\, T_e(s)\,,$$
$$f_{11} = \epsilon^4\, s\, t\, u\, T_f(s,t) - \frac{3}{4\,s\,(4\epsilon+1)}\Big[\epsilon^2\left(s^2\, T_a(s) + t^2\, T_a(t) + u^2\, T_a(u)\right) - 4\epsilon^4\left(u^2\, T_d(s,t) + s^2\, T_d(t,u) + t^2\, T_d(u,s)\right)\Big]\,,$$
$$f_{12} = \epsilon^4\, s\, t\, T_g(s,t) - \frac{3}{8\,u\,(4\epsilon+1)}\Big[\epsilon^2\left(s^2\, T_a(s) + t^2\, T_a(t) + u^2\, T_a(u)\right) - 4\epsilon^4\left(u^2\, T_d(s,t) + s^2\, T_d(t,u) + t^2\, T_d(u,s)\right)\Big]\,,$$

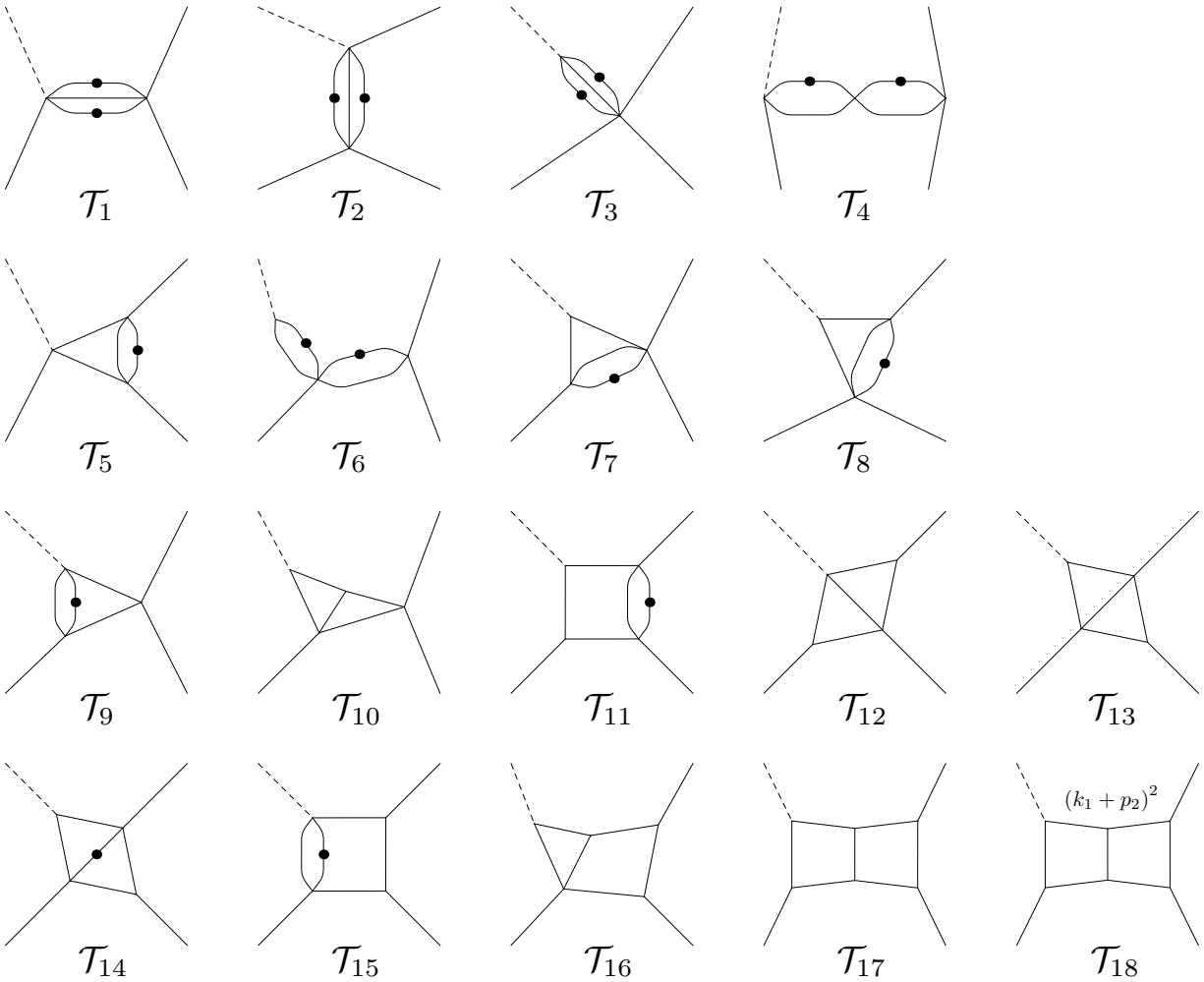
After getting rid of A_0, the g's obey a canonical DE

