Flavour symmetries in $SO(10)$ Yukawa couplings

Luís Lavoura

CFTP, Inst. Sup. Técnico, Univ. Lisboa

In collaboration with P.M. Ferreira,
W. Grimus & D. Jurčiukonis (arXiv:1510.02641),
and I.P. Ivanov (not yet finished)

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We work in the context of a supersymmetric (so that all three gauge couplings unify) renormalizable $SO(10)$ (so that all the fermion mass matrices are inter-related) GUT.

The minimal theory has only one **10** and one $\overline{\mathbf{126}}$ of scalars with (renormalizable) Yukawa couplings.

That theory is able to fit all the fermion masses and mixings, but it is self-contradictory: the scale of the neutrino masses, *i.e.* $\Delta m^2_{\text{atmospheric}} \approx 0.0025\,\text{eV}^2$, implies that the VEV of the $SU(2)_R$ triplet is in the middle of the GUT desert: $w_R \sim 10^{13}\,\text{GeV}$.

Way out: to add more scalar representations (**10**, $\overline{\mathbf{126}}$, or **120**) with Yukawa couplings.

Problem: that will add many degrees of freedom to the Yukawa couplings, which then lack predictive power. Our solution: to introduce some flavour symmetry which restricts the Yukawa matrices.

First paper: one **10**, one $\overline{\mathbf{126}}$, and one **120**, with an *Abelian* symmetry. Second paper: which symmetries is it possible in general to have?

FIRST PAPER

One $\overline{\mathbf{126}}$ which couples with (symmetric) matrix F, one $\mathbf{120}$ which couples with (anti-symmetric) matrix G, and one $\mathbf{10}$ which couples with (symmetric) matrix H.

$$\begin{aligned} M_d &= k_d\, H + \kappa_d\, G + v_d F, \\ M_\ell &= k_d\, H + \kappa_\ell\, G - 3 v_d\, F, \end{aligned}$$

$$\begin{aligned} M_u &= k_u\, H + \kappa_u\, G + v_u\, F, \\ M_D &= k_u\, H + \kappa_D\, G - 3 v_u\, F, \end{aligned}$$

Type-I and type-II seesaw mechanisms:

$$\begin{aligned} \mathcal{M}_\nu &= w_L\, F - \frac{1}{w_R}\, M_D\, F^{-1} M_D^T \\ &= \frac{v_d}{w_R} \left[\frac{w_L w_R}{v_d^2} \left(v_d F \right) - M_D\, \left(v_d F \right)^{-1} M_D^T \right]. \end{aligned}$$

Our assumptions: $F \neq 0$, $G \neq 0$, $H \neq 0$; $\det F \neq 0$; No generation decouples.

Our question: which flavour symmetries are available to constrain F, G, and H? Answer: 14 of them.

Because of the specific $H\, f^T \Gamma f$ form
of the Yukawa couplings in an $SO(10)$ GUT,
a flavour symmetry reads

$$\begin{array}{rcl} W^T F W & = & e^{-i\gamma} F, \\ W^T G W & = & e^{-i\beta} G, \\ W^T H W & = & e^{-i\alpha} H. \end{array}$$

The flavour symmetry is necessarily Abelian, because
there is only one of each type of scalar representation.

W may be $\mathrm{diag}\,(+1,\ +1,\ -1)$ (Z_2 symmetry).
Cases A, B, and C:

$$F,\, H \sim \left(\begin{array}{ccc} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{array} \right), \quad G \sim \left(\begin{array}{ccc} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{array} \right).$$

$$F \sim \left(\begin{array}{ccc} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{array} \right), \quad G,\, H \sim \left(\begin{array}{ccc} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{array} \right).$$

$$F,\, G \sim \left(\begin{array}{ccc} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{array} \right), \quad H \sim \left(\begin{array}{ccc} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{array} \right).$$

W may be $\mathrm{diag}\left(1,\ \omega,\ \omega^2\right)$ with $\omega \equiv \exp(2i\pi/3)$ (Z_3 symmetry). Cases D_1, D_3, and D_2:

$$F, G \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}, \quad H \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & 0 \end{pmatrix}.$$

$$F \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}, \quad G, H \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & 0 \end{pmatrix}.$$

$$F \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}, \quad G \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & 0 \end{pmatrix}, \quad H \sim \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}.$$

W may be $\mathrm{diag}(1,\ -1,\ i)$ (Z_4 symmetry). Case A_1:

$$F \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}, \quad G \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}, \quad H \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}.$$

W may be $\mathrm{diag}\left(e^{i\sigma},\ e^{-i\sigma},\ 1\right)$ with generic σ ($U(1)$ symmetry). Case A_2:

$$F, H \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}, \quad G \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}.$$

Gives Fritzsch-type mass matrices, doesn't work.

