pNRQCD at $N^3LO$: the potential for unequal masses and the $Bc$ spectrum

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Based on the work in collaboration with A. Pineda and M. Stahlhofen:
arxiv:1511.08210
Outline

1. Introduction

2. Heavy quarkonium with different masses
   The NRQCD potential
   The $N^3$LO spectrum

3. Final remarks
Introduction
EFTs for bound states: NR systems

Non-relativistic systems fulfil the relation: $m_r \gg |p| \gg E$

when bounded by QCD, we need to take into account the relation of the scales to $\Lambda_{QCD}$

- **NR limit:** $m_r \gg \Lambda_{QCD}$

  **Strong coupling regime:** $|p| \gg E \sim \Lambda_{QCD}$

  **Weak coupling regime:** $|p| \gg E \gg \Lambda_{QCD}$
EFTs for bound states: NR systems

Non-relativistic systems fulfil the relation: \( m_r \gg |p| \gg E \)

When bounded by QCD in the weak coupling regime \( \alpha_s \sim \sqrt{s} \)

<table>
<thead>
<tr>
<th>Scales in bound state</th>
<th>Coulomb interaction</th>
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</thead>
<tbody>
<tr>
<td>Hard scale: ( m_r )</td>
<td>( m_r )</td>
</tr>
<tr>
<td>Soft scale: (</td>
<td>p</td>
</tr>
<tr>
<td>Ultrasoft scale: ( E )</td>
<td>( m_r \alpha_s^2 )</td>
</tr>
</tbody>
</table>

Scales are well separated

We can integrate out the hard and soft scales to obtain pNRQCD

It describes systems such as: \( J/\psi, \Upsilon, \bar{t}t \) near threshold, \( B_c \), etc.
Heavy quarkonium with different masses
The physics of heavy quarks

- Bound states of heavy quarks are naturally NR systems
- We focus in the situation $m_1 \sim m_2 \sim m_r$

**Extreme weak coupling regime** $mv^2 \gg \Lambda_{QCD}$

\[
\left( i\partial_0 - \frac{p^2}{2m_r} - V^{(0)}(r) \right) \phi(r) = 0
\]

+ corrections to the potential
+ interaction with other low-energy degrees of freedom

- The potentials depend on the matching procedure: on-shell, off-shell in Coulomb and Feynman gauges, with Wilson loops
Heavy quarkonium with different masses: Introduction

The physics of heavy quarks

- Bound states of heavy quarks are naturally NR systems
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**Extreme weak coupling regime** $mv^2 \gg \Lambda_{\text{QCD}}$

\[
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\]
+ corrections to the potential
+ interaction with other low-energy degrees of freedom

\[
\begin{align*}
pNRQCD. \\
V_s &= V^{(0)} + \frac{V^{(1,0)}}{m_1} + \frac{V^{(0,1)}}{m_2} + \frac{V^{(2,0)}}{m_1^2} + \frac{V^{(0,2)}}{m_2^2} + \frac{V^{(1,1)}}{m_1 m_2} + \cdots
\end{align*}
\]

- The singlet potential: $h_s(r, p, p_R, S_1, S_2) = \frac{p^2}{2m_r} + \frac{p_R^2}{2M} + V_s(r, p, p_R, S_1, S_2)$
Heavy quarkonium with different masses

The NRQCD potential
The NRQCD potential: theoretical setup

The potential

Basis for the potential \((P_R = 0)\):

- static and \(1/m\) potentials: \(V^{(a,b)} = V^{(a,b)}(r)\)
- the \(1/m^2\) potential: \(V^{(a,b)} = V_{SD}^{(a,b)} + V_{SI}^{(a,b)}\)

\[
V_{SI}^{(a,b)} = \frac{1}{2} \left\{ p^2, V_{p^2}^{(a,b)}(r) \right\} + V_{L^2}^{(a,b)}(r) \frac{L^2}{r^2} + V_r^{(a,b)}(r)
\]

The potentials are invariant under:

- charge conjugation: \(\psi \leftrightarrow \chi_c\)
- mass exchange: \(m_1 \leftrightarrow m_2\)
The NRQCD potential: theoretical setup

The potential in momentum space

\[ \tilde{V}_s \equiv \langle p' | V_s | p \rangle \]

