Neutrino Mixing, A_4, and Dark Matter

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Brief History of A_4 in Neutrino Mixing

In 1978 (37 years ago), soon after the putative discovery of the third family of leptons and quarks, it was conjectured independently by Cabibbo and Wolfenstein:

$$U_{l\nu} = U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

where $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$. In the PDG convention, this implies $s_{23} = c_{23} = 1/\sqrt{2}$, $s_{12} = c_{12} = 1/\sqrt{2}$, $s_{13} = 1/\sqrt{3}$, $c_{13} = \sqrt{2/3}$, and $\delta = \pi/2$. If $\omega \leftrightarrow \omega^2$, then $\delta = -\pi/2$.

In 2001 (14 years ago), without knowing about Cabibbo and Wolfenstein, U_ω was discovered by Ma and Rajasekaran in the context of A_4.

This non-Abelian discrete symmetry has 12 elements and 4 irreducible representations: $\underline{1}, \underline{1}', \underline{1}'', \underline{3}$. Using

$$\underline{3} \times \underline{3} = \underline{1} + \underline{1}' + \underline{1}'' + \underline{3} + \underline{3}.$$

the following decompositions are obtained:

$$\underline{1} = 11 + 22 + 33,$$
$$\underline{1}' = 11 + \omega 22 + \omega^2 33,$$
$$\underline{1}'' = 11 + \omega^2 22 + \omega 33.$$

Let $(\nu, l)_i \sim \underline{3}$, $l_i^c \sim \underline{1}, \underline{1}', \underline{1}''$, and $\Phi_i \sim \underline{3}$, then

$$\mathcal{M}_l = \begin{pmatrix} f_e v_1^* & f_\mu v_1^* & f_\tau v_1^* \\ f_e v_2^* & f_\mu \omega v_2^* & f_\tau \omega^2 v_2^* \\ f_e v_3^* & f_\mu \omega^2 v_3^* & f_\tau \omega v_3^* \end{pmatrix}$$

$$= \begin{pmatrix} v_1^* & 0 & 0 \\ 0 & v_2^* & 0 \\ 0 & 0 & v_3^* \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} f_e & 0 & 0 \\ 0 & f_\mu & 0 \\ 0 & 0 & f_\tau \end{pmatrix}.$$

For $v_1 = v_2 = v_3$, a residual Z_3 symmetry exists with $U_\omega^\dagger$ as the link between $\mathcal{M}_l$ and $\mathcal{M}_\nu$.

For many years, theoretical effort was focused on obtaining a specific form of $\mathcal{M}_\nu$ so that tribimaximal neutrino mixing is realized:

$$U_{TBM} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} =$$

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

This means that

$$\mathcal{M}_\nu = \begin{pmatrix} m_2 & 0 & 0 \\ 0 & (m_1 - m_3)/2 & (m_1 + m_3)/2 \\ 0 & (m_1 + m_3)/2 & (m_1 - m_3)/2 \end{pmatrix}.$$

Pioneer A_4 papers: Ma/Rajasekaran(2001), Ma(2002), Babu/Ma/Valle(2003), Ma(2004), Altarelli/Feruglio(2005), Babu/He(2005).
This $\mathcal{M}_\nu$ is very hard to obtain in the context of a four-dimensional renormalizable field theory, because of the basic clash (or misalignment) of the residual symmetries (Z_3 for $\mathcal{M}_l$ and Z_2 for $\mathcal{M}_\nu$) [Lam]

On March 8, 2012, Daya Bay announced that θ_{13} had been measured at 8.8°, thus ending tribimaximal mixing.

The 2014 PDG values are: $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$,
$\Delta m^2_{21} = (7.53 \pm 0.18) \times 10^{-5}$ eV^2,
$\sin^2(2\theta_{23}) = 0.999$ $(+0.001/-0.018)$,
$\Delta m^2_{32} = (2.44 \pm 0.06) \times 10^{-3}$ eV^2 (normal),
$\sin^2(2\theta_{23}) = 1.000$ $(+0.000/-0.017)$,
$\Delta m^2_{32} = (2.52 \pm 0.07) \times 10^{-3}$ eV^2 (inverted),
$\sin^2(2\theta_{13}) = (9.3 \pm 0.8) \times 10^{-2}$.