$$\partial_x\, g(\epsilon, x) = \epsilon\, \hat{A}_1(x)\, g(\epsilon, x) \qquad \hat{A}_1(x) = \frac{M_1}{x} + \frac{M_2}{1-x}$$

$$M_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{3}{2} & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3}{2} & -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6 & -6 & -\frac{9}{2} & 0 & -4 & -2 & -18 & -12 & -12 & 1 & 1 & -2 \\
\frac{3}{4} & \frac{9}{4} & -\frac{21}{4} & 3 & 2 & -3 & 12 & -6 & -18 & 0 & 0 & -2
\end{pmatrix}$$

$$M_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{3}{2} & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3}{2} & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6 & -6 & -\frac{9}{2} & 0 & -4 & -2 & -18 & -12 & -12 & 1 & 1 & -2 \\
-\frac{21}{4} & \frac{9}{4} & -\frac{27}{4} & -6 & 2 & -4 & 12 & -6 & -24 & 1 & -1 & 0
\end{pmatrix}$$

Higgs+Jet at two-loop Gehrmann, Remiddi; Di Vita, Mastrolia, Yundin, **U.S.**



Higgs+Jet at two-loop Gehrmann, Remiddi; Di Vita, Mastrolia, Yundin, **U.S.**

- dimensionless variables

$$x = \frac{s}{m_h^2} \qquad y = \frac{t}{m_h^2}$$

- ϵ-linear basis

$$\partial_x \vec{f}(x,y,\epsilon) = (A_{1,0}(x,y) + \epsilon A_{1,1}(x,y))\,\vec{f}(x,y,\epsilon)$$
$$\partial_y \vec{f}(x,y,\epsilon) = (A_{2,0}(x,y) + \epsilon A_{2,1}(x,y))\,\vec{f}(x,y,\epsilon)$$

- Canonical form with Magnus

$$\partial_x \vec{g}(x,y,\epsilon) = \epsilon \hat{A}_1(x,y)\vec{g}(x,y,\epsilon)$$
$$\partial_y \vec{g}(x,y,\epsilon) = \epsilon \hat{A}_2(x,y)\vec{g}(x,y,\epsilon)$$

- Log-form

$$\hat{A}(x,y) = M_1 \log(x) + M_2 \log(1-x) + M_3 \log(y) + M_4 \log(1-y)$$
$$+ M_5 \log(\frac{x+y}{x}) + M_6 \log(\frac{1-x-y}{1-x})$$

$$\partial_x \hat{A}(x,y) = \hat{A}_1(x,y) \qquad \partial_y \hat{A}(x,y) = \hat{A}_2(x,y)$$

- Alphabet

$$\{x, 1-x, y, 1-y, x+y, 1-x-y\}$$

Boundary condition

Four pseudo-thresholds

$$x \to 1 \ (s \to m_H^2), \ y \to 1 \ (t \to m_H^2), \ y \to -x \ (u \to m_H^2), \ y \to 1-x \ (u \to 0)$$