There are further cases with $U(1)$ symmetry — A_1', A_1'', and $\mathrm{D}_{1,2,3}'$ — but their matrices make them subcases of the above.

A case with $Z_2 \times Z_2$ symmetry is also possible.
It is simultaneously a sub-case of cases A, B, and C.

$$W_1 = \mathrm{diag}\,(+1,\ +1,\ -1)\,, \quad W_2 = \mathrm{diag}\,(+1,\ -1,\ +1)$$

$$F \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \quad G \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}, \quad H \sim \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We must now try and fit each of the cases to the data:
9 charged-fermion masses, 3 angles in the CKM matrix,
1 neutrino mass ratio, and 3 angles in the PMNS matrix.
Plus $\Delta m^2_{\mathrm{atmospheric}}$, which fixes $|w_R/v_d|$.

The numbers of parameters in $M_d M_d^\dagger$, $M_\ell M_\ell^\dagger$,
and $M_u M_u^\dagger$ for each case are the following:
A: 13 moduli and 10 phases.
B: 11 moduli and 7 phases.
C: 10 moduli and 6 phases.
Cases $\mathrm{D}_{1,2,3}$ and A_1: 9 moduli and 5 phases.

There are 3 moduli and 2 phases more in $\mathcal{M}_\nu \mathcal{M}_\nu^\dagger$:
$|w_R/v_d|$, $w_L w_R / v_d^2$, and κ_D.

The result of the fit is simple: Only cases A and B work.
But they work perfectly! No predictions seem possible.

Remarkably, $|w_R/v_d|$ turns out $\sim 5 \times 10^{14}$: excellent.

SECOND PAPER

Yukawa couplings in the $SO(10)$ GUT are of the form $H_a f^T \Gamma_a f$, where the Γ_a are either symmetric (for H_a in the **10** or in the $\overline{\mathbf{126}}$) or anti-symmetric (for H_a in the **120**) 3×3 matrices.

Symmetries take the form $U^T \Gamma_a U = \sum_b V_{ab} \Gamma_b$, where U acts on the three fermion families and V acts on the H_a pertaining to the same $SO(10)$ irrep.

There is a trivial symmetry $f \to e^{i\delta} f$, $H_a \to e^{-2i\delta} H_a$. We search for possible additional (flavour) symmetries.

We firstly search for Abelian symmetries, where $U = \mathrm{diag}\left(e^{i\alpha_1},\ e^{i\alpha_2},\ e^{i\alpha_3}\right)$ and $V_{ab} = e^{-i\psi_a} \delta_{ab}$.

For each nonzero (i, j) entry of any Yukawa-coupling matrix Γ_a, labelled by an index k, we write the equation

$$\sum_l D_{kl}\, \alpha_l = \alpha_i + \alpha_j + \psi_a = 2\pi n_k,$$

where the phases ψ_a were re-labelled as α_l for $l > 3$.

By finding the Smith normal form (SNF) of D, we may reduce it to an almost diagonal form.

The SNF is obtained by successively applying steps like
(1) changing the order of rows or columns of D,
(2) flipping the signs of rows or columns,
(3) adding one row/column to another row/column.

These manipulations are useful because they preserve the form of the system $\sum_l D_{kl}\, \alpha_l = 2\pi n_k$.

We arrive at the result that any single matrix Γ can only have the following five rephasing symmetries:
$U(1) \times U(1),\ U(1) \times \mathbb{Z}_2,\ U(1),\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2$
(or else it may have no symmetry at all).