In retrospect, the $Z_3 - Z_2$ clash should have been a warning against tribimaximal mixing.

Cobimaximal Mixing

Special Form of $\mathcal{M}_\nu$: Ma(2002), Babu/Ma/Valle(2003), Grimus/Lavoura(2004):

In the basis where the charged-lepton mass matrix is diagonal, it was written down 13 years ago that

$$\mathcal{M}_\nu = \begin{pmatrix} A & C & C^* \\ C & D^* & B \\ C^* & B & D \end{pmatrix},$$

where A, B are real. This leads to cobimaximal mixing: $\theta_{13} \neq 0$, $\theta_{23} = \pi/4$, $\delta_{CP} = \pm\pi/2$.

θ_{13} and θ_{12} are determined by $s_{13}/c_{13} = -D_I/\sqrt{2}C_R$, $s_{13}c_{13}/(c_{13}^2 - s_{13}^2) = \sqrt{2}C_I/(A - B + D_R)$, and

$$\frac{s_{12}c_{12}}{c_{12}^2 - s_{12}^2} = \frac{-\sqrt{2}(c_{13}^2 - s_{13}^2)C_R}{c_{13}[c_{13}^2(A - B - D_R) + 2s_{13}^2 D_R]}.$$

The three neutrino masses are determined by

$$m_2 + m_1 \simeq A + B + D_R + s_{13}^2(A - B + D_R),$$

$$(c_{12}^2 - s_{12}^2)(m_2 - m_1) \simeq -A + B + D_R - s_{13}^2(A - B + D_R),$$

$$m_3 \simeq -B + D_R + s_{13}^2(A - B + D_R).$$

Note that $\theta_{13} \neq 0$ and yet $\theta_{23} = \pi/4$ is maintained, together with the prediction that $\delta_{CP} = \pm\pi/2$. This pattern of $\mathcal{M}_\nu$ is protected by a symmetry, i.e. $e \to e$ and $\mu \leftrightarrow \tau$ exchange with CP conjugation.

Cobimaximal mixing is supported by present T2K data with input from reactor data which indicate a preference for $\delta_{CP} = -\pi/2$. Note that this special form predicts that $|U_{\mu i}| = |U_{\tau i}|$. This harkens back to the original U_ω of 1978, where indeed this is satisfied. It is strongly suggestive that U_ω itself must have something to do with the realization of this special form of $\mathcal{M}_\nu$.

Since 2012, many authors have incorporated this generalized CP transformation into non-Abelian discrete symmetries (some rather complicated) to pin down the other angles, i.e. θ_{12} and θ_{13}.

See for example: Mohapatra/Nishi(2012), Holthausen/Lindner/Schmidt(2013), Feruglio/Hagedorn/Ziegler(2013, 2014), Chen/Fallbacher/Mahanthappa/Ratz/Trautner(2014), Hagedorn/Meroni/Molinaro(2014), Ding/King/Neder(2014).

Typical result links θ_{12} with θ_{13}.

The simple yet crucial observation is that if $U_{l\nu} = U_\omega{}^\dagger \mathcal{O}$, where $\mathcal{O}$ is orthogonal, then $U_{2i}^* = U_{3i}$ for $i = 1, 2, 3$. Compared this to the PDG form of $U_{l\nu}$, i.e.

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

it is obvious that after rotating the phases of the third column and the second and third rows, the two matrices are identical if and only if $s_{23} = c_{23}$ and $\cos\delta = 0$, i.e. $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm\pi/2$. [F/M/T/Y(2000)]

Obviously $\mathcal{O}$ would come from diagonalizing a real mass matrix. So if $\mathcal{M}_\nu$ is somehow purely real in the A_4 basis, then

$$\mathcal{M}_\nu^{(e,\mu,\tau)} = U_\omega^\dagger \begin{pmatrix} a & c & e \\ c & d & b \\ e & b & f \end{pmatrix} U_\omega^* = \begin{pmatrix} A & C & C^* \\ C & D^* & B \\ C^* & B & D \end{pmatrix},$$

where $A = (a + 2b + 2c + d + 2e + f)/3$,
$B = (a - b - c + d - e + f)/3$,
$C = (a - b - \omega c + \omega^2 d - \omega^2 e + \omega f)/3$,
$D = (a + 2b + 2\omega c + \omega^2 d + 2\omega^2 e + \omega f)/3$.
The special form of $\mathcal{M}_\nu$ is thus automatically obtained.