Demanding their absence at the matrix level

$$\left(-\frac{3}{2} \right) - 3 \, + 3 \, - \Big|_{x \to 1} = 0$$

$$\Big|_{y \to 1} = 0$$

$$\left(3 \right) - 3 \, - \Big|_{y \to 1-x} = 0$$

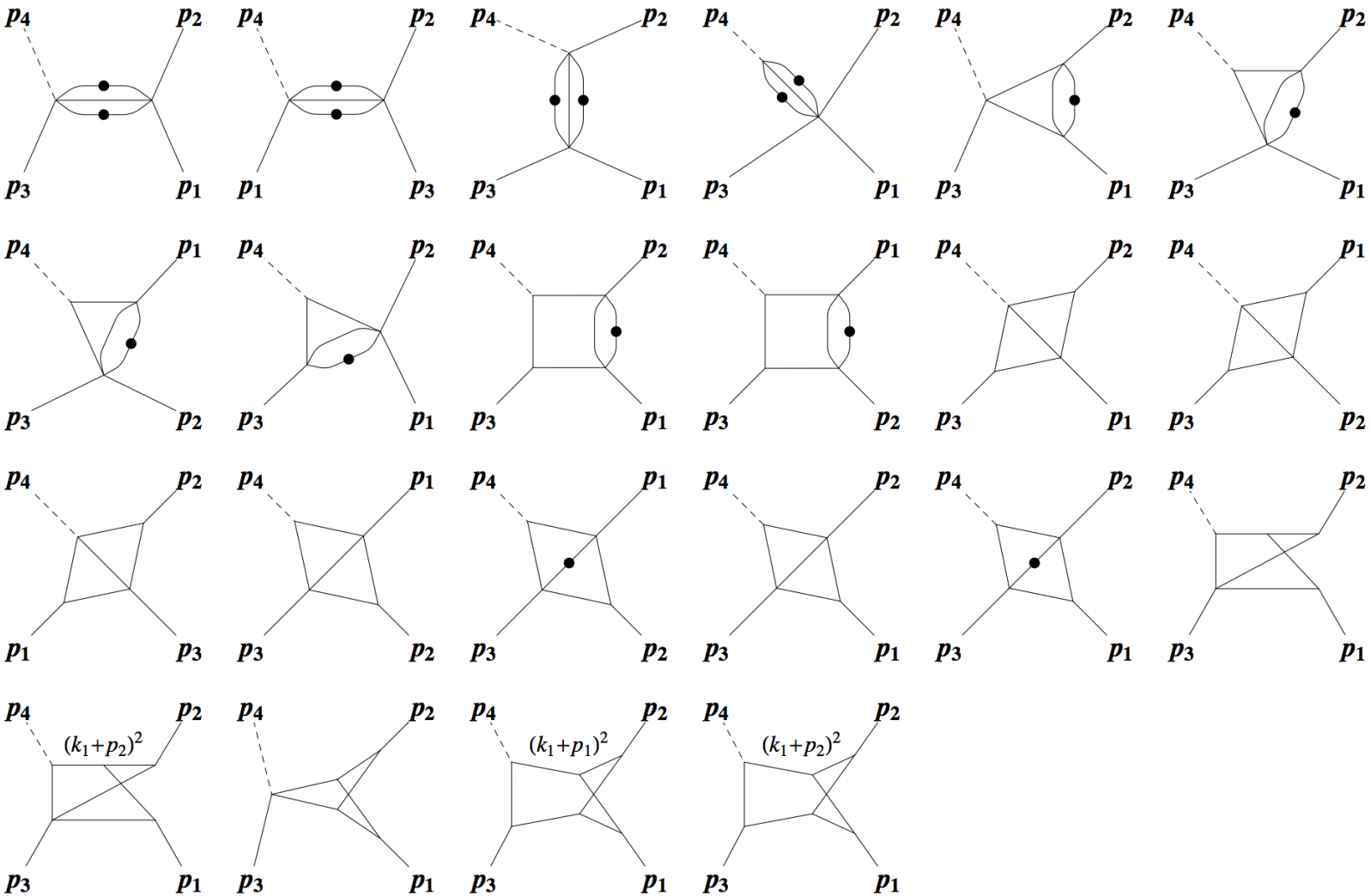
leads to relations between the boundary constants of different integrals

$$\Big|_{\epsilon^0, \text{const.}} = -\frac{1}{4} \Big|_{\epsilon^0, \text{const.}}, \qquad \Big|_{\epsilon^1, \text{const.}} = -\frac{1}{4} \Big|_{\epsilon^1, \text{const.}}$$