- **Static potential and \(1/m\) potentials:**
  \[ \tilde{V}^{(0)} = -\frac{1}{k^2} \tilde{D}^{(0)}(k), \quad \tilde{V}^{(1,0)} = \frac{1}{k} \tilde{D}^{(1,0)}(k) \]

- **\(1/m^2\) (spin-independent) potential:**
  \[ \tilde{V}^{(2,0)}_{SI} = \frac{p^2 + p'^2}{2k^2} \tilde{D}^{(2,0)}_{p^2}(k) + \tilde{D}^{(2,0)}_r(k) + \frac{(p'^2 - p^2)^2}{k^4} \tilde{D}^{(2,0)}_{\text{off}}(k) \]

The \(\tilde{D}\)-coefficients have \(D = d + 1 = 4 + 2\epsilon\) dimensions. It is the \(\tilde{D}_{\text{off}}\)-coefficients that are scheme-dependent.
The NRQCD potential: theoretical setup

Momentum vs. position space

We can relate both bases in terms of the $\tilde{D}$ coefficients.

Example

$$V^{(0)}(0) = \int \frac{d^d q}{(2\pi)^d} e^{-i q \cdot r} \tilde{V}^{(0)}(q) = -C_F \sum_{n=0}^{\infty} \frac{g_B^{2n+2}}{(4\pi)^{2n}} F_{2-2n}(r) \tilde{D}_{n+1}(\epsilon)$$

The $L^2$ operator in $d$-dimensions

$$\frac{L^2}{r^2} \equiv p^i (\delta^{ij} - \frac{r_i r^j}{r^2}) p^j$$

which is the usual $L^2$ in 3-dimensions

The transform of the off-shell potential

it contributes to the three position space structures

$$V^{(2,0)}_{\text{off}} = 4 \left( \frac{d^2 g^{(2,0)}_{\text{off}}}{dr^2} - \frac{1}{r} \frac{dg^{(2,0)}_{\text{off}}}{dr} \right) \frac{L^2}{r^2} - 2 \left\{ \frac{d^2 g^{(2,0)}_{\text{off}}}{dr^2}, p^2 \right\} + 2[p^i, [p^i, \frac{d^2 g^{(2,0)}_{\text{off}}}{dr^2}]] + h_{\text{off}}(r)$$
The NRQCD potential: theoretical setup

Field redefinition

\( \tilde{V}_1 \) and \( \tilde{V}_{\text{off}} \) are related by a field redefinition: mixing

\[
\begin{align*}
    h_s &= \frac{p^2}{2m_r} + V^{(0)}(r) + \frac{\delta V_1(r)}{m_r} + \cdots \\
    \downarrow \text{unitary transformation} \\
    h'_s &= \frac{p^2}{2m_r} + V^{(0)} + \delta \tilde{V}_{\text{FR}} + \cdots \\
    \delta \tilde{V}_{\text{FR}} &= \langle p' | \delta V_{\text{FR}} | p \rangle = \frac{1}{2m_r^2} \frac{(p'^2 - p^2)^2}{k^4} \tilde{g}(k),
\end{align*}
\]

where \( \tilde{g}(k) \sim \tilde{g}(k, V_0, \delta V_1) \)

we can exchange \( 1/m_r \) terms by \( 1/m_r^2 \) off-shell contributions
The NRQCD potential: computation

Matching with on-shell Green functions

- on-shell Green functions $\equiv$ S-matrix elements
- asymptotic quarks fulfilling the free EOM order by order

Imperfect cancellation between NRQCD and pNRQCD potential loops

$\tilde{V}_C \sim \tilde{V}_{\text{off}} \Rightarrow$ Nontrivial mass dependence in the $1/m$ potential: $\sim \frac{1}{m_1 + m_2} = \frac{m_r}{m_1 m_2}$

As expected: $\tilde{D}_{\text{off}}^{(a,b)}(k) = 0$
Matching with off-shell Green functions

- **Gauge dependent**: Coulomb and Feynman gauges
- **Freedom treating energy dependence**: different choices affect the $1/m$ and $1/m^2$ potentials

Our choice: the one that exhibits the divergence structure of the on-shell potential most “naturally”
The NRQCD potential: computation