If we allow for several matrices Γ instead of a single one, then two additional Abelian symmetries may exist:
$\mathbb{Z}_3,\ \mathbb{Z}_4$.

Because we must factor out the possibility of a general rephasing $f \to e^{i\delta} f$,
there is an additional possible Abelian symmetry:
$\mathbb{Z}_3 \times \mathbb{Z}_3 = \Delta(27) \,/\, \mathbb{Z}_3^{\mathrm{center}}$, where $\mathbb{Z}_3^{\mathrm{center}}$ is the center of $SU(3)$ and of its subgroup $\Delta(27)$.

Any non-Abelian symmetries can contain as subgroups only the above Abelian ones, *viz.*
$U(1) \times U(1),\ U(1) \times \mathbb{Z}_2,\ U(1)$ (continuous),
$\mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2,\ \mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_3,\ \mathbb{Z}_4$ (discrete).

Complicated mathematical reasonings lead us to the conclusion that the possible non-Abelian symmetries are $O(2)$, $O(2) \times U(1)$, $[U(1) \times U(1)] \rtimes S_3$, $SU(2)$, $SU(2) \times U(1)$, $SO(3)$, $SU(3)$ (continuous),
S_3, D_4, Q_4, A_4, S_4, $\Delta(54) \,/\, \mathbb{Z}_3^{\mathrm{center}}$, $\Sigma(36)$ (discrete).

It remains to construct explicit models with these symmetries. Not all the above symmetries can be realized in practice, though—sometimes accidental symmetries cannot be avoided
(*v.g.* $Q_4 \to Q_4 \times U(1)$).

$O(2)$: $\begin{pmatrix} f & 0 & 0 \\ 0 & 0 & g \\ 0 & g & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & h & 0 \\ h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & h \\ 0 & 0 & 0 \\ h & 0 & 0 \end{pmatrix}$.

D_4: same matrices as $O(2)$, plus $\begin{pmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}$

$S_3 = D_3$: $\begin{pmatrix} f & 0 & 0 \\ 0 & 0 & g \\ 0 & g & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & h & 0 \\ h & 0 & 0 \\ 0 & 0 & t \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & h \\ 0 & t & 0 \\ h & 0 & 0 \end{pmatrix}$.

S_4: $\begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{pmatrix}$ and

$$\begin{pmatrix} 0 & g & 0 \\ g & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & g \\ 0 & 0 & 0 \\ g & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & g \\ 0 & g & 0 \end{pmatrix}.$$

A_4: $\begin{pmatrix} f & 0 & 0 \\ 0 & \omega f & 0 \\ 0 & 0 & \omega^2 f \end{pmatrix}$ and $\begin{pmatrix} g & 0 & 0 \\ 0 & \omega^2 g & 0 \\ 0 & 0 & \omega g \end{pmatrix}$ and

$$\begin{pmatrix} 0 & h & 0 \\ h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & h \\ 0 & 0 & 0 \\ h & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & h \\ 0 & h & 0 \end{pmatrix}.$$

$\Delta(54)\,/\,\mathbb{Z}_3^{\text{center}}$: $\begin{pmatrix} f & 0 & 0 \\ 0 & 0 & g \\ 0 & g & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & g \\ 0 & f & 0 \\ g & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & g & 0 \\ g & 0 & 0 \\ 0 & 0 & f \end{pmatrix}$

and $\begin{pmatrix} h & 0 & 0 \\ 0 & 0 & t \\ 0 & t & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & t \\ 0 & h & 0 \\ t & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & t & 0 \\ t & 0 & 0 \\ 0 & 0 & h \end{pmatrix}.$

$\Sigma(36)$: same as $\Delta(54)\,/\,\mathbb{Z}_3^{\text{center}}$ but with, additionally, $\left[g/f = \left(-1+\sqrt{3}\right)/\,2 \quad \text{or} \quad g/f = \left(-1-\sqrt{3}\right)/\,2\right]$ and $\left[t/h = \left(-1+\sqrt{3}\right)/\,2 \quad \text{or} \quad t/h = \left(-1-\sqrt{3}\right)/\,2\right]$.

It remains to be seen whether any of the models with a non-Abelian symmetry may have any practical usefulness and is able to fit the data and, maybe, even have some predictive power.