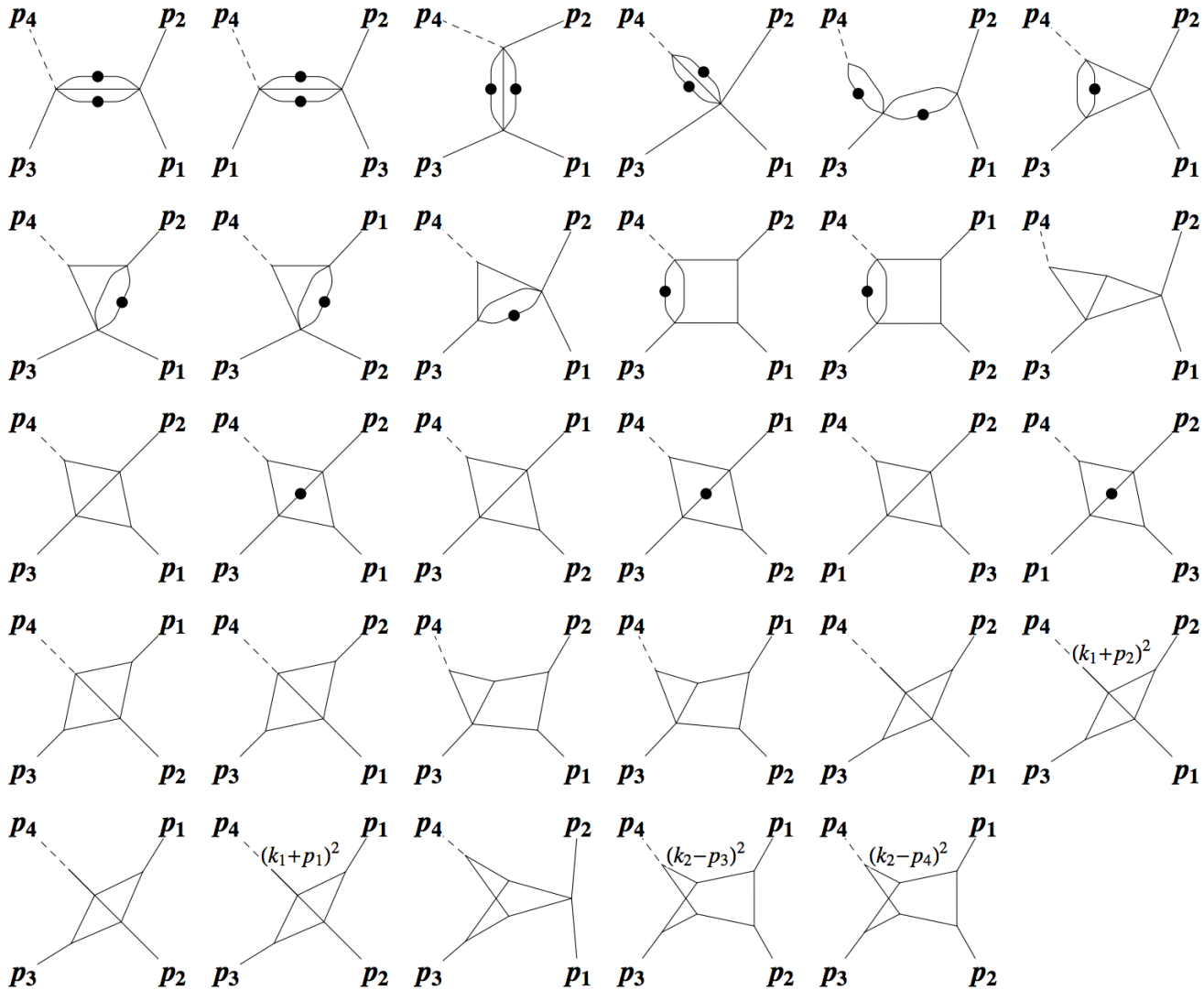
$$\Big|_{\epsilon^2, \text{const.}} = -\frac{1}{4} \Big|_{\epsilon^2, \text{const.}} - \frac{\zeta_2}{2} \Big|_{\epsilon^2, \text{const.}}$$