Matching with off-shell Green functions

The $1/m^2$ potential in the Coulomb gauge:

- Less diagrams
- Efficient way to perform the computation
- Natural choice of treating energies
The NRQCD potential: computation

Matching with off-shell Green functions

The $1/m^2$ potential in the Feynman gauge:

- More diagrams
- Heavy dependence on the energies
- Computations are easily automated
The NRQCD potential: computation

Matching with Wilson loops

\[ W_\Box \equiv \text{P} \exp \left\{ -ig \oint_{r \times T_W} dz^\mu A_\mu(z) \right\} \]

- Green functions in position space and setting the time of the quark and anti-quark equal
- Gauge independent

The "quasi-static" energy \( E_s \)

\[
\frac{p^2}{2m_r} + \frac{P_R^2}{2M} + E^{(0)} + E^{(1,0)}_{m_1} + E^{(0,1)}_{m_2} + E^{(2,0)}_{m_1} + E^{(0,2)}_{m_2} + E^{(1,1)}_{m_1 m_2} + \cdots
\]

Example

\[
E^{(1,1)}_{L^2}(r) = \frac{i}{d-1} \left( \delta^{ij} - d \frac{r^i r^j}{r^2} \right) \lim_{T \to \infty} \int_0^T dt \, t^2 \langle \langle g E^i_1(t) g E^j_2(0) \rangle \rangle_c
\]

where \( \langle \ldots \rangle \equiv \langle \ldots W_\Box \rangle / \langle W_\Box \rangle \)
The NRQCD potential: computation

The $1/m^2$ potential with Wilson loops

- We obtain: $V_{s,W}(r) = E_s(r)|_{\text{soft}}$

- Feynman rules for chromoelectric insertions

**Example:**

$$V_{L^2,W}^{(1,1)}(r) = \frac{ig_0^2 C_F}{(d-1)} \left( \delta^{ij} - \frac{d \cdot r^i r^j}{r^2} \right) \lim_{T \to \infty} \int_0^T dt \ t^2 \int \frac{d^D k}{(2\pi)^D} e^{i(k_0 t - kr)} \frac{ik_0^2}{k^2 + i0} P_{ij}(k)$$
The NRQCD potential: computation

The $1/m^2$ potential with Wilson loops

- We obtain: $V_{s,W}(r) = E_s(r)|_{\text{soft}}$

Only need compute off-shell potential

- Diagrams showing various contributions to the potential

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pNRQCD at $N^3LO$: the potential for unequal masses and the Bc spectrum
The NRQCD potential: computation

**The $1/m$ potential**

- Profit from previous computation: on-shell, for equal masses and to $O(\epsilon)$ (Kniehl et al.)
- The $O(1/m^2)$ scheme difference is off-shell: $\tilde{V}_{s,X} = \tilde{V}_{s,\text{on-shell}} + \delta\tilde{V}_X^{(2)}$

We exploit the relation:

\[
\left. \frac{\tilde{V}_X^{(1,0)}}{m} + \frac{\tilde{V}_X^{(0,1)}}{m} + \delta \tilde{V}_X^{(1)} \right|_{m=m_1=m_2} = \left[ \frac{\tilde{V}_{\text{on-shell}}^{(1,0)}}{m} + \frac{\tilde{V}_{\text{on-shell}}^{(0,1)}}{m} \right]_{m=m_1=m_2}
\]

- We compute the $O(\alpha_s^3/m)$ potential for different masses in all previous schemes to $O(\epsilon)$
The NRQCD potential: computation

The renormalised potential

The (singlet) heavy quarkonium self-energy

$$\Sigma_B(1 - \text{loop}) = -g_B^2 C_F V_A^2(1 + \epsilon) \frac{\Gamma(2 + \epsilon) \Gamma(-3 - 2\epsilon)}{\pi^{2+\epsilon}} r \left( h_s - E + \Delta V \right)^{3+2\epsilon} r$$

where $\Delta V \equiv V_o^{(0)} - V^{(0)}$

- Its UV divergences ($\delta V_s$) **cancel** the US divergences of the soft potential:
  $$V_s^{\overline{\text{MS}}} + \delta V_s = V_s$$

- $V_s^{\overline{\text{MS}}}$ produces **finite physical results**

- $\delta V_s$ is **ambiguous**: we choose it so that the 4-dimensional potentials are finite
Poincaré invariance