In the tribimaximal basis, the neutrino mass matrix is

$$\mathcal{M}_B = \begin{pmatrix} m_1 & m_6 & m_4 \\ m_6 & m_2 & m_5 \\ m_4 & m_5 & m_3 \end{pmatrix},$$

where $m_1 = b + (d + f)/2$, $m_2 = a$, $m_3 = b - (d + f)/2$, $m_4 = i(f - d)/2$, $m_5 = i(e - c)/\sqrt{2}$, $m_6 = (e + c)/\sqrt{2}$.

To obtain tribimaximal mixing, $c = e = 0$ and $f = d$ are required. The remaining three parameters (a, b, d) are in general complex. If (a, b, c, d, e, f) are all real, cobimaximal mixing is obtained.

Whereas θ_{13} and θ_{12} are not predicted, there are two generic results (even if all parameters are complex):

$$m_5 = m_6 = 0 \Rightarrow \tan^2\theta_{12} = \frac{1}{2 - 3\sin^2\theta_{13}}.$$

$$m_4 = m_6 = 0 \Rightarrow \tan^2\theta_{12} = \frac{1}{2}(1 - 3\sin^2\theta_{13}).$$

If $\delta_{CP} = 0$ as well as $\theta_{23} = \pi/4$ are imposed, then both m_4 and m_5 should be nonzero. [Ma/Wegman(2011)]

If $m_4 = m_6 = 0$, but $m_{1,2,3,5}$ are complex, then $\delta_{CP} \neq 0$ is obtained as a function of $\theta_{23} \neq \pi/4$ for a fixed $\theta_{13} \neq 0$. [Ishimori/Ma(2012)]

Some Simple Examples

In the A_4 basis, let

$$\mathcal{M}_A = \begin{pmatrix} a & -e & e \\ -e & d & b \\ e & b & d \end{pmatrix},$$

then in the tribimaximal basis,

$$\mathcal{M}_B = \begin{pmatrix} b+d & 0 & 0 \\ 0 & a & i\sqrt{2}e \\ 0 & i\sqrt{2}e & b-d \end{pmatrix}.$$

Let $\mathcal{M}_B$ be diagonalized by

$$U_E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & is \\ 0 & is & c \end{pmatrix},$$

where $s = \sin\theta_E$ and $c = \cos\theta_E$, then

$$\frac{sc}{c^2 - s^2} = \frac{e\sqrt{2}}{a+b-d}, \quad \text{with } s = \sqrt{3}\sin\theta_{13}$$

and the three neutrino mass eigenvalues are $m'_1 = b + d$, $m'_2 = [c^2 a + s^2(b-d)]/(c^2 - s^2)$, $m'_3 = [s^2 a + c^2(b-d)]/(c^2 - s^2)$.

Three simple examples:

(I) $d = a$, hence $m'_1 = b + a$, $m'_2 = a + 0.08335b$, $m'_3 = -a + 1.08336b$, where $s = 0.2673$ has been used. Now for $\Delta m^2_{21} = 7.53 \times 10^{-5}\ \mathrm{eV}^2$ and $\Delta m^2_{32} = 2.44 \times 10^{-3}\ \mathrm{eV}^2$, the solution $b/a = -1.714$, $a = 0.0183$ eV and $e/a = -0.3642$ implies $m_{ee} = |A| = |a + 2b/3| = 2.6 \times 10^{-3}$ eV.

(II) $d = b$, hence $m'_1 = 2b$, $m'_2 = 1.08336a$, $m'_3 = 0.08336a$. The solution $a = 0.0465$ eV, $b = 0.0248$ eV, and $e = 0.0099$ eV implies $m_{ee} = |(a + 4b)/3| = 0.0486$ eV.