After solving all relations only **one** input integral is needed

Higgs+Jet at two-loop Gehrmann, Remiddi; Di Vita, Mastrolia, Yundin, **U.S.**

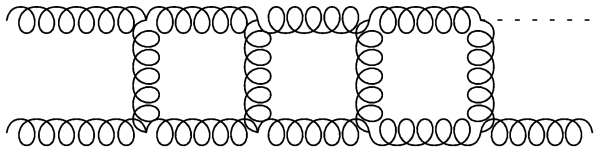


Higgs+Jet at two-loop Gehrmann, Remiddi; Di Vita, Mastrolia, Yundin, **U.S.**

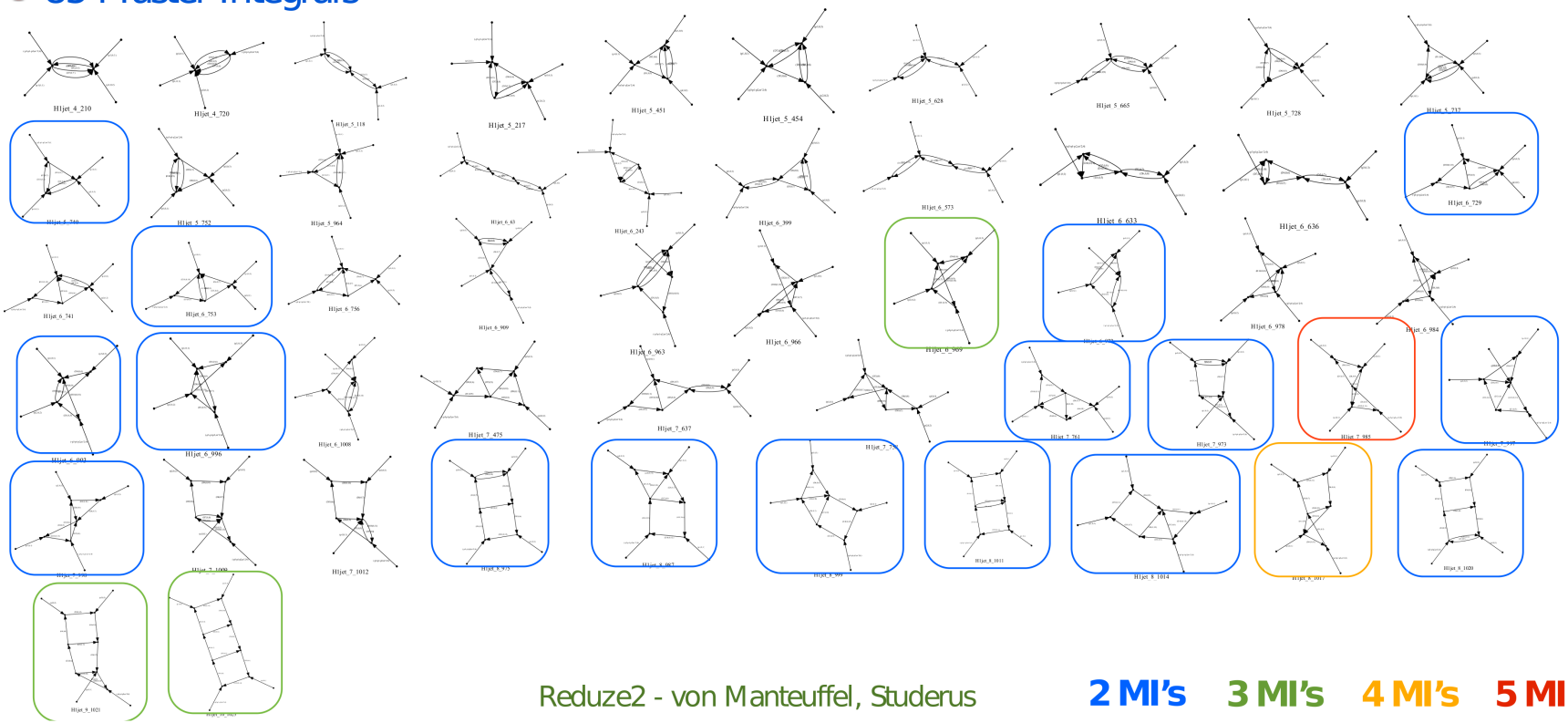


Higgs+Jet at three-loop

Di Vita, Mastrolia, Yundin, **U.S.** (2014)



- 85 Master Integrals



Reduze2 - von Manteuffel, Studerus

2 MI's **3 MI's** **4 MI's** **5 MI's**

taken from Pierpaolo's slides at Amplitudes 2014

Higgs+Jet at three-loop

Di Vita, Mastrolia, Yundin, **U.S.**

- ϵ-linear basis

$$\partial_x \vec{f}(x,y,\epsilon) = \left(A_{1,0}(x,y) + \epsilon A_{1,1}(x,y)\right) \vec{f}(x,y,\epsilon)$$
$$\partial_y \vec{f}(x,y,\epsilon) = \left(A_{2,0}(x,y) + \epsilon A_{2,1}(x,y)\right) \vec{f}(x,y,\epsilon)$$

- Canonical form with Magnus

$$\partial_x \vec{g}(x,y,\epsilon) = \epsilon \hat{A}_1(x,y) \vec{g}(x,y,\epsilon)$$
$$\partial_y \vec{g}(x,y,\epsilon) = \epsilon \hat{A}_2(x,y) \vec{g}(x,y,\epsilon)$$

- Log-form

$$\hat{A}(x,y) = M_1 \log(x) + M_2 \log(1-x) + M_3 \log(y) + M_4 \log(1-y)$$
$$+ M_5 \log\left(\frac{x+y}{x}\right) + M_6 \log\left(\frac{1-x-y}{1-x}\right)$$

$$\partial_x \hat{A}(x,y) = \hat{A}_1(x,y) \qquad \partial_y \hat{A}(x,y) = \hat{A}_2(x,y)$$

- Alphabet

$$\{x, 1-x, y, 1-y, x+y, 1-x-y\}$$

same as in the two-loop case!