Poincaré invariance constrains for potentials with an exact expansion on the masses:

\[ 2V_{L^2}^{(2,0)} - V_{L^2}^{(1,1)} + \frac{r}{2} \frac{dV^{(0)}(r)}{dr} = 0 \]
\[ -4V_{p^2}^{(2,0)} + 2V_{p^2}^{(1,1)} - V^{(0)}(r) + r \frac{dV^{(0)}(r)}{dr} = 0 \]

- Our bare and renormalized potentials fulfil them
- They are **not affected by field redefinitions**: they produce
  \[ \delta V_{L^2}^{(1,1)} = 2\delta V_{L^2}^{(2,0)}, \quad \delta V_{p^2}^{(1,1)} = 2\delta V_{p^2}^{(2,0)} \]
Example of potential result

The $1/m_1^2$ renormalized potential with Wilson loops:

$$V^{(2,0),\overline{\text{MS}}}_{r, W}(r) = \frac{C_F \alpha_s}{8} \left( c_D^{(1)} + \frac{\alpha_s}{\pi} \left\{ -\frac{5}{9} \left( c_D^{(1)} + c_1^{hl(1)} \right) T_F n_f + \left( \frac{13}{36} c_F^{(1)\ 2} + \frac{8}{3} \right) C_A + \left( \left( \frac{4}{3} + \frac{5}{6} c_F^{(1)\ 2} \right) C_A - \frac{2}{3} \left( c_D^{(1)} + c_1^{hl(1)} \right) T_F n_f \right) \ln(\nu) \right\} \right) 4\pi\delta^{(3)}(r)$$

$$+ \frac{C_F \alpha_s^2}{8\pi} \left\{ \left( \frac{4}{3} + \frac{5}{6} c_F^{(1)\ 2} \right) C_A - \frac{2}{3} \left( c_D^{(1)} + c_1^{hl(1)} \right) T_F n_f \right\} \text{reg} \frac{1}{r^3},$$

$$V^{(2,0),\overline{\text{MS}}}_{L^2, W}(r) = \frac{C_A C_F \alpha_s^2}{4\pi r} \left( \frac{11}{3} - \frac{8}{3} \ln(r \nu e^{\gamma_E}) \right),$$

$$V^{(2,0),\overline{\text{MS}}}_{p^2, W}(r) = -\frac{C_A C_F \alpha_s^2}{\pi r} \left( \frac{2}{3} + \frac{1}{3} \ln(r \nu e^{\gamma_E}) \right).$$
Heavy quarkonium with different masses

The N³LO spectrum
The N$^3$LO spectrum

The $B_c$ spectrum

**US energy correction**

\[
\delta E_{n\ell}^{US} = -E_n^C \frac{\alpha_s^3}{\pi} \left[ \frac{2}{3} C_F L_{n\ell}^E + \frac{1}{3} C_A \left( L_\nu - L_{US} + \frac{5}{6} \right) \left( \frac{C_A^2}{2} + \frac{4 C_A C_F}{(2\ell + 1)n} \right) 
+ 2 C_F^2 \left( \frac{8}{(2\ell + 1)n} - \frac{1}{n^2} \right) \right] + \frac{8\delta l_0}{3n} C_F^2 \left( C_F - \frac{C_A}{2} \right) \left( L_\nu - L_{US} + \frac{5}{6} \right),
\]

where $L_{n\ell}^E$ is the Bethe logarithm

**Energy correction associated to the static potential**

\[
\left. \delta E(n, l, s, j) \right|_{V(0)} = E_n^C \left( 1 + \frac{\alpha_s}{\pi} P_1(L_\nu) + \left( \frac{\alpha_s}{\pi} \right)^2 P_2^c(L_\nu) + \left( \frac{\alpha_s}{\pi} \right)^3 P_3^c(L_\nu) \right),
\]

(Kiyo, Sumino)
The $N^3$LO spectrum

The $B_c$ spectrum

Energy correction associated to the relativistic potentials

$$\delta E(n, l, s, j) = E_n^C \left[ \left( \frac{\alpha_s}{\pi} \right)^2 c_{2n}^{nc} + \left( \frac{\alpha_s}{\pi} \right)^3 c_{3n}^{nc} \right]$$