(III) $d = -b = 2a$: $m_1' = 0$,

$$m_2' = \left(\frac{c^2 - 4s^2}{c^2 - s^2}\right) a = 0.75a,$$

$$m_3' = \left(\frac{s^2 - 4c^2}{c^2 - s^2}\right) a = -4.25a.$$

$$\frac{\Delta m_{21}^2}{\Delta m_{31}^2} = \left(\frac{1 - 15\sin^2\theta_{13}}{4 - 15\sin^2\theta_{13}}\right)^2 = 0.031,$$

which is exactly the experimental central value. Here $a = 0.0116$ eV, $e/a = -0.6375$, and $m_{ee} = |a/3| = 3.9 \times 10^{-3}$ eV.

Connection to Dark Matter

The Majorana neutrino mass matrix is in general complex, so how does one guarantee it to be real for cobimaximal mixing? The answer was already there in a radiative model [Fraser/Ma/Popov(2014), Ma/Natale/Popov(2015)], where the origin of the neutrino mass matrix is that of a real scalar mass-squared matrix. Actually the neutrino mass eigenvalues may pick up phases from the parameters involved in the loop, but to obtain $|U_{\mu i}| = |U_{\tau i}|$, all that is required is for $\mathcal{M}_\nu$ to be diagonalized by $\mathcal{O}$.

Radiative Inverse Seesaw through Dark Matter:

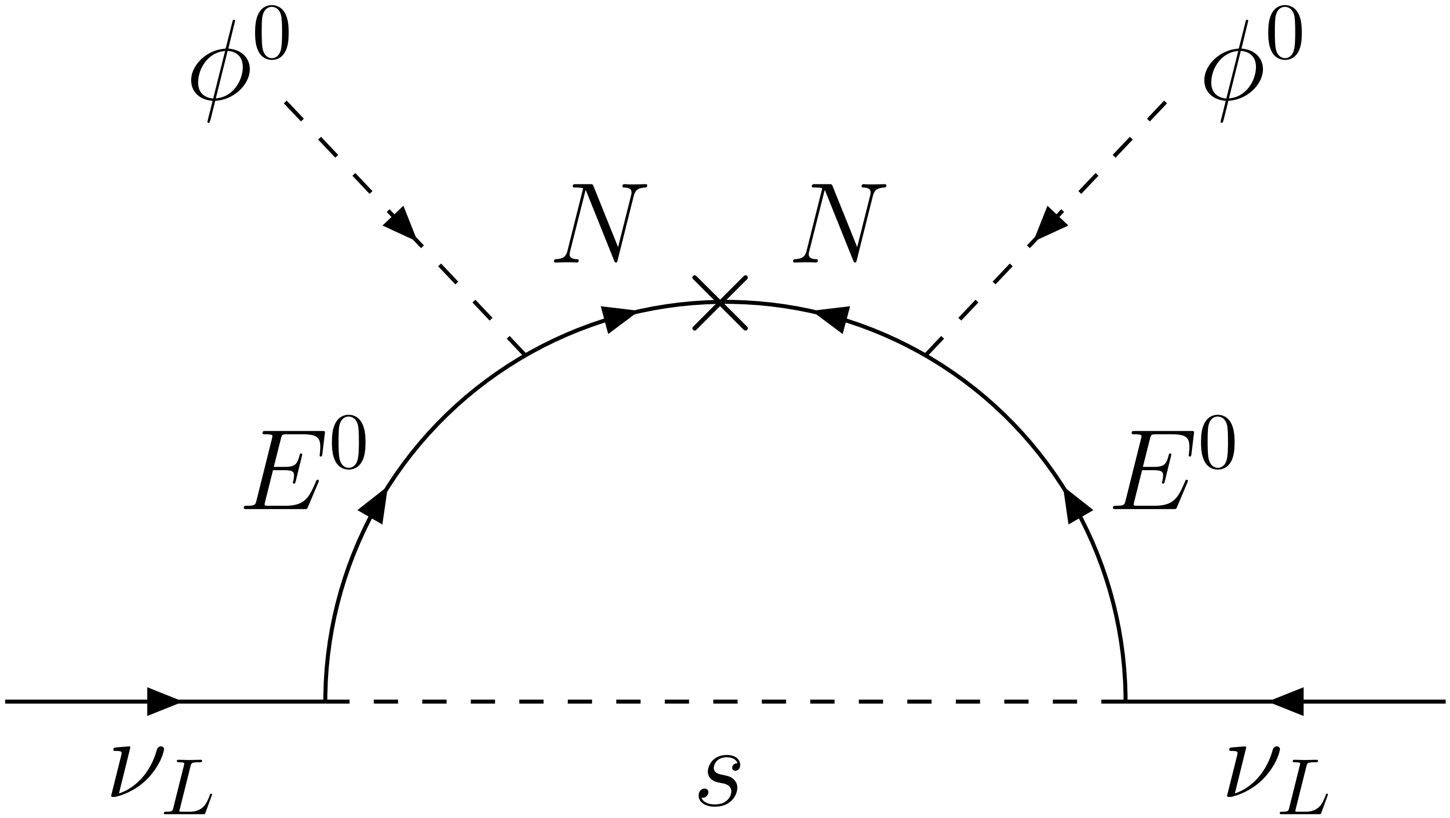
Under A_4, let the three families of leptons transform as

