Simplified differential equations approach for NNLO calculations

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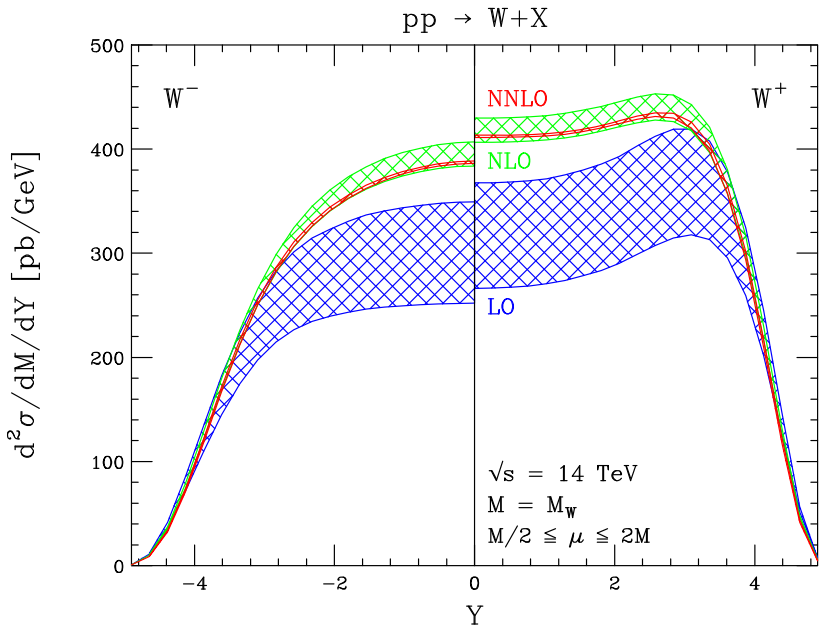
NCSR-D, January 17, 2014

QCD beyond LO

(N)NLO needed in order to properly interpret the data at the LHC

- LO: shape
- NLO: shape+normalization
- NNLO: shape+normalization+uncertainty



C. Anastasiou, L. J. Dixon, K. Melnikov and F. Petriello, Phys. Rev. D **69** (2004) 094008 [hep-ph/0312266].

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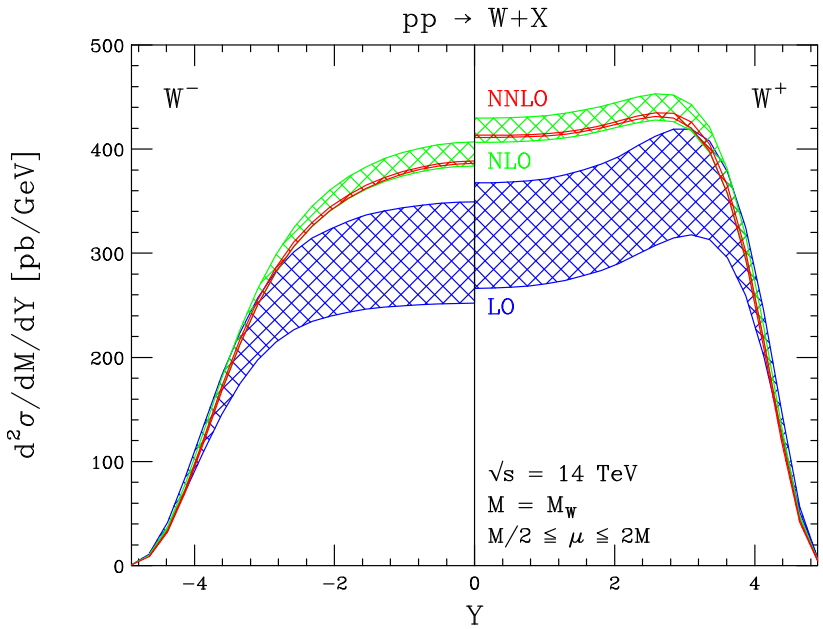

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QCD beyond LO

(N)NLO needed in order to properly interpret the data at the LHC

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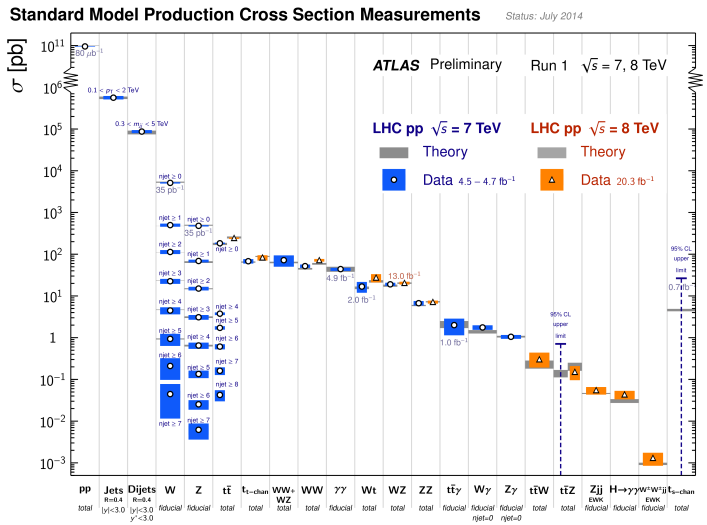
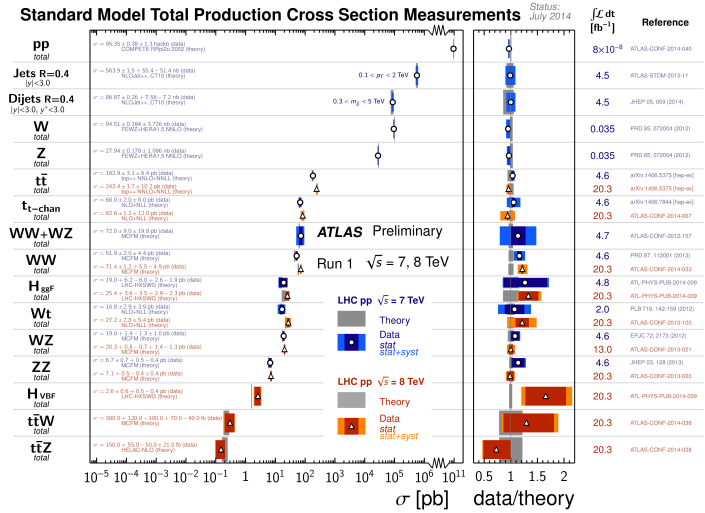
(SM) Physics from Run 1



Physics from Run 1

…in most cases, good agreement with SM predictions (at NLO and higher).
The SM will be tested more stringently (with hopefully BSM physics discovered) in Run 2. We need to have the predictions available to test data vs theory.



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Realistic wishlist

- Was developed at Les Houches in 2005, and expanded in 2007 and 2009
- Calculations that are important for the LHC AND do-able in finite time
- In 2009, we added tttt, Wbbj, W/Z+4j plus an extra column for each process indicating the level of precision required by the experiments
  - to see for example if EW corrections may need to be calculated
- In order to be most useful, decays for final state particles (t,W,H) need to be provided in the codes as well
- With the calculation of tttt, all processes on the wishlist have been calculated
- The wishlist has been retired since new techniques allow for the semi-automatic generation of new (reasonable) NLO cross sections

Process ($V \in \{Z, W, \gamma\}$)	Comments
Calculations completed since Les Houches 2005	
1. $pp \to VV$jet	WWjet completed by Dittmaier/Kallweit/Uwer [4,5]; Campbell/Ellis/Zanderighi [6]. ZZjet completed by Binoth/Gleisberg/Karg/Kauer/Sanguinetti [7]
2. $pp \to$ Higgs+2jets note we didn't even think Higgs+3 jets possible	NLO QCD to the gg channel completed by Campbell/Ellis/Zanderighi [8]; NLO QCD+EW to the VBF channel completed by Ciccolini/Denner/Dittmaier [9, 10]
3. $pp \to VVV$	ZZZ completed by Lazopoulos/Melnikov/Petriello [11] and WWZ by Hankele/Zeppenfeld [12] (see also Binoth/Ossola/Papadopoulos/Pittau [13])
4. $pp \to t\bar{t}b\bar{b}$	relevant for $t\bar{t}H$ computed by Bredenstein/Denner/Dittmaier/Pozzorini [14, 15] and Bevilacqua/Czakon/Papadopoulos/Pittau/Worek [16]
5. $pp \to V$+3jets	calculated by the Blackhat/Sherpa [17] and Rocket [18] collaborations
Calculations remaining from Les Houches 2005	
6. $pp \to t\bar{t}$+2jets	relevant for $t\bar{t}H$ computed by Bevilacqua/Czakon/Papadopoulos/Worek [19]
7. $pp \to VVb\bar{b}$,	relevant for VBF $\to H \to VV$, $t\bar{t}H$
8. $pp \to VV$+2jets	relevant for VBF $\to H \to VV$ VBF contributions calculated by (Bozzi/)Jäger/Oleari/Zeppenfeld [20–22]
NLO calculations added to list in 2007	
9. $pp \to b\bar{b}b\bar{b}$	$q\bar{q}$ channel calculated by Golem collaboration [23]
NLO calculations added to list in 2009	
10. $pp \to V$+4 jets	top pair production, various new physics signatures
11. $pp \to Wb\bar{b}j$	top, new physics signatures
12. $pp \to t\bar{t}t\bar{t}$	various new physics signatures
Calculations beyond NLO added in 2007	
13. $gg \to W^*W^* \mathcal{O}(\alpha^2\alpha_s^3)$	backgrounds to Higgs
14. NNLO $pp \to t\bar{t}$	normalization of a benchmark process
15. NNLO to VBF and Z/γ+jet	Higgs couplings and SM benchmark
Calculations including electroweak effects	
16. NNLO QCD+NLO EW for W/Z	precision calculation of a SM benchmark

Table 1: The updated experimenter's wishlist for LHC processes

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Realistic wishlist

- 4 top final state

arXiv:1206.3064v1 [hep-ph] 14 Jun 2012

Constraining BSM Physics at the LHC: Four top final states with NLO accuracy in perturbative QCD

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[b] *Theoretische Physik, Fachbereich C, Bergische Universität Wuppertal, Gauss Str. 20, D-42097 Wuppertal, Germany*

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Abstract: Many theories, from Supersymmetry to models of Strong Electroweak Symmetry Breaking, look at the production of four top quarks as an interesting channel to evidentiate signals of new physics beyond the Standard Model. The production of four-top final states requires large partonic energies, above the $4m_t$ threshold, that are available at the CERN Large Hadron Collider and will become more and more accessible with increasing energy and luminosity of the proton beams. A good theoretical control on the Standard Model background is a fundamental prerequisite for a correct interpretation of the possible signals of new physics that may arise in this channel. In this paper we report on the calculation of the next-to-leading order QCD corrections to the Standard Model process $pp \to t\bar{t}t\bar{t} + X$. As it is customary for such studies, we present results for both integrated and differential cross sections. A judicious choice of a dynamical scale allows us to obtain nearly constant K-factors in most distributions.