- $c_{3n}^{nc}$ involves the use of perturbation theory

$$\delta E_{nlj}^{V \times V} = \langle \psi_{nlj} | V \frac{1}{(E_n^C - h)^'} V | \psi_{nlj} \rangle$$

$$= \int dr_2 dr_1 \psi_{nlj}^*(r_2) V(r_2) G_{nl}(r_1, r_2) V(r_1) \psi_{nlj}(r_1)$$
The $N^3LO$ spectrum

The $B_c$ spectrum

Complete $N^3LO$ spectrum for the $B_c$

\[ E(n, l, s, j) = E_n^C \left( 1 + \frac{\alpha_s}{\pi} P_1(L_\nu) + \left( \frac{\alpha_s}{\pi} \right)^2 P_2(L_\nu) + \left( \frac{\alpha_s}{\pi} \right)^3 P_3(L_\nu) \right), \]

\[ P_1(L_\nu) = \beta_0 L_\nu + \frac{a_1}{2}, \]

\[ P_2(L_\nu) = \frac{3}{4} \beta_0^2 L_\nu^2 + \left( -\frac{\beta_0^2}{2} + \frac{\beta_1}{4} + \frac{3\beta_0 a_1}{4} \right) L_\nu + c_2, \]

\[ P_3(L_\nu) = \frac{1}{2} \beta_0^3 L_\nu^3 + \left( -\frac{7\beta_0^3}{8} + \frac{7\beta_0 \beta_1}{16} + \frac{3}{4} \beta_0^2 a_1 \right) L_\nu^2 \]

\[ + \left( \frac{\beta_0^3}{4} - \frac{\beta_0 \beta_1}{4} + \frac{\beta_2}{16} - \frac{3}{8} \beta_0^2 a_1 + 2\beta_0 c_2 + \frac{3\beta_1 a_1}{16} \right) L_\nu + c_3 \]

where $c_i = c_i^c + c_i^{nc}$
Final remarks
Heavy quarkonium for different masses: conclusions

Summary of the results

- we develop the $N^3\text{LO}$ potential in pNRQCD for different masses
- The potentials obtained are valid for $mv \gg \Lambda_{\text{QCD}}$

The $O(\alpha_s/m^2)$ potential in different matching schemes
- all schemes are feasible
- they are related by a field redefinition

The $O(\alpha_s^3/m)$ potential in different matching schemes
- no $C_F^2 T_F n_F$ in off-shell scheme
- no $C_F^2 T_F n_F$ or $C_F^2 C_A$ in Wilson loop scheme
- The US contribution is valid for $mv^2 \gg \Lambda_{\text{QCD}}$

- We computed the full $N^3\text{LO}$ spectrum for different masses
Heavy quarkonium for different masses: conclusions

**Future perspectives**

- Study the $B_c$ spectrum and decays
  - obtain the charm mass
- Compute **US contribution with Wilson loops**
  - comparison with lattice predictions
- Explicitly compute $1/m$ potential with **Wilson loops**
  - few color structures
  - check on previous computations
- Compute higher order contributions: $\mathcal{O}(m_r \alpha_s^6 \ln(\alpha_s))$
Thank you!
The spin-dependent and the static potentials

<table>
<thead>
<tr>
<th>The static potential</th>
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<tbody>
<tr>
<td>• gauge independent order-by-order</td>
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<tr>
<td>• already computed to $\mathcal{O}(\alpha^4)$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>The spin-dependent potential</th>
</tr>
</thead>
<tbody>
<tr>
<td>• renormalized result for different masses computed in 1986 by Pantaleone et al.</td>
</tr>
<tr>
<td>• bare potential needs prescription for the $D$-dimensional Levi-Civita symbol</td>
</tr>
<tr>
<td>• the renormalized potential is enough to compute the spectrum</td>
</tr>
</tbody>
</table>
The problematic of QCD

Quantum Chromodynamics describes the interaction of quarks and gluons

- It is asymptotically free: predictions can be made at high energies
- The strong interaction grows at long distances: quarks are confined in hadrons at low energies
The problematic of QCD