$$(\nu_i, l_i)_L \sim \underline{3}, \quad l_{iR} \sim \underline{1}, \underline{1}', \underline{1}''.$$

Add the following new particles, all assumed odd under an exactly conserved discrete Z_2 (dark) symmetry, whereas all SM particles are even:

$$(E^0, E^-)_{L,R} \sim \underline{1}, \quad N_{L,R} \sim \underline{1}, \quad s_i \sim \underline{3},$$

where (E^0, E^-) is a fermion doublet, N a neutral fermion singlet, and $s_{1,2,3}$ are real neutral scalar singlets.


ϕ^0
ϕ^0
N
N
E^0
E^0
ν_L
s
ν_L

The mass matrix linking $(\bar{N}_L, \bar{E}_L^0)$ to (N_R, E_R^0) is given by

$$\mathcal{M}_{N,E} = \begin{pmatrix} m_N & m_D \\ m_F & m_E \end{pmatrix},$$

where m_N, m_E are invariant mass terms, and m_D, m_F come from the respective Higgs couplings with $\langle \phi^0 \rangle = v/\sqrt{2}$. As a result, N and E^0 mix to form two Dirac fermions of masses $m_{1,2}$ with mixing angles

$m_D m_E + m_F m_N = \sin\theta_L \cos\theta_L (m_1^2 - m_2^2)$,

$m_D m_N + m_F m_E = \sin\theta_R \cos\theta_R (m_1^2 - m_2^2)$.

The mass terms $(m_{L,R}/2) N_{L,R} N_{L,R}$ also exist.

$$m_\nu = f^2 m_R s_R^2 c_R^2 (m_1^2 - m_2^2)^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^2 - m_s^2)} \frac{1}{(k^2 - m_1^2)^2} \frac{1}{(k^2 - m_2^2)^2}$$

$$+ f^2 m_L m_1^2 s_R^2 c_L^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m_s^2)} \frac{1}{(k^2 - m_1^2)^2}$$

$$+ f^2 m_L m_2^2 s_L^2 c_R^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m_s^2)} \frac{1}{(k^2 - m_2^2)^2}$$

$$- 2 f^2 m_L m_1 m_2 s_L s_R c_L c_R \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m_s^2)} \frac{1}{(k^2 - m_1^2)} \frac{1}{(k^2 - m_2^2)},$$

where s is a mass eigenstate. If A_4 is unbroken, then $\mathcal{M}_\nu$ is proportional to the identity matrix. However, if A_4 is softly broken by the necessarily real $s_i s_j$ terms, then

$$\mathcal{M}_\nu = \mathcal{O} \begin{pmatrix} m_{\nu 1} & 0 & 0 \\ 0 & m_{\nu 2} & 0 \\ 0 & 0 & m_{\nu 3} \end{pmatrix} \mathcal{O}^T,$$