Keywords: NLO Computations, Heavy Quark Physics, Standard Model, Beyond Standard Model

WUB/12-12, TTK-12-22

NNLO needed in order to properly interpret the data at the LHC: Whishlist

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NNLO QCD+NLO EW wishlist

Higgs

Process	known	desired	details
H	$d\sigma$ @ NNLO QCD $d\sigma$ @ NLO EW finite quark mass effects @ NLO	$d\sigma$ @ NNNLO QCD + NLO EW MC@NNLO finite quark mass effects @ NNLO	H branching ratios and couplings
H + j	$d\sigma$ @ NNLO QCD (g only) $d\sigma$ @ NLO EW finite quark mass effects @ LO	$d\sigma$ @ NNLO QCD + NLO EW finite quark mass effects @ NLO	H p_T
H + 2j	σ_{tot}(VBF) @ NNLO(DIS) QCD $d\sigma$(gg) @ NLO QCD $d\sigma$(VBF) @ NLO EW	$d\sigma$ @ NNLO QCD + NLO EW	H couplings
H + V	$d\sigma$ @ NNLO QCD $d\sigma$ @ NLO EW	with $H \to b\bar{b}$ @ same accuracy	H couplings
$t\bar{t}H$	$d\sigma$(stable tops) @ NLO QCD	$d\sigma$(top decays) @ NLO QCD + NLO EW	top Yukawa coupling
HH	$d\sigma$ @ LO QCD (full m_t dependence) $d\sigma$ @ NLO QCD (infinite m_t limit)	$d\sigma$ @ NLO QCD (full m_t dependence) $d\sigma$ @ NNLO QCD (infinite m_t limit)	Higgs self coupling

Table 1: Wishlist part 1 – Higgs (V = W, Z)

justify the requested precision based on current/extrapolated experimental errors

S. Dittmaier, N. Glover, J. Huston

NNLO needed in order to properly interpret the data at the LHC: Whishlist

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NNLO QCD + NLO EWK wishlist

heavy quarks, photons, jets

Process	known	desired	details
$t\bar{t}$	σ_{tot} @ NNLO QCD $d\sigma$(top decays) @ NLO QCD $d\sigma$(stable tops) @ NLO EW	$d\sigma$(top decays) @ NNLO QCD + NLO EW	precision top/QCD, gluon PDF, effect of extra radiation at high rapidity, top asymmetries
$t\bar{t}+j$	$d\sigma$(NWA top decays) @ NLO QCD	$d\sigma$(NWA top decays) @ NNLO QCD + NLO EW	precision top/QCD top asymmetries
single-top	$d\sigma$(NWA top decays) @ NLO QCD	$d\sigma$(NWA top decays) @ NNLO QCD (t channel)	precision top/QCD, V_{tb}
dijet	$d\sigma$ @ NNLO QCD (g only) $d\sigma$ @ NLO weak	$d\sigma$ @ NNLO QCD + NLO EW	Obs.: incl. jets, dijet mass $\rightarrow$ PDF fits (gluon at high x) $\rightarrow \alpha_s$ CMS http://arxiv.org/abs/1212.6660
3j	$d\sigma$ @ NLO QCD	$d\sigma$ @ NNLO QCD + NLO EW	Obs.: $R3/2$ or similar $\rightarrow \alpha_s$ at high scales dom. uncertainty: scales CMS http://arxiv.org/abs/1304.7498
$\gamma+j$	$d\sigma$ @ NLO QCD $d\sigma$ @ NLO EW	$d\sigma$ @ NNLO QCD +NLO EW	gluon PDF $\gamma+b$ for bottom PDF

Table 2: Wishlist part 2 – jets and heay quarks

NNLO needed in order to properly interpret the data at the LHC: Whishlist

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NNLO QCD + NLO EWK wishlist

Vector bosons

Process	known	desired	details
V	$d\sigma$(lept. V decay) @ NNLO QCD $d\sigma$(lept. V decay) @ NLO EW	$d\sigma$(lept. V decay) @ NNNLO QCD + NLO EW MC@NNLO	precision EW, PDFs
V + j	$d\sigma$(lept. V decay) @ NLO QCD $d\sigma$(lept. V decay) @ NLO EW	$d\sigma$(lept. V decay) @ NNLO QCD + NLO EW	Z + j for gluon PDF W + c for strange PDF
V + jj	$d\sigma$(lept. V decay) @ NLO QCD	$d\sigma$(lept. V decay) @ NNLO QCD + NLO EW	study of systematics of H + jj final state
VV′	$d\sigma$(V decays) @ NLO QCD $d\sigma$(stable V) @ NLO EW	$d\sigma$(V decays) @ NNLO QCD + NLO EW	off-shell leptonic decays TGCs
gg → VV	$d\sigma$(V decays) @ LO QCD	$d\sigma$(V decays) @ NLO QCD	bkg. to $H \to VV$ TGCs
Vγ	$d\sigma$(V decay) @ NLO QCD $d\sigma$(PA, V decay) @ NLO EW	$d\sigma$(V decay) @ NNLO QCD + NLO EW	TGCs
Vbb̄	$d\sigma$(lept. V decay) @ NLO QCD massive b	$d\sigma$(lept. V decay) @ NNLO QCD massless b	bkg. for VH → bb̄
VV′γ	$d\sigma$(V decays) @ NLO QCD	$d\sigma$(V decays) @ NLO QCD + NLO EW	QGCs
VV′V″	$d\sigma$(V decays) @ NLO QCD	$d\sigma$(V decays) @ NLO QCD + NLO EW	QGCs, EWSB
VV′ + j	$d\sigma$(V decays) @ NLO QCD	$d\sigma$(V decays) @ NLO QCD + NLO EW	bkg. to H, BSM searches
VV′ + jj	$d\sigma$(V decays) @ NLO QCD	$d\sigma$(V decays) @ NLO QCD + NLO EW	QGCs, EWSB
γγ	$d\sigma$ @ NNLO QCD		bkg to $H \to \gamma\gamma$

Table 3: Wishlist part 3 – EW gauge bosons (V = W, Z)

Dyson-Schwinger Recursive Equations

Automated computation era: 1994 Madgraph

- 1999 HELAC: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

Dyson-Schwinger Recursive Equations

Automated computation era: 1994 Madgraph

From Feynman Diagrams to recursive equations: taming the $n!$

- 1999 HELAC: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

Dyson-Schwinger Recursive Equations

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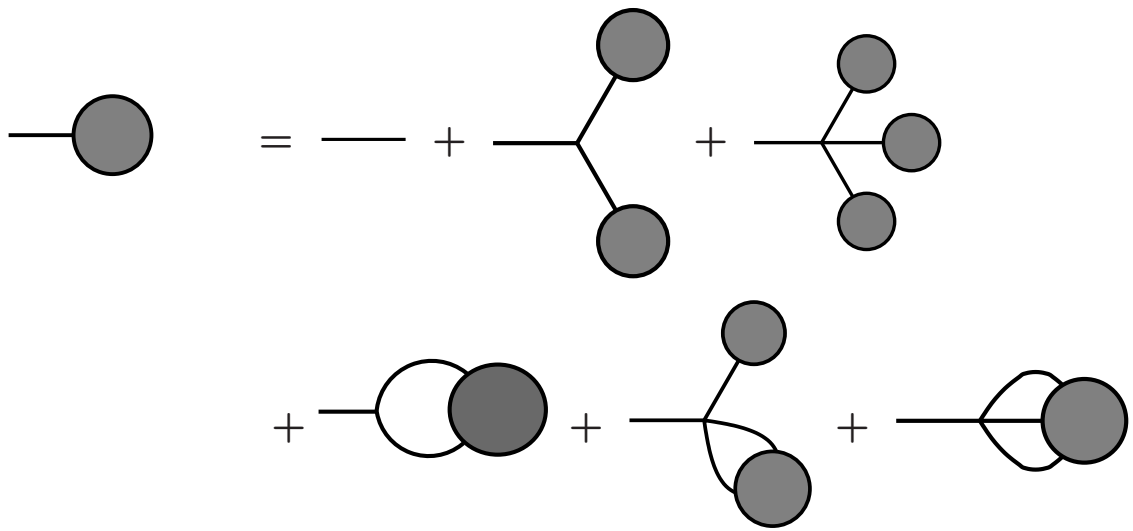
From Feynman Diagrams to recursive equations: taming the n!

- 1999 HELAC: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306 [arXiv:hep-ph/0002082].

F. A. Berends and W. T. Giele, Nucl. Phys. B **306** (1988) 759.

F. Caravaglios and M. Moretti, Phys. Lett. B **358** (1995) 332.



Unfortunately not so much on the second line !

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m |M_m^{(0)}|^2 J_m(\Phi) \\
&+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \\
&+ \int_{m+1} d\Phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)
\end{aligned}$$

$J_m(\Phi)$ jet function: Infrared safeness $J_{m+1} \to J_m$

Perturbative QCD at NLO

What do we need for an NLO calculation ?

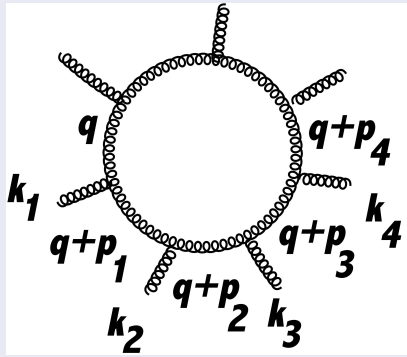
$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m^{D=4}(|M_m^{(0)}|^2 + 2Re(M_m^{(0)*} M_m^{(CT)}(\epsilon_{UV})))J_m(\Phi) \\
&+ \int_m d\Phi_m^{D=4} 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR}))J_m(\Phi) \\
&+ \int_{m+1} d\Phi_{m+1}^{D=4-2\epsilon_{IR}} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)
\end{aligned}$$

IR and UV divergencies, Four-Dimensional-Helicity scheme; scale dependence μ_R QCD factorization$-\mu_F$ Collinear counter-terms when PDF are involved

One-loop amplitudes

Any m-point one-loop amplitude can be written as



$$\int d^D q\, A(\bar{q}) = \int d^D q\, \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

A bar denotes objects living in $n = 4 + \epsilon$ dimensions

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2$$

$$\bar{q}^2 = q^2 + \tilde{q}^2$$

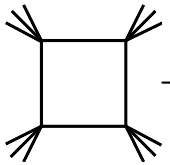
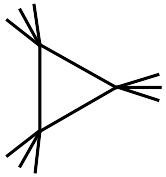
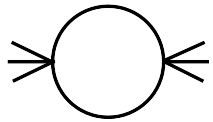
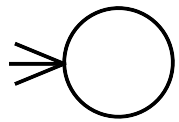
$$\bar{D}_i = D_i + \tilde{q}^2$$

The one loop paradigm

basis of scalar integrals:

G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217 [arXiv:hep-ph/9403226].

$$\mathcal{A} = \sum d_{i_1 i_2 i_3 i_4}$$  $$+ \sum c_{i_1 i_2 i_3}$$  $$+ \sum b_{i_1 i_2}$$  $$+ \sum a_{i_1}$$  $$+ R$$

$a, b, c, d \rightarrow$ cut-constructible part $\qquad$ $R \rightarrow$ rational terms

$$\mathcal{A} = \sum_{I \subset \{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)} d^d q}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i \in I} \bar{D}_i(\bar{q})}$$

The old "master" formula

$$
\begin{aligned}
\int A &= \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3) \\
&+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) C_0(i_0 i_1 i_2) \\
&+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) B_0(i_0 i_1) \\
&+ \sum_{i_0}^{m-1} a(i_0) A_0(i_0) \\
&+ \text{rational terms}
\end{aligned}
$$

D_0, C_0, B_0, A_0, scalar one-loop integrals: 't Hooft and Veltman
`QCDLOOP` Ellis & Zanderighi ; `OneLOop` A. van Hameren

The old "master" formula

$$\int \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} = \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}}$$
$$+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}}$$
$$+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1}}$$
$$+ \sum_{i_0}^{m-1} a(i_0) \int \frac{1}{\bar{D}_{i0}}$$
$$+ \text{rational terms}$$

Remove the integration !