Quantum Chromodynamics describes the interaction of quarks and gluons

- It is asymptotically free: predictions can be made at high energies
- The strong interaction grows at long distances: quarks are confined in hadrons at low energies

We exploit the EFT tools to overcome our limitation to describe low energy QCD

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pNRQCD at $N^3LO$: the potential for unequal masses and the Bc spectrum
The spectrum functions

\[ \xi_{\text{SD}} = \frac{2}{3n \ m_1 m_2} \left\{ \frac{-3(1 - \delta_{l0})}{l(l+1)(2l+1)} \left( D_S + X_{LS} + \frac{m_1}{m_2} X_{LS_2} + \frac{m_2}{m_1} X_{LS_1} \right) \right\} \]

\[ - 4S_{12} \delta_{l0} \left[ 2 + 3 \frac{m_1 m_2}{m_2^2 - m_1^2} \ln \left( \frac{m_1^2}{m_2^2} \right) \right] \}, \]

\[ \xi_{\text{FFnf}} = \frac{2m_r^2}{9n^2 m_1 m_2} \left\{ \frac{1 - \delta_{l0}}{l(l+1)(2l+1)} \left[ 2n(4S_{12} - D_S) \right. \right. \]

\[ + 6 \left( D_S + \frac{m_2}{m_1} X_{LS_1} + \frac{m_1}{m_2} X_{LS_2} + 2X_{LS} \right) \left( \frac{3n}{2l+1} + \frac{n}{2l(l+1)(2l+1)} + l + \frac{1}{2} \right) \]

\[ + 2n \left\{ S_1(l + n) + S_1(2l - 1) - 2S_1(2l + 1) - l(\Sigma_1^{(k)} + \Sigma_1^{(m)}) + n\Sigma_b - \Sigma_1^{(m)} + \frac{1}{6} \right\} \bigg] \]

\[ + 8\delta_{l0} S_{12} \left[ 1 + 4n \left( \frac{11}{12} - \frac{1}{n} - S_1(n - 1) - S_1(n) + nS_2(n) \right) \right] \bigg] } \},
Logarithmic functions

\[ E_n^C = -\frac{C_F^2 \alpha_s^2 m_r}{2 n^2} \]

\[ L_\nu = \ln \left( \frac{n\nu}{2 m_r C_F \alpha_s} \right) + S_1(n + l) \]

\[ L_{US} = \ln \left( \frac{C_F \alpha_s n}{2} \right) + S_1(n + l) \]

\[ L_H = \ln \left( \frac{n}{C_F \alpha_s} \right) + S_1(n + l) \]

The Bethe logarithm is defined as

\[ L_n^E = \frac{1}{(C_F \alpha_s)^2 E_n^C} \int_0^\infty \frac{d^3 k}{(2\pi)^3} |\langle r \rangle_{kn}|^2 \left( E_n^C - \frac{k^2}{2 m_r} \right)^3 \ln \frac{E_1^C}{E_n^C - \frac{k^2}{2 m_r}} \]
Energy dependence in the Coulomb gauge

\[ - \frac{i g_B^4}{3} \frac{k^2 e - 4 \csc(\pi e)}{2^4 e + 4 \pi e + \frac{1}{2} \Gamma(\epsilon + \frac{5}{2})} \left[ 3 C_F T_F n_f \epsilon (1 + \epsilon) \right. \\
- \frac{C_A C_F}{4} \left. ((\epsilon + 1)(\epsilon(56 \epsilon + 121) + 60) - \frac{5 \Gamma\left(\epsilon + \frac{3}{2}\right)^2}{\sqrt{\pi} \Gamma\left(2\epsilon + \frac{5}{2}\right)} 4^{\epsilon + 1}(\epsilon + 1)(2 \epsilon + 3)(4 \epsilon + 3) \right] \\
= - \frac{i g_B^4}{3 m_i} C_A C_F \frac{k^2 e - 4 (\epsilon + 1)(2 \epsilon + 1) \Gamma(1 - \epsilon) \Gamma(2 \epsilon)}{4^2 e + 1 \pi e + \frac{3}{2} \Gamma\left(2\epsilon + \frac{3}{2}\right)} \\
\left[ E_i \left( k^2 - (p'^2 - p^2)^2 \right) + E'_i \left( k^2 + (p'^2 - p^2)^2 \right) \right] \]