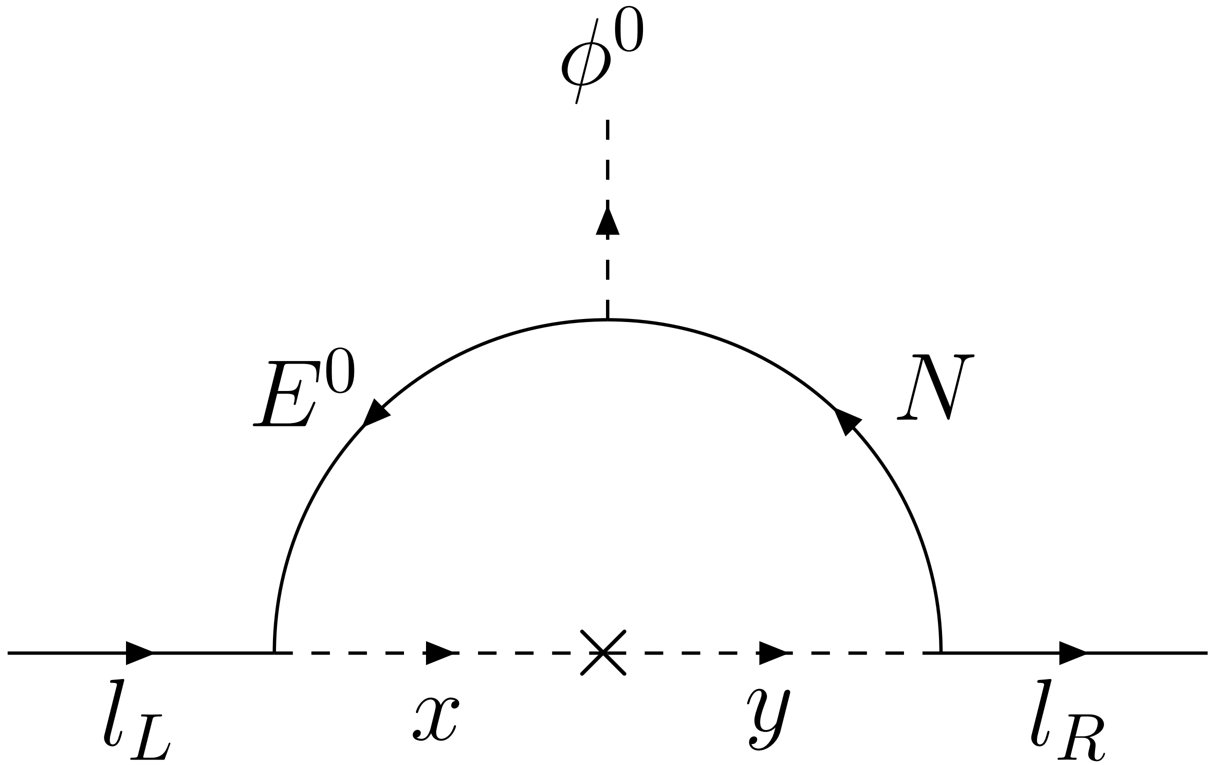
where $\mathcal{O}$ is an orthogonal matrix. Whereas f, m_L, m_R may be complex, only the relative phase between m_L and m_R appears in the two relative intrinsic Majorana phases of the neutrino mass eigenstates from the different s masses. Thus the desired form of $U_{l\nu}$ is obtained with $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm\pi/2$, once U_ω is applied.

Radiative Lepton Mass with Dark Matter:

Instead of using three Higgs doublets $\Phi_i \sim \underline{3}$ to obtain U_ω in the charged-lepton sector as in the original A_4 model of 2001, a radiative model of lepton mass is proposed. Again the fermion doublet (E^0, E^-) and singlet N are used, but now in conjunction of two sets of charged scalars which are also odd under dark Z_2, i.e.

$$x_i^- \sim \underline{3}, \quad y_i^- \sim \underline{1}, \underline{1}', \underline{1}''.$$

To connect x with y, the soft scalar term $x_i y_j^*$ is assumed to break A_4 to Z_3.