The new "master" formula

$$
\begin{aligned}
\frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i0} \bar{D}_{i1}} \\
&+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i0}} \\
&+ \text{rational terms}
\end{aligned}
$$

OPP "MASTER" FORMULA

Equation in a from "solvable" à la "unitarity"; not the only way

General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

A next to simple example

- Not only tensor integrals need reduction!

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}}$$

$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \dots D_{i_{m-1}}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}} \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \dots D_{i_{m-1}}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)$$

$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left(\prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^+)} + \prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^-)} \right)$$

A next to simple example

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$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \dots D_{i_{m-1}}$$

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$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)$$

$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left(\prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^+)} + \prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^-)} \right)$$

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$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}}$$

$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \dots D_{i_{m-1}}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}} \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \dots D_{i_{m-1}}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)$$

$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left(\prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^+)} + \prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^-)} \right)$$

A next to simple example

- Not only tensor integrals need reduction!

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}}$$

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Rational Terms

Numerically treat $D = 4 - 2\epsilon$, means $4 \oplus 1$

Expand in D-dimensions ?

$$\bar{D}_i = D_i + \tilde{q}^2$$

$$\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\
&+ \sum^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2) \right] \prod^{m-1} \bar{D}_i + \tilde{P}(q) \prod^{m-1} \bar{D}_i
\end{aligned}$$

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$$
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&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2) \right] \prod_{i \neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i
\end{aligned}
$$

$$m_i^2 \to m_i^2 - \tilde{q}^2$$

Rational Terms

Numerically treat $D = 4 - 2\epsilon$, means $4 \oplus 1$

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$$
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N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2) \right] \prod_{i \neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i
\end{aligned}
$$

$$
m_i^2 \to m_i^2 - \tilde{q}^2
$$

Rational Terms

In practice, once the 4-dimensional coefficients have been determined, one can redo the fits for different values of $\tilde{q}^2$, in order to determine $b^{(2)}(ij)$, $c^{(2)}(ijk)$ and $d^{(2m-4)}$.

$$\begin{aligned} \mathrm{R}_1 &= -\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0 i_1 i_2) \\ &- \frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0 i_1) \left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3} \right) . \end{aligned}$$

G. Ossola, C. G. Papadopoulos and R. Pittau,arXiv:0802.1876 [hep-ph]

Rational Terms - R_2

A different source of Rational Terms, called R_2, can also be generated from the ϵ-dimensional part of $N(q)$

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, \epsilon; q)$$

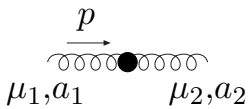
$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4} \int d^n \bar{q} \frac{\tilde{N}(\tilde{q}^2, \epsilon; q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \equiv \frac{1}{(2\pi)^4} \int d^n \bar{q}\, \mathcal{R}_2$$

$$\begin{aligned} \bar{q} &= q + \tilde{q}\,, \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}\,, \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}\,. \end{aligned}$$

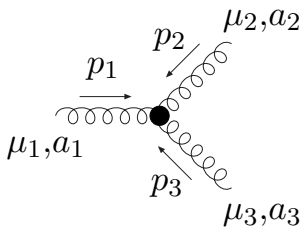
New vertices/particles or GKMZ-approach

HELAC R2 terms

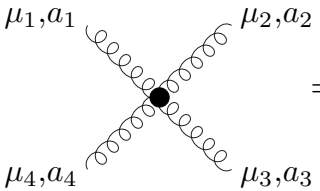
Contribution from $d-$dimensional parts in numerators:



$$= \frac{ig^2 N_{col}}{48\pi^2}\,\delta_{a_1a_2}\left[\frac{p^2}{2}g_{\mu_1\mu_2} + \lambda_{HV}\left(g_{\mu_1\mu_2}p^2 - p_{\mu_1}p_{\mu_2}\right) + \frac{N_f}{N_{col}}\left(p^2 - 6\,m_q^2\right)g_{\mu_1\mu_2}\right]$$



$$= -\frac{g^3 N_{col}}{48\pi^2}\left(\frac{7}{4} + \lambda_{HV} + 2\frac{N_f}{N_{col}}\right) f^{a_1a_2a_3}\,V_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)$$



$$= -\frac{ig^4 N_{col}}{96\pi^2}\sum_{P(234)}\left\{\left[\frac{\delta_{a_1a_2}\delta_{a_3a_4} + \delta_{a_1a_3}\delta_{a_4a_2} + \delta_{a_1a_4}\delta_{a_2a_3}}{N_{col}} + 4\,Tr(t^{a_1}t^{a_3}t^{a_2}t^{a_4} + t^{a_1}t^{a_4}t^{a_2}t^{a_3})\,(3+\lambda_{HV}) - Tr(\{t^{a_1}t^{a_2}\}\{t^{a_3}t^{a_4}\})\,(5+2\lambda_{HV})\right]g_{\mu_1\mu_2}g_{\mu_3\mu_4} + 12\frac{N_f}{N_{col}}Tr(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\left(\frac{5}{3}g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_2\mu_3}g_{\mu_1\mu_4}\right)\right\}$$

HELAC R2 terms

$$
\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m^{D=4}(|M_m^{(0)}|^2 + 2Re(M_m^{(0)*} M_m^{(CT)}(\epsilon_{UV})))J_m(\Phi) \\
&+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR}))J_m(\Phi) \\
&+ \int_{m+1} d\Phi_{m+1}|M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)
\end{aligned}
$$

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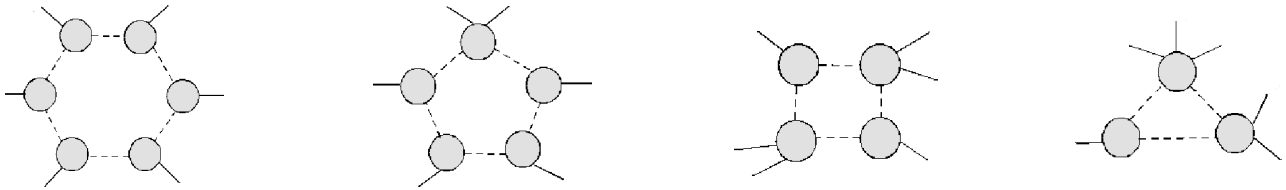
&+ \boxed{\int_m d\Phi_m^{D=4}\left(2Re(M_m^{(0)*} M_m^{(R_2)})\right) J_m(\Phi)} \\
&+ \int_m d\Phi_m^{D=4} 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR}))J_m(\Phi) \\
&+ \int_{m+1} d\Phi_{m+1}^{D=4-2\epsilon_{IR}} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)
\end{aligned}
$$

The one-loop calculation in a nutshell

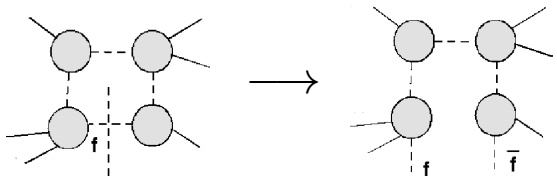
The computation of $pp(p\bar{p}) \to e^+\nu_e\mu^-\bar{\nu}_\mu b\bar{b}$ involves up to six-point functions.
The most generic integrand has therefore the form

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}}_{} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}}_{} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}}_{} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}}_{} + \cdots$$



In order to apply the OPP reduction, `HELAC` evaluates numerically the numerators $N_i^6(q), N_i^5(q), \ldots$ with the values of the loop momentum q provided by `CutTools`

- generates all inequivalent partitions of 6,5,4,3... blobs attached to the loop, and check all possible flavours (and colours) that can be consistently running inside
- hard-cuts the loop (q is fixed) to get a $n+2$ *tree-like* process



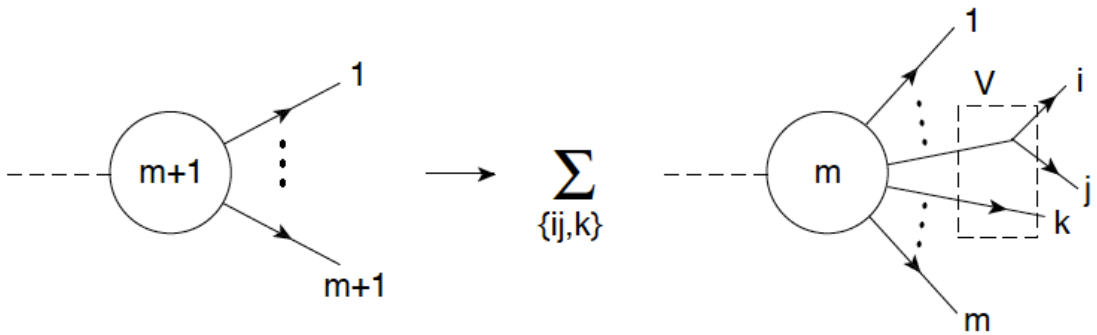
The R_2 contributions (rational terms) are calculated in the same way as the tree-order amplitude, taking into account *extra vertices*

Real corrections

Real corrections: $D \to 4$ dimensions (Catani & Seymour)

$$\int_{m+1} d\sigma^R + \int_m d\sigma^V$$

$$\sigma^{NLO} = \int_{m+1} \left[\left(d\sigma^R\right)_{\epsilon=0} - \left(d\sigma^A\right)_{\epsilon=0} \right] + \int_m \left[d\sigma^V + \int_1 d\sigma^A \right]_{\epsilon=0}$$