1 Introduction

2 Differential Equations
- Basics
- Matrix Approach
- Solution
- Boundary Conditions

3 Examples
- massive QED vertex at two-loop
- $2 \to 2$ non-planar massless box
- Higgs+Jet at two-loop
- Higgs+Jet at three-loop

4 Open Problems

Finding the Canonical Form

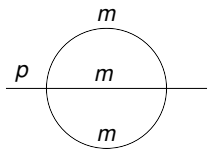
No algorithm for finding the canonical form

But first steps have been done:

- Magnus Exponential
  DEQ has to be linear in ϵ
  Magnus series might not converge
  $$B(x) = e^{\Omega[A_0](x)}$$
- Reducing DEQ
  Not all DEQ are reducible to a Fuchsian Form
  $$\tilde{A}(x) = \sum_{n=1}^{k} \frac{\mathcal{S}_n}{x - x_n}$$
  $\rightarrow$ Related to the negative answer of the 21st Hilbert Problem
  Eigenvalues of $\mathcal{S}_n$ have to be linear in ϵ

Equal-mass sunrise

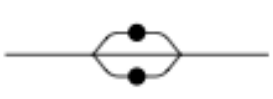
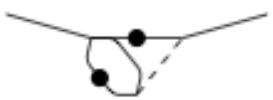
Laporta, Remiddi; Bloch, Vanhove; Adams, Bogner, Weinzierl



DEQ is known to evaluate to elliptic functions

Easy to find a DEQ $\partial_s \vec{f} = A(s,\epsilon)\vec{f}$ which is Fuchsian and linear in $\epsilon = \frac{(D-4)}{2}$

$$A(s,\epsilon) = \frac{1}{s}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -\epsilon & -1+\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}-\epsilon & -1+\epsilon \end{pmatrix} + \frac{1}{s-m^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ \epsilon & -2\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\epsilon}{2} & \frac{3}{4}+\frac{3}{2}\epsilon & -1+2\epsilon \end{pmatrix} + \frac{1}{s-9m^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2\epsilon & -1-2\epsilon & 4 \\ 0 & \frac{\epsilon}{2} & \frac{1}{4}+\frac{1}{2}\epsilon & 1 \end{pmatrix}$$

$f^T =$ (, ,  , )

$\rightarrow$ Let's apply Magnus and Lees algorithm

Magnus

Split ϵ^0-part into diagonal and off-diagonal part and construct transformation

$$A_0 = D_0 + N_0$$

$$D_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{s} & 0 & 0 \\ 0 & 0 & -\frac{1}{s-9m^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{s} - \frac{1}{s-m^2} + \frac{1}{s-9m^2} \end{pmatrix}, \quad B_1 = e^{\int dx\, D_0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{s} & 0 & 0 \\ 0 & 0 & -\frac{1}{s-9m^2} & 0 \\ 0 & 0 & 0 & \frac{8}{s-m^2} - \frac{9}{s} \end{pmatrix}$$

Transformed DEQ

$$\hat{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\epsilon}{s-m^2} & \frac{\epsilon}{s} - \frac{2\epsilon}{s-m^2} & 0 & 0 \\ 0 & -\frac{2\epsilon}{s} & -\frac{2\epsilon}{s-9m^2} & \frac{36}{s} - \frac{32}{s-m^2} \\ 0 & -\frac{4m\epsilon}{(s-9m^2)^2} & -\frac{3m^2(1+2\epsilon)}{2(s-9m^2)^2} - \frac{18\left(2\epsilon m^4+m^4\right)}{(s-9m^2)^3} & \frac{\epsilon}{s} - \frac{2\epsilon}{s-m^2} \end{pmatrix}$$

Transform with the ϵ^0 part of $\hat{A}$

$$\hat{N}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{36}{s} - \frac{32}{s-m^2} \\ 0 & 0 & -\frac{18m^4}{(s-9m^2)^3} - \frac{3m^2}{2(s-9m^2)^2} & 0 \end{pmatrix}$$