ϕ^0
E^0
N
l_L
x
y
l_R

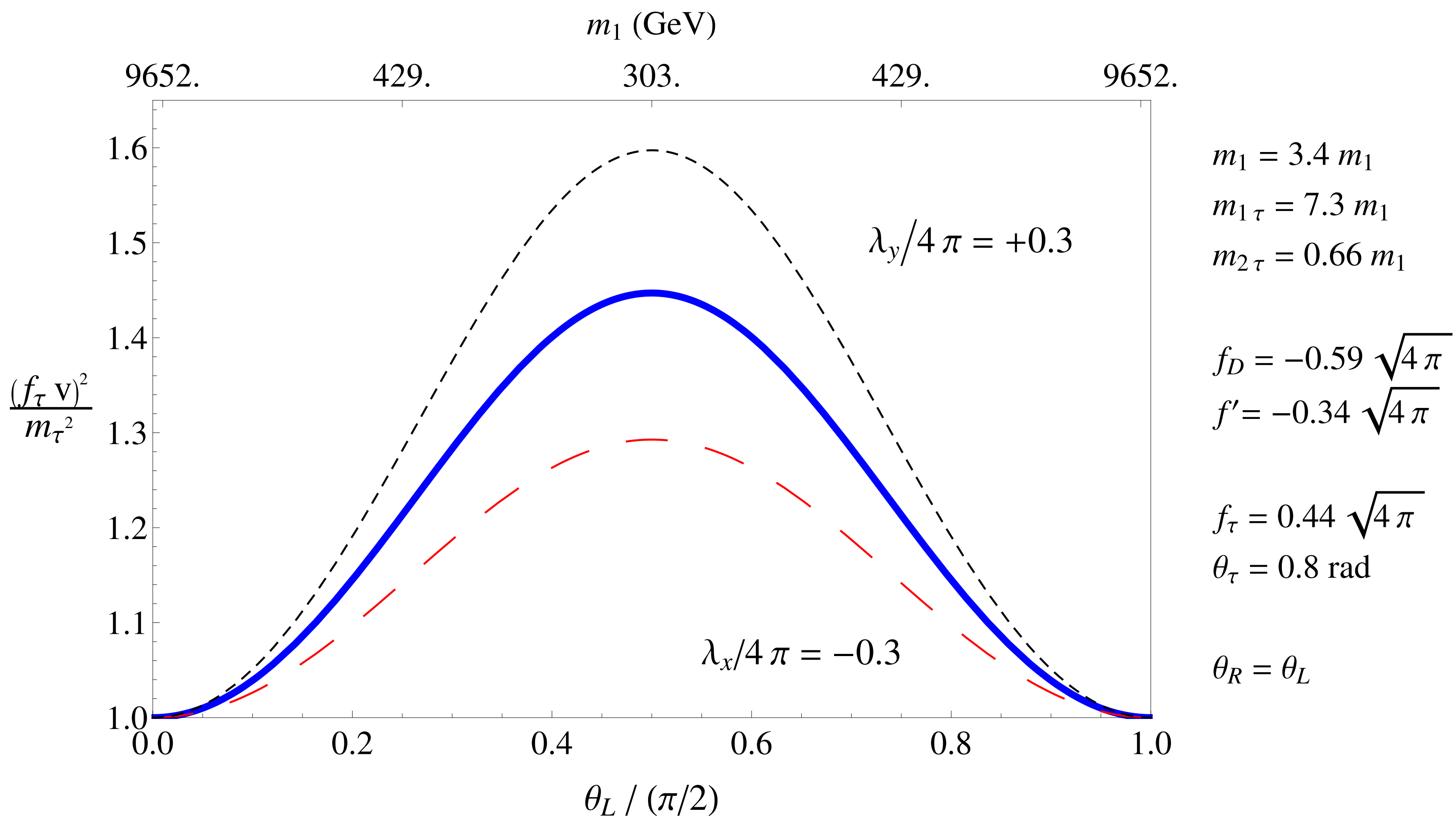
Each lepton mass is then given by

$$m_l = f' f_l \mu_l^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m_{1l}^2)(k^2 - m_{2l}^2)} \left[\frac{m_1 c_R s_L}{k^2 - m_1^2} - \frac{m_2 c_L s_R}{k^2 - m_2^2} \right],$$