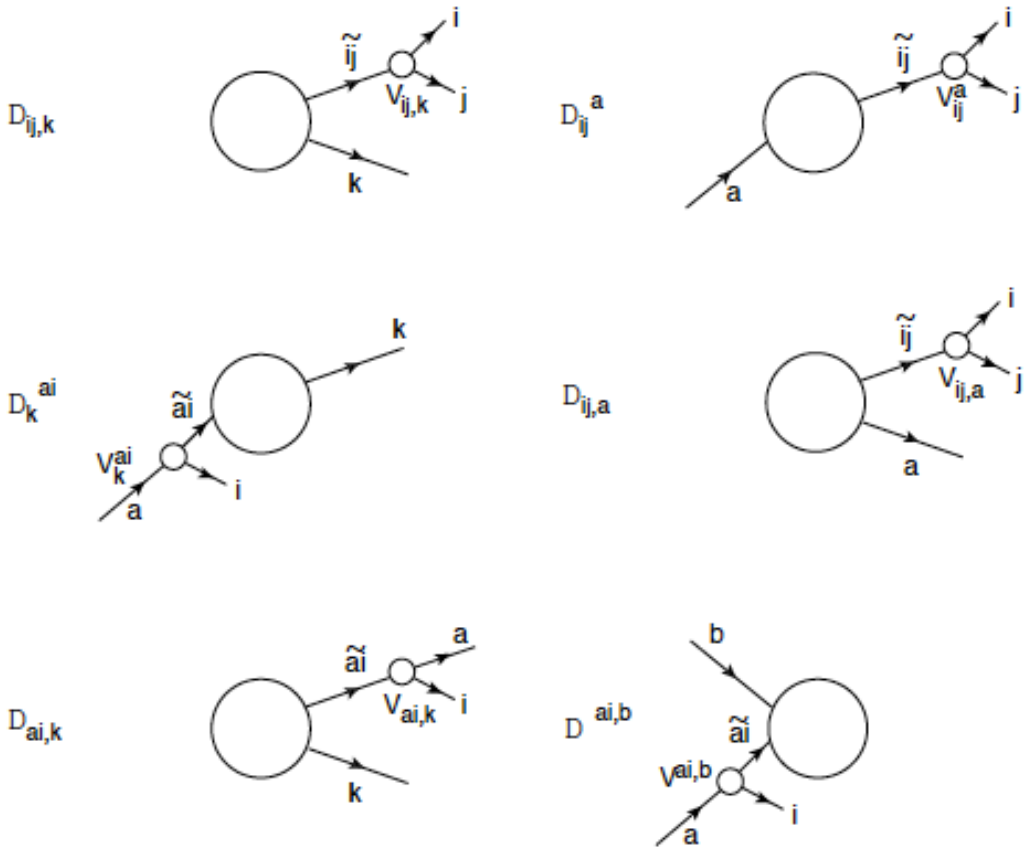
$$\tilde{p}_k^\mu = \frac{1}{1-y_{ij,k}}\, p_k^\mu \quad , \quad \tilde{p}_{ij}^\mu = p_i^\mu + p_j^\mu - \frac{y_{ij,k}}{1-y_{ij,k}}\, p_k^\mu$$

$$d\phi(p_i,p_j,p_k;Q) = \frac{d^d p_i}{(2\pi)^{d-1}}\,\delta_+(p_i^2)\;\frac{d^d p_j}{(2\pi)^{d-1}}\,\delta_+(p_j^2)\;\frac{d^d p_k}{(2\pi)^{d-1}}\,\delta_+(p_k^2)\;\;(2\pi)^d\,\delta^{(d)}(Q-p_i-p_j-p_k)$$

$$d\phi(p_i,p_j,p_k;Q) = d\phi(\tilde{p}_{ij},\tilde{p}_k;Q)\;\;[dp_i(\tilde{p}_{ij},\tilde{p}_k)]$$

Real corrections

Dipoles in real life



Dipoles in real life: the formulae

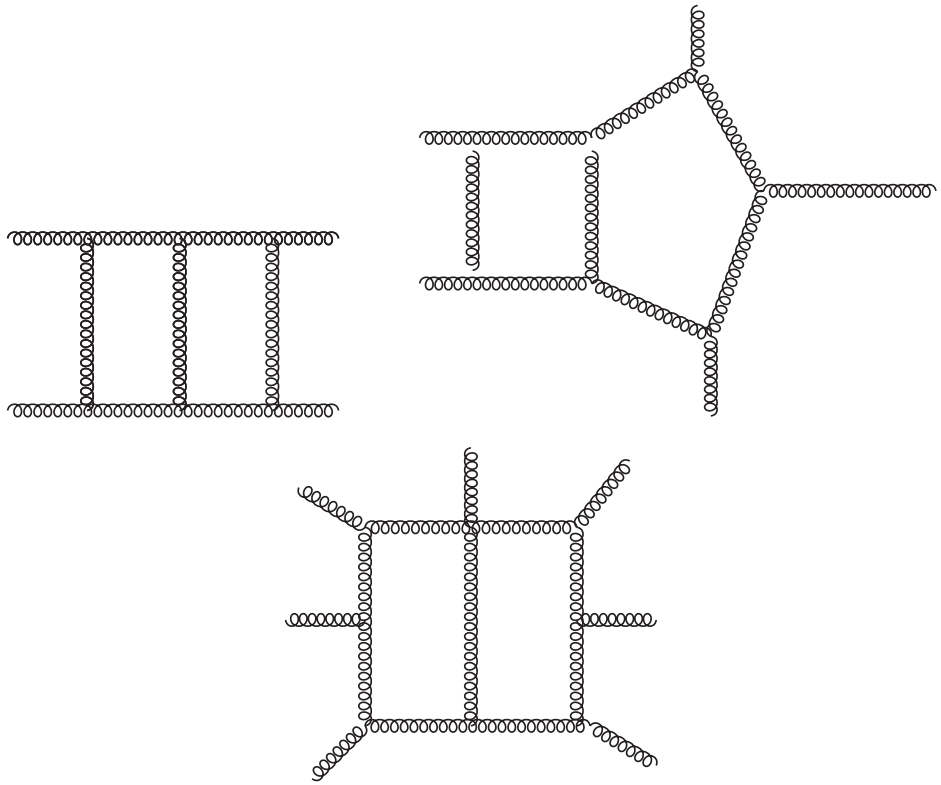
$$d\sigma^A = \mathcal{N}_{in} \sum_{\{m+1\}} d\phi_{m+1}(p_1,...,p_{m+1};Q)\ \frac{1}{S_{\{m+1\}}} \cdot \sum_{\substack{\text{pairs}\\ \text{i,j}}} \sum_{k\neq i,j} \mathcal{D}_{ij,k}(p_1,...,p_{m+1})\ F_J^{(m)}(p_1,..\tilde{p}_{ij},\tilde{p}_k,..,p_{m+1})$$

$$\mathcal{D}_{ij,k}\ (p_1,...,p_{m+1}) = -\frac{1}{2p_i\cdot p_j} \cdot \ {}_m\!<1,..,\tilde{ij},..,\tilde{k},..,m+1|\ \frac{\boldsymbol{T}_k\cdot\boldsymbol{T}_{ij}}{\boldsymbol{T}_{ij}^2}\ \boldsymbol{V}_{ij,k}\ |1,..,\tilde{ij},..,\tilde{k},..,m+1>_m$$

$$d\sigma^R - d\sigma^A = \mathcal{N}_{in} \sum_{\{m+1\}} d\phi_{m+1}(p_1,...,p_{m+1};Q)\ \frac{1}{S_{\{m+1\}}} \cdot \Big\{|\mathcal{M}_{m+1}(p_1,...,p_{m+1})|^2\ F_J^{(m+1)}(p_1,...,p_{m+1}) - \sum_{\substack{\text{pairs}\\ \text{i,j}}} \sum_{k\neq i,j} \mathcal{D}_{ij,k}(p_1,...,p_{m+1})\ F_J^{(m)}(p_1,..\tilde{p}_{ij},\tilde{p}_k,..,p_{m+1})\Big\}$$

$$\int_{m+1} d\sigma^A = -\int_m \mathcal{N}_{in} \sum_{\{m\}} d\phi_m(p_1,..,p_m;Q)\ \frac{1}{S_{\{m\}}}\ F_J^{(m)}(p_1,...,p_m) \cdot \sum_i \sum_{k\neq i} |\mathcal{M}_m^{i,k}(p_1,...,p_m)|^2\ \frac{\alpha_S}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\left(\frac{4\pi\mu^2}{2p_i\cdot p_k}\right)^{\epsilon}\ \frac{1}{\boldsymbol{T}_i^2}\ \mathcal{V}_i(\epsilon)\ ,$$

Perturbative QCD at NNLO



Repeat the one-loop "success story" ?

What do we need for an NNLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NNLO} \quad \to \quad & \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left| M_m^{(1)} \right|^2 \right) J_m(\Phi) \qquad VV \\
+ \quad & \int_{m+1} d\Phi_{m+1} \left(2Re \left(M_{m+1}^{(0)*} M_{m+1}^{(1)} \right) \right) J_{m+1}(\Phi) \qquad RV \\
+ \quad & \int_{m+2} d\Phi_{m+2} \left| M_{m+2}^{(0)} \right|^2 J_{m+2}(\Phi) \qquad RR
\end{aligned}$$

Reduction at the integrand level

Over the last few years very important activity to extend unitarity and integrand level reduction ideas beyond one loop

J. Gluza, K. Kajda and D. A. Kosower, "Towards a Basis for Planar Two-Loop Integrals," Phys. Rev. D **83** (2011) 045012 [arXiv:1009.0472 [hep-th]].

D. A. Kosower and K. J. Larsen, "Maximal Unitarity at Two Loops," Phys. Rev. D **85** (2012) 045017 [arXiv:1108.1180 [hep-th]].

P. Mastrolia and G. Ossola, "On the Integrand-Reduction Method for Two-Loop Scattering Amplitudes," JHEP **1111** (2011) 014 [arXiv:1107.6041 [hep-ph]].

S. Badger, H. Frellesvig and Y. Zhang, "Hepta-Cuts of Two-Loop Scattering Amplitudes," JHEP **1204** (2012) 055 [arXiv:1202.2019 [hep-ph]].

Y. Zhang, "Integrand-Level Reduction of Loop Amplitudes by Computational Algebraic Geometry Methods," JHEP **1209** (2012) 042 [arXiv:1205.5707 [hep-ph]].

P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, "Integrand-Reduction for Two-Loop Scattering Amplitudes through Multivariate Polynomial Division," arXiv:1209.4319 [hep-ph].

P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, "Multiloop Integrand Reduction for Dimensionally Regulated Amplitudes," arXiv:1307.5832 [hep-ph].