This matrix is not nilpotent

$\Rightarrow$ Dyson/Magnus Series will not converge

Lees Algorithm

Eigenvalues of pole matrices

$$s \to 0 : \{0, 0, -1+\epsilon, -1+\epsilon\} \quad s \to m^2 : \{0, 0, -1+2\epsilon, -2\epsilon\}$$
$$s \to 9m^2 : \{0, 0, 0, -2\epsilon\} \quad s \to \infty : \{0, 1+\epsilon, 1+\epsilon, 1+2\epsilon\}$$

Use balance transformation to shift eigenvalues

$$\mathcal{B}(\mathbb{P}, x_1, x_2|x) = \bar{\mathbb{P}} + c\frac{x-x_2}{x-x_1}\mathbb{P}, \quad \mathbb{P} = u(\lambda_1)v^\dagger(\lambda_2), \quad \bar{\mathbb{P}} = 1 - \mathbb{P}$$

After two transformations we arrive at

$$s \to 0 : \{0, 0, \epsilon, \epsilon\} \quad s \to m^2 : \{0, 0, -1+2\epsilon, -2\epsilon\}$$
$$s \to 9m^2 : \{0, 0, 0, -2\epsilon\} \quad s \to \infty : \{0, 1+\epsilon, \epsilon, 2\epsilon\}$$

Eigenvalues $-1+2\epsilon$ and $1+\epsilon$ can not be balanced since there eigenvectors are orthogonal to each other

$\Rightarrow$ Both results indicate that the ϵ^0-part is related to the elliptic functions

Conclusion

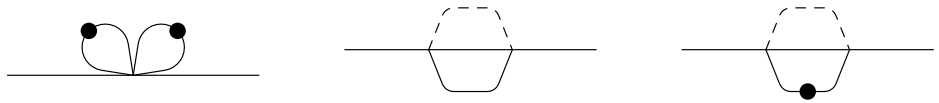
- New ideas stimulated the Differential Equations method
  -Work at matrix level
  -Bring DEQ in canonical form

$$\vec{g} = \epsilon A(x)\vec{g}$$

- Magnus Series finds a canonical basis for a DEQ which is linear in ϵ
- Has been applied to a variety of known and unknown processes
  -QED Vertex at two-loop
  -$2 \to 2$ massless box
  -Higgs+jet at two-loop
  -Ladder topology for Higgs+jet at three loop

More are coming so stay tuned

Back up slides



$$\partial_x \vec{f} = A(x)\vec{f} \qquad \text{with} \qquad A(x) = \frac{S_{-1}(\epsilon)}{x+1} + \frac{S_0(\epsilon)}{x} + \frac{S_1(\epsilon)}{x-1}$$

Eigenvalues of the pole matrices:

$$S_{-1}(\epsilon) : \{0, 0, 3-6\epsilon\} \qquad S_0(\epsilon) : \{0, \epsilon, -1+2\epsilon\}$$
$$S_1(\epsilon) : \{0, 1-2\epsilon, -2+2\epsilon\} \qquad S_\infty(\epsilon) : \{0, \epsilon, -1+2\epsilon\}$$

with $S_\infty(\epsilon) = -S_{-1}(\epsilon) - S_0(\epsilon) - S_1(\epsilon)$

use balance transformations $\mathcal{B}(\mathbb{P}, x_1, x_2|x) = \bar{\mathbb{P}} + c\frac{x-x_2}{x-x_1}\mathbb{P}$ to make all Eigenvalues $\propto \epsilon$

$$\tilde{S}_{-1}(\epsilon) : \{0, 0, -6\epsilon\} \qquad \tilde{S}_0(\epsilon) : \{0, \epsilon, 2\epsilon\}$$
$$\tilde{S}_1(\epsilon) : \{0, -2\epsilon, 2\epsilon\} \qquad \tilde{S}_\infty(\epsilon) : \{0, \epsilon, 2\epsilon\}$$