where f' is the $E_L^0 l_L x^*$ Yukawa coupling, f_l is the $N_R l_R y^*$ Yukawa coupling, μ_l^2 is the scalar xy^* mass-squared term, and $m_{1l,2l}$ are the mass eigenvalues of the 2×2 mass-squared matrix

$$\mathcal{M}_{xy}^2 = \begin{pmatrix} m_x^2 & \mu_l^2 \\ \mu_l^2 & m_y^2 \end{pmatrix},$$

with $\mu_l^2 = \sin\theta_l \cos\theta_l (m_{1l}^2 - m_{2l}^2)$. This means that the $h\bar{l}l$ coupling will differ from $m_l/(246 \text{ GeV})$ without the usual $16\pi^2$ suppression.


m_1 (GeV)
9652.
429.
303.
429.
9652.
1.6
1.5
1.4
1.3
1.2
1.1
1.0
$\frac{(f_\tau v)^2}{m_\tau^2}$
0.0
0.2
0.4
0.6
0.8
1.0
θ_L / (π/2)
$\lambda_y/4\pi = +0.3$
$\lambda_x/4\pi = -0.3$
$m_1 = 3.4\ m_1$
$m_{1\tau} = 7.3\ m_1$
$m_{2\tau} = 0.66\ m_1$
$f_D = -0.59\ \sqrt{4\pi}$
$f' = -0.34\ \sqrt{4\pi}$
$f_\tau = 0.44\ \sqrt{4\pi}$
$\theta_\tau = 0.8$ rad
$\theta_R = \theta_L$

There is a one-to-one correlation of the neutrino mass eigenstates to the $s_{1,2,3}$ mass eigenstates, the lightest of which is dark matter.

It is also clear that all three neutrino masses are expected to be of the same order of magnitude and their mass-squared differences are related to the scalar mass differences. Using the most recent cosmological data

$$\sum m_\nu < 0.23 \text{ eV},$$

the effective neutrino mass m_{ee} in neutrinoless double beta decay is bounded below 0.07 eV for normal ordering and 0.08 eV for inverted ordering.

Ma(2015):

The dark matter parity of this model is also derivable from lepton parity.

Under lepton parity, let the new particles $(E^0, E^-), N$ be even and s, x, y be odd, then the same Lagrangian is obtained. As a result, dark parity is simply given by $(-1)^{L+2j}$, which is odd for all the new particles and even for all the SM particles.

Note that the tree-level Yukawa coupling $\bar{l}_L l_R \phi^0$ would be allowed by lepton parity alone, but is forbidden here because of the A_4 symmetry.

The radiative lepton mass matrix is diagonal because of the Z_3 residual symmetry. This means that the muon anomalous magnetic moment Δa_μ gets a significant contribution from xy exchange, but not $\mu \to e\gamma$. Because Δa_μ is now of order m_μ^2/m_E^2 instead of the usual $(16\pi^2)^{-1} m_\mu^2/m_E^2$, a large $m_E \sim 1$ TeV is possible for the explanation of the experimental-theoretical discrepancy instead of the usual $m_E \sim 200$ GeV.

As for $\mu \to e\gamma$, it will come from s exchange in the analog diagram to radiative neutrino mass. For $m_E \sim 1$ TeV, it will be suitably suppressed.

Personal Remarks

Neutrino theory attempts to answer several fundamental questions, foremost is the scale of new physics responsible for neutrino mass and mixing. A possible hint is that they may also be connected to dark matter at the mass scale of 1 TeV. In the scotogenic framework, the A_4 transformation U_ω may be used to obtain a desirable form of $U_{l\nu}$, i.e. $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm\pi/2$, automatically if the origin of the neutrino mass matrix is a set of real scalars $s_i \sim \underline{3}$ under A_4. Only one Higgs doublet with $\langle\phi^0\rangle$ accounting for all of electroweak symmetry breaking is required. New particles $(E^0, E^-), N, x^-, y^-, s$ are predicted and may be observed.