OPP at two loops

- Write the "OPP-type" equation at two loops

$$\frac{N\left(l_1, l_2; \{p_i\}\right)}{D_1 D_2 \dots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \dots i_m}\left(l_1, l_2; \{p_i\}\right)}{D_{i_1} D_{i_2} \dots D_{i_m}}$$

$S_{m;n}$ stands for all subsets of m indices out of the n ones

Multivariate Division and Groebner Basis

D. Cox, J. Little, D. O'Shea *Ideals, Varieties and Algorithms*

- Given any set of polynomials π_i, the ideal I, $f = \sum_i \pi_i h_i$, we can define a unique Groebner basis up to ordering $< g_1, \ldots, g_s >$

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multivariate polynomial division

$$f = h_1 g_1 + \ldots + h_n g_n + r$$

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Strategy:
• Start with a set of denominators, pick up a $4d$ parametrisation, define an ideal, $I =< D_1, \ldots, D_n >$, even D_i^2

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Strategy:
• Start with a set of denominators, pick up a $4d$ parametrisation, define an ideal, $I =< D_1, \ldots, D_n >$, even D_i^2
• Find the GB of I, $G =< g_1, \ldots, g_s >$

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$$f = h_1 g_1 + \ldots + h_n g_n + r$$

Strategy:

• Start with a set of denominators, pick up a $4d$ parametrisation, define an ideal, $I =< D_1, \ldots, D_n >$, even D_i^2

• Find the GB of I, $G =< g_1, \ldots, g_s >$

• Perform the division of an arbitrary polynomial N

$$N = h_1 g_1 + \ldots + h_n g_s + v$$

• Express back g_i in terms of D_i

$$N = \tilde{h}_1 D_1 + \ldots + \tilde{h}_n D_n + v$$

Physicist's point of view

The simplest case: $n \to n-1$ reduction

The general strategy consists in finding polynomials $\Pi_j \equiv \Pi_j(l_1, l_2)$

$$\sum_{j=1}^{n_1} \Pi_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} \Pi_j D(l_1 + l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} \Pi_j D(l_2 + p_j) = 1 \ .$$

Is this possible at all ?

R. H. P. Kleiss, I. Malamos, C. G. Papadopoulos and R. Verheyen, JHEP **1212** (2012) 038 [arXiv:1206.4180 [hep-ph]].

Physicist's point of view

$$\sum_{j=1}^{n_1} \Pi_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} \Pi_j D(l_1 + l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} \Pi_j D(l_2 + p_j) = 1 \ .$$

Hilbert's Nullstellensatz theorem

Hilbert's Nullstellensatz (German for "theorem of zeros," or more literally, "zero-locus-theorem" see Satz) is a theorem which establishes a fundamental relationship between geometry and algebra. This relationship is the basis of algebraic geometry, an important branch of mathematics. It relates algebraic sets to ideals in polynomial rings over algebraically closed fields. This relationship was discovered by David Hilbert who proved Nullstellensatz and several other important related theorems named after him (like Hilbert's basis theorem).

$$1 = g_1 f_1 + \cdots + g_s f_s \quad g_i, f_i \in k[x_1, \ldots, x_n]$$

Janos Kollar, J. Amer. Math. Soc., Vol. 1, No. 4. (Oct., 1988), pp 963-975

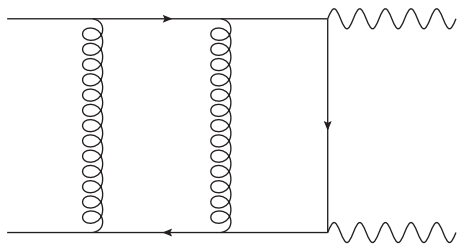
$$\deg g_i f_i \leq \max\{3, d\}^n \quad d = \max \deg f_i \quad 3^8 = 6561$$

M. Sombra, Adv. in Appl. Math. 22 (1999), 271-295

$$\deg g_i f_i \leq 2^{n+1} \quad 2^9 = 512$$

OPP AT TWO LOOPS

As an example reduce a two-loop 7-propagator graph contributing to $q\bar{q} \to \gamma^*\gamma^*$



with $l_1^\mu = \sum_{i=1}^{3} z_i v_i^\mu + z_4 \omega^\mu$, with $z_i = l_1 \cdot p_i, i = 1 \ldots, 3$ (l_2, with w_i replacing z_i).

OPP AT TWO LOOPS

Basis

$$v_i^\mu \sim p_i^\mu (i = 1 \dots 3) \quad \omega^\mu = \frac{\varepsilon^{\mu p_1 p_2 p_3}}{\sqrt{\Delta}}$$

with

$$\Delta = \delta^{p_1 p_2 p_3}_{p_1 p_2 p_3} = \varepsilon^{p_1 p_2 p_3} \varepsilon_{p_1 p_2 p_3} = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & p_1 \cdot p_3 \\ p_2 \cdot p_1 & p_2 \cdot p_2 & p_2 \cdot p_3 \\ p_3 \cdot p_1 & p_3 \cdot p_2 & p_3 \cdot p_3 \end{vmatrix}$$

OPP AT TWO LOOPS

$\{0,0,0,0\}$: $4\,\text{chi}\,(\text{chi}+1)^2\,s^3$

$\{0,1,0,0\}$: $16\,\text{chi}\,(\text{chi}+1)\,s^2$

$\{0,2,0,0\}$: $16\,(\text{chi}+1)\,s$

$\{0,3,0,0\}$: $\dfrac{32}{\text{chi}+1}$

$\{1,0,0,0\}$: $-16\,\text{chi}\,(\text{chi}+1)\,s^2$

$\{1,1,0,0\}$: $-16\,(4\,\text{chi}\,s+s)$

$\{1,2,0,0\}$: $-64-\dfrac{32}{\text{chi}+1}$

$\{1,3,0,0\}$: $-\dfrac{64\,(2\,\text{chi}+1)}{\text{chi}\,(\text{chi}+1)^2\,s}$

$\{2,0,0,0\}$: $16\,\text{chi}\,s$

$\{2,1,0,0\}$: $64-\dfrac{32}{\text{chi}+1}$

$\{2,2,0,0\}$: $\dfrac{64\,(\text{chi}+2)}{(\text{chi}+1)^2\,s}$

$\{2,3,0,0\}$: $\dfrac{128}{\text{chi}\,(\text{chi}+1)^2\,s^2}$

k, q loop momenta

$$\frac{N}{D_1 D_2 \dots D_7} = \frac{\Delta}{D_1 D_2 \dots D_7} + \dots$$

$$\{n_1, n_2, n_3, n_4\} \to (k\cdot p_3)^{n_1}(q\cdot p_2)^{n_2}(k\cdot\omega)^{n_3}(q\cdot\omega)^{n_4}$$

$\text{chi} \to \chi = \frac{t}{s}$

OPP AT TWO LOOPS

1	:	$-8\,(1+\text{chi})^2\,s^2$
qd2	:	$-24\,(1+\text{chi})\,s$
qd2^2	:	$-\frac{16\,(2+\text{chi})}{\text{chi}}$
qd2^3	:	$-\frac{32}{\text{chi}\,(1+\text{chi})\,s}$
kd3	:	$16\,(1+\text{chi})\,s$
kd3*qd2	:	$16\left(3+\frac{1}{\text{chi}}\right)$
kd3*qd2^2	:	$\frac{32\,(3+\text{chi})}{\text{chi}\,(1+\text{chi})\,s}$
kd2*qd2	:	$\frac{8\,(1+\text{chi})}{\text{chi}}$
kd2*qd2^2	:	$\frac{16}{\text{chi}\,s}$

$$\frac{N_2}{D_1\widehat{D_2}\dots D_7} = \frac{\Delta_2}{D_1\widehat{D_2}\dots D_7} + \dots$$

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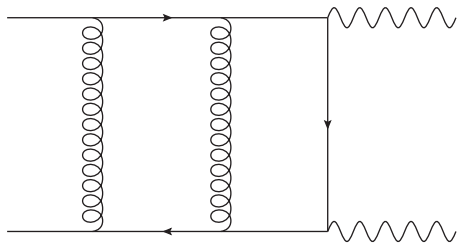

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$$\sum \frac{\Pi\left(\{z_i\}, \{w_j\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}} \to \text{spurious} \oplus \text{nonscalar integrals}$$

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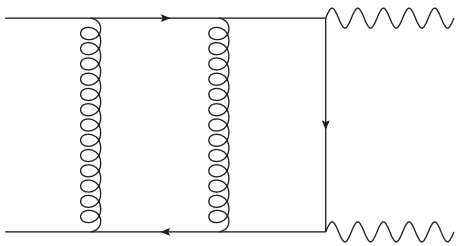

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- IBPI to Master Integrals

OPP at two loops - Rational terms

S. Badger, talk in Amplitudes 2013

- Rational terms

$$I_1 \to I_1 + I_1^{(2\varepsilon)},\ I_2 \to I_2 + I_2^{(2\varepsilon)},\ I_{1,2} \cdot I_{1,2}^{(2\varepsilon)} = 0$$

$$\left(I_1^{(2\varepsilon)}\right)^2 = \mu_{11},\quad \left(I_2^{(2\varepsilon)}\right)^2 = \mu_{22},\quad I_1^{(2\varepsilon)} \cdot I_2^{(2\varepsilon)} = \mu_{12}$$

$$\left\{I_1^{(4)}, I_2^{(4)}\right\} \to \left\{I_1^{(4)}, I_2^{(4)}, \mu_{11}, \mu_{22}, \mu_{12}\right\}$$

Welcome: $I = \sqrt{I}$ prime ideal

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Welcome: $I = \sqrt{I}$ prime ideal

- R_2 terms: hopefully *à la* 1 loop

Master Integrals: The current approach

- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \dots D_N$, allows to express all scalar products $D_i = (\{k, l\} + p_i)^2 - M_i^2$
- $F[a_1, \dots, a_N]$

$$\int d^d k d^d l \; \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1} \dots D_N^{a_N}} \right) = 0$$

- IBP Laporta: `FIRE`, `AIR`, `Reduze` reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations

Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

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Differential Equations Approach

- Library of MI à la one-loop
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K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831

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A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

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Differential Equations Approach

- Let us consider a simple example

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_0 D_1 \dots D_{n-1}}$$

with $D_i = (k + p_0 + \dots + p_i)^2$ and take for convenience $p_0 = 0$. It can be considered as a function of the external momenta p_i.