Find the similarity transformation which brings us to the canonical form through solving

$$\frac{\tilde{S}_i(\epsilon)}{\epsilon} T = T \frac{\tilde{S}_i(\mu)}{\mu}$$

Our new masters are

$$\vec{g} = T\vec{f}$$

$$T = \begin{pmatrix} (1-2\epsilon)(2-3\epsilon)(1-3\epsilon) & 0 & 0 \\ \frac{2(4\epsilon-1)}{(x-1)^2} + \frac{2(3\epsilon-1)}{x-1} - \frac{\epsilon}{x} & +\frac{8\epsilon}{x-1} + \frac{2\epsilon}{x} + 4\epsilon - 2x\epsilon & -\frac{4(1-4\epsilon)}{(x-1)^2} - \frac{4(1-12\epsilon)}{x-1} + \frac{6\epsilon}{x} - 2(2-17\epsilon) - 10x\epsilon \\ \frac{-12\epsilon^2+11\epsilon-2}{(x-1)^2} + \frac{-9\epsilon^2+9\epsilon-2}{x-1} & \frac{4\epsilon(2-3\epsilon)}{x-1} + 2\epsilon(2-3\epsilon) & -\frac{2\left(12\epsilon^2-11\epsilon+2\right)}{(x-1)^2} - \frac{2\left(36\epsilon^2-27\epsilon+2\right)}{x-1} - 2 + 29\epsilon - 39\epsilon^2 \end{pmatrix}$$

which satisfy a canonical DEQ

$$\vec{g} = \epsilon\tilde{A}(x)$$

$$\tilde{A}(x) = \frac{1}{x+1}\begin{pmatrix} 0 & 0 & 0 \\ \frac{3}{2} & -6 & -24 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{x}\begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & -15 \\ -1 & 2 & 7 \end{pmatrix} \frac{1}{x-1}\begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & -2 & -16 \\ 1 & 0 & 2 \end{pmatrix}$$

After Integrating and fixing the boundary we find non uniform transcendental(UT) masters

$$f_2 = \frac{9\epsilon^2}{8} + \epsilon^3\left(\frac{1}{4}G(0;x) + \frac{117}{16}\right) + \epsilon^4\left(\frac{13}{8}G(0;x) - \frac{3}{2}G(-1,0;x) - G(0,0;x) - \frac{1}{2}G(1,0;x) + \frac{1}{96}\left(3105 - 4\pi^2\right)\right)$$

$\Rightarrow$ canonical form does **not** imply UT masters
Boundary conditions can spoil UT

massive Sunrise with Lee

Starting from DEQ

$$A(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\epsilon}{x-1} - \frac{\epsilon}{x} & \frac{\epsilon-1}{x} - \frac{2\epsilon}{x-1} & 0 & 0 \\ 0 & \frac{2\epsilon}{x-9} & \frac{-2\epsilon-1}{x-9} & \frac{4}{x-9} \\ 0 & \frac{\epsilon}{2(x-9)} - \frac{\epsilon}{2(x-1)} & \frac{-2\epsilon-1}{4(x-9)} + \frac{-2\epsilon-1}{2x} + \frac{3(2\epsilon+1)}{4(x-1)} & \frac{-2\epsilon-1}{x-1} + \frac{\epsilon-1}{x} + \frac{1}{x-9} \end{pmatrix}$$

with $x = \frac{s}{m^2}$
Transformation generated by shifting eigenvalues twice

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{x} & -\frac{x+1}{2x} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{(x+1)\epsilon}{2x} & -\frac{1}{x} \end{pmatrix}$$

Resulting DEQ

$$\hat{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{\epsilon}{x-1} & \frac{10\epsilon}{9(x-9)} - \frac{2\epsilon}{x-1} + \frac{8\epsilon}{9x} & -\frac{2\epsilon}{x-1} + \frac{5\epsilon}{9x} + \frac{5(8\epsilon+9)}{9(x-9)} & \frac{20}{9(x-9)} - \frac{2}{9x} \\ 0 & \frac{2\epsilon}{9x} - \frac{2\epsilon}{9(x-9)} & \frac{-8\epsilon-9}{9(x-9)} - \frac{\epsilon}{9x} & \frac{4}{9x} - \frac{4}{9(x-9)} \\ 0 & \frac{\epsilon^2}{9x} - \frac{\epsilon}{2(x-1)} + \frac{9\epsilon-20\epsilon^2}{18(x-9)} & -\frac{5\epsilon^2}{9x} + \frac{-160\epsilon^2-108\epsilon+81}{36(x-9)} + \frac{8\epsilon^2-4\epsilon-3}{4(x-1)} & \frac{9-20\epsilon}{9(x-9)} + \frac{-2\epsilon-1}{x-1} + \frac{11\epsilon}{9x} \end{pmatrix}$$