- It belongs to the topology defined by

$$G_{a_1 \dots a_n} = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_0^{a_1} D_1^{a_2} \dots D_{n-1}^{a_n}}$$

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Differential Equations Approach

The integral is a function of external momenta, so one can set-up differential equations by differentiating with respect to these

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \ldots, a_n] \to \sum G[a_1', \ldots, a_n']$$

$$p_1^\mu \frac{\partial}{\partial p_1^\mu} (k + p_1)^2 = 2 (k + p_1) \cdot p_1 = (k + p_1)^2 + p_1^2 - k^2$$

- Find the proper parametrization; Bring the system of equations in a form suitable to express the MI in terms of GPs
- Boundary conditions: expansion by regions

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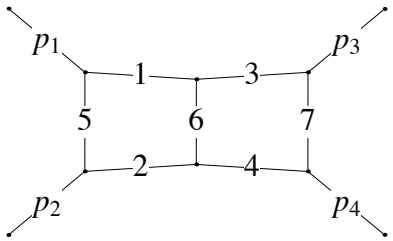
Differential Equations Approach

J. M. Henn, K. Melnikov and V. A. Smirnov, arXiv:1402.7078 [hep-ph].



$$S = (q_1+q_2)^2 = (q_3+q_4)^2, \quad T = (q_1-q_3)^2 = (q_2-q_4)^2, \quad U = (q_1-q_4)^2 = (q_2-q_3)^2;$$

$$\frac{S}{M_3^2} = (1+x)(1+xy), \quad \frac{T}{M_3^2} = -xz, \quad \frac{M_4^2}{M_3^2} = x^2 y.$$

$$d\,\vec{f}(x,y,z;\epsilon) = \epsilon\, d\,\tilde{A}(x,y,z)\,\vec{f}(x,y,z;\epsilon)$$

$$\tilde{A} = \sum_{i=1}^{15} \tilde{A}_{\alpha_i}\ \log(\alpha_i)$$

$$\alpha = \{x, y, z, 1+x, 1-y, 1-z, 1+xy, z-y, 1+y(1+x)-z, xy+z,$$
$$1+x(1+y-z), 1+xz, 1+y-z, z+x(z-y)+xyz, z-y+yz+xyz\}.$$

The Simplified Differential Equations Approach

C. G. Papadopoulos, arXiv:1401.6057 [hep-ph].

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

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$$G_{11\ldots1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)\,(k + x\,p_1)^2\,(k + p_1 + p_2)^2 \ldots (k + p_1 + p_2 + \ldots + p_n)^2}$$

- Now the integral becomes a function of x, which allows to define a differential equation with respect to x, schematically given by

$$\frac{\partial}{\partial x} G_{11\ldots1}(x) = -\frac{1}{x} G_{11\ldots1}(x) + x p_1^2 G_{12\ldots1} + \frac{1}{x} G_{02\ldots1}$$

- and using IBPI we obtain

$$\begin{aligned} m_1 x G_{121} + \tfrac{1}{x} G_{021} \quad &= \left(\tfrac{1}{x-1} + \tfrac{1}{x - m_3/m_1}\right)\left(\tfrac{d-4}{2}\right) G_{111} \\ &+ \tfrac{d-3}{m_1 - m_3}\left(\tfrac{1}{x-1} - \tfrac{1}{x - m_3/m_1}\right)\left(\tfrac{G_{101} - G_{110}}{x}\right) \end{aligned}$$

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The Simplified Differential Equations Approach

- The integrating factor M is given by

$$M = x\,(1-x)^{\frac{4-d}{2}}\,(-m_3+m_1 x)^{\frac{4-d}{2}}$$

- and the DE takes the form, $d = 4-2\varepsilon$,

$$\frac{\partial}{\partial x} M G_{111} = c_\Gamma \frac{1}{\varepsilon}\,(1-x)^{-1+\varepsilon}\,(-m_3+m_1 x)^{-1+\varepsilon}\left(\left(-m_1 x^2\right)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)$$

  - Integrating factors $\epsilon = 0$ do not have branch points
  - DE can be straightforwardly integrated order by order $\rightarrow$ GPs.

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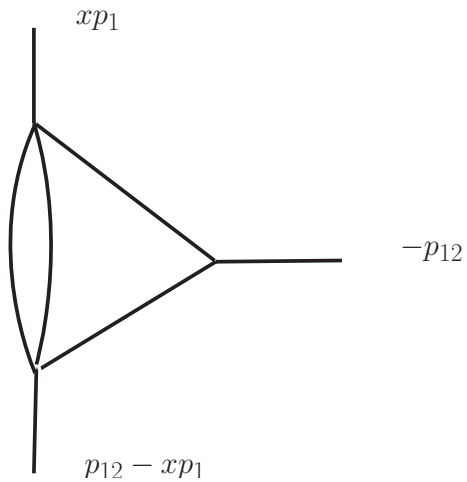
The Simplified Differential Equations Approach

$$G_{111} = \frac{c_\Gamma}{(m_1 - m_3)x}\mathcal{I}$$

$$
\begin{aligned}
\mathcal{I} = {} & \frac{-(-m_1)^{-\varepsilon} + (-m_3)^{-\varepsilon} + \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)x_1}{\varepsilon^2} \\
& + \frac{\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)x_1 G\left(\frac{m_3}{m_1},1\right) - \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)\left(G\left(\frac{m_3}{m_1},1\right) - G\left(\frac{m_3}{m_1},x\right)\right)}{\varepsilon} \\
& + \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)\left(G\left(\frac{m_3}{m_1},1\right)G\left(\frac{m_3}{m_1},x\right) - G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) - G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},x\right)\right) + x_1\left(-2G(0,1,x)(-m_1)^{-\varepsilon}\right. \\
& \quad + 2G\left(0,\frac{m_3}{m_1},x\right)(-m_1)^{-\varepsilon} + 2G\left(\frac{m_3}{m_1},1,x\right)(-m_1)^{-\varepsilon} + G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)(-m_1)^{-\varepsilon} - G\left(\frac{m_3}{m_1},x\right)\log(1-x)(-m_1)^{-\varepsilon} \\
& \quad - 2G\left(\frac{m_3}{m_1},x\right)\log(x)(-m_1)^{-\varepsilon} + 2\log(1-x)\log(x)(-m_1)^{-\varepsilon} - 2(-m_3)^{-\varepsilon}G\left(\frac{m_3}{m_1},1,x\right) - (-m_3)^{-\varepsilon}G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) \\
& \quad \left. - \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)G\left(\frac{m_3}{m_1},1\right)\left(G\left(\frac{m_3}{m_1},x\right) - \log(1-x)\right) + (-m_3)^{-\varepsilon}G\left(\frac{m_3}{m_1},x\right)\log(1-x)\right) \\
& + \varepsilon\left(\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)\left(G\left(\frac{m_3}{m_1},x\right)G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) - G\left(\frac{m_3}{m_1},1\right)G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},x\right) - G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)\right.\right. \\
& \quad \left.\left. + G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},\frac{m_3}{m_1},x\right)\right) + \frac{1}{2}x_1\left(\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)G\left(\frac{m_3}{m_1},1\right)\left(\log^2(1-x) + 2G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},x\right)\right)\right. \\
& \quad + G\left(\frac{m_3}{m_1},x\right)\left(4\log^2(x) - 2\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) + 2\left((-m_3)^{-\varepsilon} - (-m_1)^{-\varepsilon}\right)G\left(\frac{m_3}{m_1},1\right)\log(1-x)\right) \\
& \quad \left.\left. + 2\left(G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)(-m_1)^{-\varepsilon} + G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)\log(1-x)(-m_1)^{-\varepsilon} - 2\log(1-x)\log^2(x) - 4G(0,0,1,x)\right.\right. \right. \\
& + 4G\left(0,0,\frac{m_3}{m_1},x\right) - 2G(0,1,1,x) + 4G\left(0,\frac{m_3}{m_1},1,x\right) - 2G\left(0,\frac{m_3}{m_1},\frac{m_3}{m_1},x\right) + 2G\left(\frac{m_3}{m_1},0,1,x\right) - 2G\left(\frac{m_3}{m_1},0,\frac{m_3}{m_1},x\right) \\
& - (-m_3)^{-\varepsilon}G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) - (-m_3)^{-\varepsilon}G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)\log(1-x) + \log^2(1-x)\log(x) \\
& \left.\left.\left. + 4G(0,1,x)\log(x) - 4G\left(0,\frac{m_3}{m_1},x\right)\log(x) - 4G\left(\frac{m_3}{m_1},1,x\right)\log(x) + 2G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},x\right)\log(x)\right)\right)\right)
\end{aligned}
$$

The Simplified Differential Equations Approach

The two-loop 3-mass triangle



The Simplified Differential Equations Approach

We are interested in $G_{0101011}$. The DE involves also the MI $G_{0201011}$, so we have a system of two coupled DE, as follows:

$$\frac{\partial}{\partial x}\left(M_{0101011}\,G_{0101011}\right) = \frac{A_3(2-3\varepsilon)(1-x)^{-2\varepsilon}x^{\varepsilon-1}(m_1x-m_3)^{-2\varepsilon}}{2\varepsilon(2\varepsilon-1)} + \frac{m_1\varepsilon(1-x)^{-2\varepsilon}(m_1x-m_3)^{-2\varepsilon}}{2\varepsilon-1}g(x)$$

$$\frac{\partial}{\partial x}\left(M_{0201011}\,G_{0201011}\right) = \frac{A_3(3\varepsilon-2)(3\varepsilon-1)(-m_1)^{-2\varepsilon}(1-x)^{2\varepsilon-1}x^{-3\varepsilon}(m_1x-m_3)^{2\varepsilon-1}}{2\varepsilon^2} + (2\varepsilon-1)(3\varepsilon-1)(1-x)^{2\varepsilon-1}(m_1x-m_3)^{2\varepsilon-1}f(x)$$

where $f(x) \equiv M_{0101011}\,G_{0101011}$ and $g(x) \equiv M_{0201011}\,G_{0201011}$, $M_{0201011} = (1-x)^{2\varepsilon}x^{\varepsilon+1}(m_1x-m_3)^{2\varepsilon}$ and $M_{0101011} = x^{\varepsilon}$

The Simplified Differential Equations Approach

The singularity at $x = 0$ is proportional to $x^{-1-\varepsilon}$ and can easily be integrated by the following decomposition

$$\int_0^x dt\ t^{-1-\varepsilon} F(t) = F(0) \int_0^x dt\ t^{-1-\varepsilon} + \int_0^x dt\ \frac{F(t)-F(0)}{t} t^{-\varepsilon}$$
$$= F(0)\, \frac{x^{-\varepsilon}}{(-\varepsilon)} + \int_0^x dt\ \frac{F(t)-F(0)}{t} \left(1 - \varepsilon \log(t) + \tfrac{1}{2}\varepsilon^2 \log^2(t) + ...\right)$$

No boundary term $x = 0$

The Simplified Differential Equations Approach

General setup

m: number of denominators

$$\partial_x G_{m+1} = H\left(\{s_{ij}\}, \epsilon; x\right) G_{m+1} + \sum_{m' \geq m_0}^{m} R\left(\{s_{ij}\}, \epsilon; x\right) G_{m'},$$

$m_0 = 3$ in the case of two loops

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$$\partial_x (MG_{m+1}) = M \sum_{m' \geq m_0}^{m} R\left(\{s_{ij}\}, \epsilon; x\right) G_{m'}.$$

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$$M \sum_{m' \geq m_0}^{m} R\left(\{s_{ij}\}, \epsilon; x\right) G_{m'} =: \sum_i x^{-1+\beta_i \epsilon} \tilde{I}^{(i)}_{sin}(\{s_{ij}\}, \epsilon) + \tilde{I}_{reg}(\{s_{ij}\}, \epsilon; x).$$

The Simplified Differential Equations Approach

$$MG_{m+1} = C(\{s_{ij}\},\epsilon) + \sum_i \frac{x^{\beta_i \epsilon}}{\beta_i \epsilon} \tilde{I}^{(i)}_{sin}(\{s_{ij}\},\epsilon) + \int_0^x dx' \tilde{I}_{reg}(\{s_{ij}\},\epsilon;x'),$$

- Integrating factors M rational functions of x in the limit $\epsilon \to 0$
- *Sufficient condition* DE solvable in terms of GPs.
- All re-summed parts at $x \to 0 \to$ fully determined by the one-scale MI involved in the system
- Two-point integrals, two three-point integrals and double one-loop integrals $\to$ homogenous differential equations.
- $C(\{s_{ij}\},\epsilon) = 0$: *no independent calculation of boundary terms needed*.

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The Simplified Differential Equations Approach

When the DE are coupled

$$\partial_x \vec{G}_{m+1} = \mathbf{H}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m+1} + \sum_{m' \geq m_0}^{m} \mathbf{R}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m'},$$

- $\mathbf{M}_D$: $\partial_x \mathbf{M}_D = -\mathbf{M}_D \mathbf{H}_D$, where $\mathbf{H}_D$ is the diagonal part of $\mathbf{H}$.
- $\tilde{\mathbf{H}} =: \mathbf{M}_D \left(\mathbf{H} - \mathbf{H}_D\right) \mathbf{M}_D^{-1}$ of the reduced system of DE is then a *strictly triangular matrix* at order ϵ^0 and the system becomes effectively uncoupled.
- Problem: In very few specific cases, $x^{-2+\beta_i \epsilon}$ appears in the matrix $\tilde{\mathbf{H}}$,
- Solution: $x \to 1/x$ back to $x^{-1+\beta_i \epsilon}$ in the *inhomogeneous part* of the DE.

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Two-loop, four-point, two off-shell legs

$$p(q_1)p'(q_2) \to V_1(-q_3)V_2(-q_4), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = M_4^2.$$

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$$q_1 = xp_1, \quad q_2 = xp_2, \quad q_3 = p_{123} - xp_{12}, \quad q_4 = -p_{123}, \quad p_i^2 = 0,$$
$$s_{12} := p_{12}^2, \quad s_{23} := p_{23}^2, \quad q := p_{123}^2,$$

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$$S = (q_1 + q_2)^2 \quad T = (q_1 + q_3)^2$$

$$S = s_{12}x^2, \quad T = q - (s_{12} + s_{23})x, \quad M_3^2 = (1-x)(q - s_{12}x), \quad M_4^2 = q.$$

$$U = (q_1 + q_4)^2 : S + T + U = M_3^2 + M_4^2.$$

Two-loop, four-point, two off-shell legs

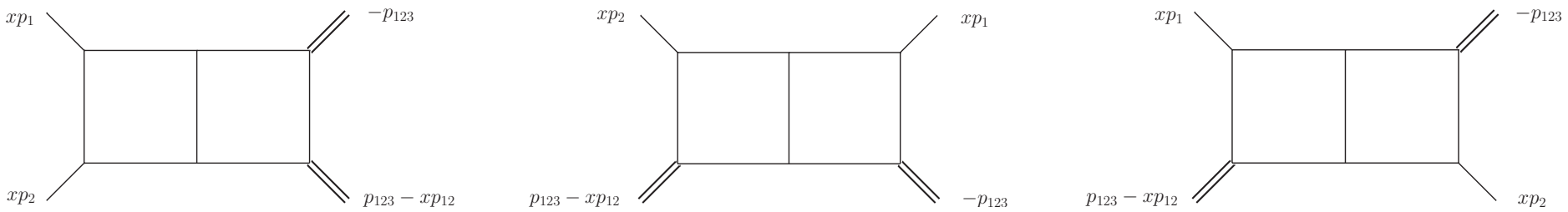

Figure : The parametrization of external momenta for the three planar double boxes of the families P_{12} (left), P_{13} (middle) and P_{23} (right) contributing to pair production at the LHC. All external momenta are incoming.

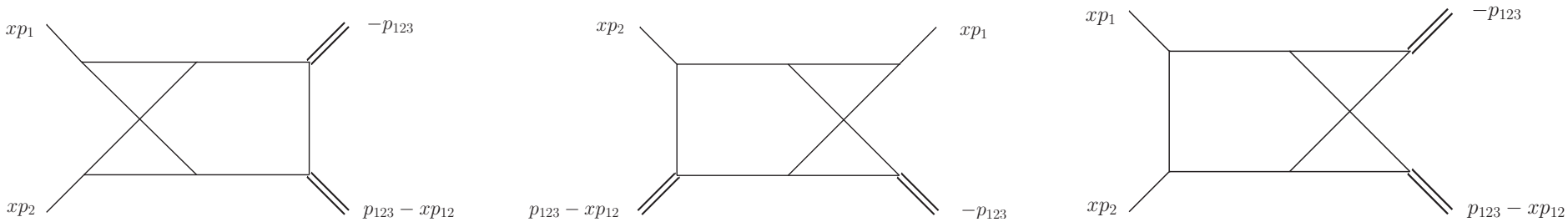

Figure : The parametrization of external momenta for the three non-planar double boxes of the families N_{12} (left), N_{13} (middle) and N_{34} (right) contributing to pair production at the LHC. All external momenta are incoming.

Two-loop, four-point, two off-shell legs

Triangle rule:

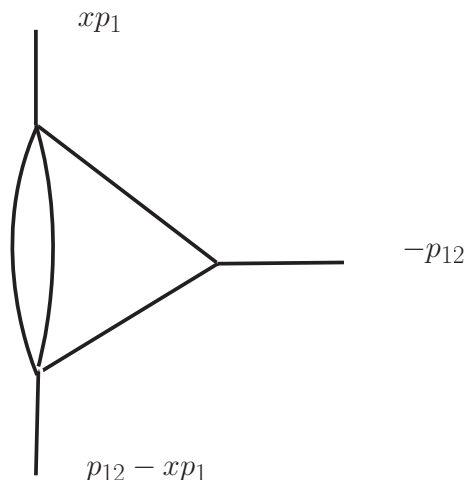

Figure : Required parametrization for off mass-shell triangles after possible pinching of internal line(s).

Planar topologies

$$G^{P_{12}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon} \int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-xp_{12})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}},$$

$$G^{P_{13}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon} \int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-p_{12})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}},$$

$$G^{P_{23}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon} \int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+p_{123}-xp_2)^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-p_1)^{2a_6}(k_2+xp_2-p_{123})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}},$$

Planar topologies

P_{12} : {010000011, 001010001, 001000011, 100000011, 101010010, 101010100, 101000110, 010010101, 101000011, 101000012, 100000111, 100000112, 001010011, 001010012, 010000111, 010010011, 101010110, 111000011, 101000111, 101010011, 011010011, 011010012, 110000111, 110000112, 010010111, 010010112, 111010011, 111000111, 111010111, 111m10111, 11101m111},

P_{13} : {000110001, 001000011, 001010001, 001101010, 001110010, 010000011, 010101010, 010110010, 001001011, 001010011, 001010012, 001011011, 001101001, 001101011, 001110001, 001110002, 001110011, 001111001, 001111011, 001211001, 010010011, 010110001, 010110011, 011010011, 011010021, 011110001, 011110011, 011111011, m11111011},

P_{23} : {001010001, 001010011, 010000011, 010000101, 010010011, 010010101, 010010111, 011000011, 011010001, 011010010, 011010011, 011010012, 011010100, 011010101, 011010111, 011020011, 012010011, 021010011, 100000011, 101000011, 101010010, 101010011, 101010100, 110000111, 111000011, 111010011, 111010111, 111m10111}.

Non-planar topologies

$$G^{N_{12}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-p_{123})^{2a_7}(k_1+k_2+xp_2)^{2a_8}(k_1+k_2)^{2a_9}},$$

$$G^{N_{13}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-xp_{12})^{2a_6}(k_2-p_{123})^{2a_7}(k_1+k_2+xp_1)^{2a_8}(k_1+k_2)^{2a_9}},$$

$$G^{N_{34}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}} \times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-p_{123})^{2a_7}(k_1+k_2+xp_{12}-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}.$$

Non-planar topologies

N_{12} : {100001010, 000110010, 000110001, 000101010, 000101001, 101010010, 100110010, 100101020, 100101010, 100101001, 001110010, 001110002, 001110001, 001101001, 101110020, 101110010, 101101002, 101101001, 100111020, 100111010, 100102011, 100101011, 001120011, 001111002, 001111001, 001110011, 000111011, 101011011, 100111011, 1m0111011, 001111011, 0m1111011, 101111011, 1m1111011, 1m1111m11},

N_{13} : {010000110, 000110010, 001000101, 001000110, 001010001, 010110100, 001110100, 001010102, 001110002, 000110110, 001010101, 001010110, 001100110, 001110001, 001110010, 010100110, 010110101, 002010111, 001120011, 001210110, 011010102, 001110120, 001010111, 001110210, 001110011, 001110101, 001110110, 002110110, 011000111, 011010101, 011100110, 011110001, 011110110, m11010111, 010110111, m01110111, 0m1110111, 00111m111, 001110111, 011010111, 011110101, 011110111, m11110111},

N_{34} : {001001010, 001010010, 010010010, 100000110, 100010010, 000010111, 010010110, 001010102, 001010101, 010010101, 001020011, 010000111, 001010011, 010010011, 101010020, 101010010, 101010100, 101000011, 110010120, 110010110, 010010112, 010010121, 010010111, 010020111, 020010111, 011010102, 001010111, 011010101, 110000211, 011020011, 110000111, 011010011, 111000101, 111010010, 101010101, 101010011, 111010110, 111010101, 101010111, 11m010111, 110m10111, 11001m111, 110010111, m11010111, 011m10111, 01101m111, 011010111, 111000111, 111010011, 111010111, 111m10111}.

GP-indices

$$I(P_{12}) = \left\{0, 1, \frac{q}{s_{12}}, \frac{s_{12}}{q}, \frac{q}{q-s_{23}}, 1-\frac{s_{23}}{q}, 1+\frac{s_{23}}{s_{12}}, \frac{s_{12}}{s_{12}+s_{23}}\right\},$$

$$I(P_{13}) = \left\{0, 1, \frac{q}{s_{12}}, \frac{s_{12}+s_{23}}{s_{12}}, \frac{q}{q-s_{23}}, \xi_-, \xi_+, \frac{q(q-s_{23})}{q^2-(q+s_{12})s_{23}}\right\},$$

$$I(P_{23}) = \left\{0, 1, \frac{q}{s_{12}}, 1+\frac{s_{23}}{s_{12}}, \frac{q}{q-s_{23}}, \frac{q}{s_{12}+s_{23}}, \frac{q-s_{23}}{s_{12}}\right\},$$

$$\xi_\pm = \frac{qs_{12} \pm \sqrt{qs_{12}s_{23}(-q+s_{12}+s_{23})}}{qs_{12}-s_{12}s_{23}}.$$

$$I(N_{12}) = I(P_{23}),$$

$$I(N_{34}) = I(P_{12}) \cup I(P_{23}) \cup \left\{\frac{s_{12}}{q-s_{23}}, \frac{s_{12}+s_{23}}{q}, \frac{q^2-qs_{23}-s_{12}s_{23}}{s_{12}(q-s_{23})}, \frac{s_{12}^2+qs_{23}+s_{12}s_{23}}{s_{12}(s_{12}+s_{23})}\right\},$$

$$I(N_{13}) = I(P_{23}) \cup \left\{\xi_-, \xi_+, 1+\frac{q}{s_{12}}+\frac{q}{-q+s_{23}}\right\}.$$

Example

$$G^{P_{13}}_{011111011}(x,s,\epsilon) = \frac{A_3(\epsilon)}{x^2 s_{12}(-q+x(q-s_{23}))^2}\left\{\frac{-1}{2\epsilon^4}+\frac{1}{\epsilon^3}\left(-GP\left(\frac{q}{s_{12}};x\right)+2\,GP\left(\frac{q}{q-s_{23}};x\right)\right.\right.$$
$$+2\;GP(0;x)-GP(1;x)+\log(-s_{12})+\frac{9}{4}\Big)+\frac{1}{4\epsilon^2}\left(18\;GP\left(\frac{q}{s_{12}};x\right)-36\;GP\left(\frac{q}{q-s_{23}};x\right)\right.$$
$$\left.-8\;GP\left(0,\frac{q}{s_{12}};x\right)+16\;GP\left(0,\frac{q}{q-s_{23}};x\right)+8\;GP\left(\frac{s_{23}}{s_{12}}+1,\frac{q}{q-s_{23}};x\right)+\cdots\right)$$
$$+\frac{1}{\epsilon}\left(9\left(GP\left(0,\frac{q}{s_{12}};x\right)+GP(0,1;x)\right)-4\left(GP\left(0,0,\frac{q}{s_{12}};x\right)+GP(0,0,1;x)\right)+\cdots\right)$$
$$\left.+6\left(GP\left(0,0,1,\xi_-;x\right)+GP\left(0,0,1,\xi_+;x\right)\right)-2\,GP\left(0,0,\frac{q}{q-s_{23}},\frac{q\,(q-s_{23})}{q^2-s_{23}\,(q+s_{12})};x\right)+\cdots\right\}.$$

$$A_3(\epsilon) = -e^{2\gamma_E\epsilon}\,\frac{\Gamma(1-\epsilon)^3\Gamma(1+2\epsilon)}{\Gamma(3-3\epsilon)}.$$

C. G. Papadopoulos, D. Tommasini and C. Wever, arXiv:1409.6114 [hep-ph].

Physical region

$$S > \left(\sqrt{M_3^2} + \sqrt{M_4^2}\right)^2, \quad T < 0, \quad U < 0,$$

$$M_3^2 > 0, \quad M_4^2 > 0, \quad q_\perp^2 = \frac{TU - M_3^2 M_4^2}{S} > 0,$$

$$x > 1, \quad \frac{q - s_{12}}{s_{23}} > 1, \quad x s_{12} > q, \quad q > 0.$$

$$x > 1, \quad \begin{cases} s_{23} < 0, & s_{12} + s_{23} > q, & q > 0 \\ s_{23} > 0, & s_{12} + s_{23} < q, & s_{12} > q/x. \end{cases}$$

Analytic continuation

- Feynman propagator

$$D \to D + i\epsilon$$

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s_{ij} (s_{12}, s_{23} and q in the present study) and the parameter x,
$s_{ij} \to s_{ij} + i\delta_{s_{ij}}\eta$, $x \to x + i\delta_x\eta$, with $\eta \to 0$.

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- $\delta_{s_{ij}}$ and δ_x are determined as follows:

1) Input data:

$$G^{P12}_{001000011} \sim (-(-1+x)(-q+s_{12}x))^{1-2\epsilon} \sim (1-x)^{1-2\epsilon}\,(1-xs_{12}/q)^{1-2\epsilon}\,(-q)^{1-2\epsilon}$$

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2) Second graph polynomial: $\mathcal{F}$

C. Bogner and S. Weinzierl, Int. J. Mod. Phys. A **25** (2010) 2585 [arXiv:1002.3458 [hep-ph]].

in terms of s_{ij} and x, should acquire a definite-negative imaginary part in the limit $\eta \to 0$.

The Simplified Differential Equations Approach

Summary & Outlook

- One-loop up to 5-point at order ϵ: 6 scales, GP-weight 3(4) (look forward for pentaboxes)
- Two-loop triangles and 4-point MI
- Double boxes with two external off-shell legs (more than 100 MI) $\rightarrow$ P12 P13 P23 N12 N13 N34 topologies completed and tested!
- Completing the list of all MI with arbitrary off-shell legs ($m = 0$).

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NNLO in the future

- The NNLO automation to come
  - In a few years the new "wish list" should be completed
    $pp \to t\bar{t}$, $pp \to W^+W^-$, $pp \to W/Z + nj$, $pp \to H + nj$, ...
  - Virtual amplitudes: Reduction at the integrand level $\oplus$ IBP

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  - Virtual-Real — A.van Hameren, OneLoop MI
  - Real-Real — STRIPPER, M. Czakon, Phys. Lett. B **693** (2010) 259 [arXiv:1005.0274 [hep-ph]].
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- 2-loop triangle: $G^{N34}_{101010010}$

$$I(x) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^2 (k_1 + x p_{12})^2 k_2^2 (k_1 + k_2 - p_{123} + x p_{12})^2}.$$

$M(x) = x^{3\epsilon}$
$\lim_{x\to 0} M(x) I(x) = \lim_{x\to 0} x^{3\epsilon} \sum_{R\in\mathcal{R}} I^{(R)}$.

$$\lim_{x\to 0} (x^{3\epsilon} I^{(\mathcal{H},\mathcal{H})}) = x^{3\epsilon} G^{N34}_{002010010} \overset{IBP}{=} \frac{x^{3\epsilon}(1-2\epsilon)(2-3\epsilon)}{\epsilon q} G^{N34}_{001010010}$$
$$= \frac{(2-3\epsilon) A_3(\epsilon) (-q)^{-2\epsilon}}{2\epsilon^2} x^{3\epsilon},$$
$$\lim_{x\to 0} (x^{3\epsilon} I^{(\mathcal{S},\mathcal{H})}) = x^{3\epsilon} G^{N34}_{101010100} = \frac{(-q)^{-\epsilon} (-s_{12})^{-\epsilon} c_\Gamma(\epsilon)^2}{(1-2\epsilon)^2 \epsilon^2} x^{\epsilon}.$$

where $\mathcal{H} := k \sim p_{123}$.

$I_2(x) := G^{N34}_{101010020}$:

$$\partial_x(M(x)I(x)) = \frac{(2-3\epsilon)G^{N34}_{100010010}}{q(1-x)x^{1-3\epsilon}(1-xs_{12}/q)} - \frac{q\epsilon\,(M_2(x)I_2(x))}{(1-2\epsilon)(1-x)^{2\epsilon}x^{2-4\epsilon}(1-xs_{12}/q)^{2\epsilon}},$$

$$\partial_x(M_2(x)I_2) = \frac{(1-2\epsilon)(2-3\epsilon)(1-3\epsilon)G^{N34}_{001010010}}{\epsilon q^2(1-x)^{1-2\epsilon}x^{\epsilon}(1-xs_{12}/q)^{1-2\epsilon}} - \frac{(1-2\epsilon)(1-3\epsilon)\,(M(x)I(x))}{q(1-x)^{1-2\epsilon}x^{4\epsilon}(1-xs_{12}/q)^{1-2\epsilon}},$$

where $M_2(x) = (1-x)^{2\epsilon}x^{1-\epsilon}(1-xs_{12}/q)^{2\epsilon}$ is the integrating factor of $I_2(x)$. Moreover

$$G^{N34}_{100010010} = -\frac{A_3(\epsilon)}{2\epsilon(1-2\epsilon)}(-q)^{1-2\epsilon}(1-x)^{1-2\epsilon}(1-xs_{12}/q)^{1-2\epsilon}$$

and

$$G^{N34}_{001010010} = -\frac{A_3(\epsilon)}{2\epsilon(1-2\epsilon)}(-q)^{1-2\epsilon}.$$

Back-up slides

$$\lim_{x\to 0} I_2^{(\mathcal{H},\mathcal{H})} = G^{N34}_{002010020} \overset{IBP}{=} \frac{2(1-2\epsilon)(2-3\epsilon)(-1+3\epsilon)}{\epsilon q^2} G^{N34}_{001010010}$$
$$= \frac{(2-3\epsilon)(1-3\epsilon)A_3(\epsilon)(-q)^{-1-2\epsilon}}{\epsilon^2},$$
$$\lim_{x\to 0} I_2^{(\mathcal{S},\mathcal{H})} = G^{N34}_{101020100} \overset{IBP}{=} \frac{(1-2\epsilon)}{-q} G^{N34}_{101010100} = \frac{(-q)^{-1-\epsilon}(-s_{12})^{-\epsilon} c_\Gamma(\epsilon)^2}{(1-2\epsilon)\epsilon^2} x^{-2\epsilon}.$$

- $(\mathcal{S},\mathcal{H})$ of I: $G^{N34}_{101010100}$, not in inhomogeneous of $\partial_x(MI)$ and lies in the homogeneous term $\sim I_2$.
- $G^{N34}_{101010100}$ cannot be reproduced from the graph $G^{N34}_{101010010}$ by pinching internal lines.

$$I(1/x) = x^{4\epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^2 (k_1 - p_{12})^2 (k_2 + x p_{123})^2 (k_1 + k_2)^2}.$$

$$\mathcal{R} = \{(\mathcal{H}, \mathcal{H}), (\mathcal{S}, \mathcal{S})\}.$$

$$M(1/x) \sim x^{-3\epsilon} \text{ as } x \to 0$$

$$\lim_{x\to 0} MI(1/x) = \lim_{x\to 0} x^{-3\epsilon} \sum_{R\in\mathcal{R}} I^{(R)}.$$

Performing the expansion, it can be shown by power counting that

$$\lim_{x\to 0}(x^{-3\epsilon} I^{(\mathcal{S},\mathcal{S})}) = 0.$$

$$\lim_{x\to 0}(x^{-3\epsilon} I^{(\mathcal{H},\mathcal{H})}) = x^{-3\epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^2 (k_1 - p_{12})^2 k_2^2 (k_1 + k_2)^2}$$

$$\overset{IBP}{=} \frac{x^{2-3\epsilon}(-2+3\epsilon)}{s_{12}\epsilon} G^{N34}_{100000110}(1/x) = \lim_{x\to 0} \frac{x^{2-3\epsilon}(-2+3\epsilon)}{s_{12}\epsilon} G^{N34}_{100010010}(1/x)$$

$$= -\frac{(2-3\epsilon) A_3(\epsilon) (-s_{12})^{-2\epsilon} x^{\epsilon}}{2\epsilon^2 (1-2\epsilon)}. \tag{1}$$

with

$$G^{N34}_{100010010}(1/x) = -\frac{A_3(\epsilon)}{2\epsilon(1-2\epsilon)}(-s_{12})^{1-2\epsilon}(1-x)^{1-2\epsilon}(1-xq/s_{12})^{1-2\epsilon}x^{-2+4\epsilon}.$$

Contrary to the case of $I(x)$, in this case the resulting integrals $G^{N34}_{100000110}(1/x)$ and $G^{N34}_{100010010}(1/x)$ can be found from $I(1/x)$ by pinching internal lines. In fact, the singular part in the inhomogeneous term of the DE of $MI(1/x)$ is exactly given by the above single scale 2-point MI $G^{N34}_{100010010}$