pNRQCD at N^3LO: the potential for unequal masses and the Bc spectrum

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Outline

1. Introduction

2. Heavy quarkonium with different masses

   The NRQCD potential

   The N^3LO spectrum

3. Final remarks

Introduction

EFTs for bound states: NR systems

Non-relativistic systems fulfil the relation: $m_r \gg |\mathbf{p}| \gg E$

when bounded by QCD, we need to take into account the relation of the scales to $\Lambda_{\rm QCD}$

- **NR limit:** $m_r \gg \Lambda_{\rm QCD}$

  Strong coupling regime: $|\mathbf{p}| \gg E \sim \Lambda_{\rm QCD}$

  Weak coupling regime: $|\mathbf{p}| \gg E \gg \Lambda_{\rm QCD}$

EFTs for bound states: NR systems

Non-relativistic systems fulfil the relation: $m_r \gg |\mathbf{p}| \gg E$

When bounded by QCD in the weak coupling regime $\alpha_s \sim v$

Scales in bound state		Coulomb interaction
Hard scale: m_r	$\longrightarrow$	m_r
Soft scale: $\lvert\mathbf{p}\rvert$	$\longrightarrow$	$m_r\alpha_s$
Ultrasoft scale: E	$\longrightarrow$	$m_r\alpha_s^2$

Scales are well separated

We can integrate out the hard and soft scales to obtain **pNRQCD**

It describes systems such as: J/ψ, Υ, $\bar{t}t$ near threshold, B_c, etc.

Heavy quarkonium with different masses

The physics of heavy quarks

- Bound states of heavy quarks are naturally NR systems
- We focus in the situation $m_1 \sim m_2 \sim m_r$

Extreme weak coupling regime $mv^2 \gg \Lambda_{\mathrm{QCD}}$

$$\left.\begin{array}{l} \left(i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V^{(0)}(r)\right)\phi(\mathbf{r}) = 0 \\ + \text{ corrections to the potential} \\ + \text{ interaction with other low-energy degrees of freedom} \end{array}\right\} \text{pNRQCD.}$$

- The potentials depend on the matching procedure: on-shell, off-shell in Coulomb and Feynman gauges, with Wilson loops

The physics of heavy quarks

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- The singlet potential: $h_s(\mathbf{r}, \mathbf{p}, \mathbf{P_R}, \mathbf{S}_1, \mathbf{S}_2) = \frac{\mathbf{p}^2}{2\,m_r} + \frac{\mathbf{P_R}^2}{2\,M} + V_s(\mathbf{r}, \mathbf{p}, \mathbf{P_R}, \mathbf{S}_1, \mathbf{S}_2)$

$$V_s = V^{(0)} + \frac{V^{(1,0)}}{m_1} + \frac{V^{(0,1)}}{m_2} + \frac{V^{(2,0)}}{m_1^2} + \frac{V^{(0,2)}}{m_2^2} + \frac{V^{(1,1)}}{m_1 m_2} + \cdots$$

Heavy quarkonium with different masses

The NRQCD potential

The potential

Basis for the potential ($\mathbf{P_R} = 0$):

- static and $1/m$ potentials: $V^{(a,b)} = V^{(a,b)}(r)$
- the $1/m^2$ potential: $V^{(a,b)} = V_{SD}^{(a,b)} + V_{SI}^{(a,b)}$

$$V_{SI}^{(a,b)} = \frac{1}{2}\left\{\mathbf{p}^2, V_{\mathbf{p}^2}^{(a,b)}(r)\right\} + V_{\mathbf{L}^2}^{(a,b)}(r)\frac{\mathbf{L}^2}{r^2} + V_r^{(a,b)}(r)$$

The potentials are invariant under:

- charge conjugation: $\psi \leftrightarrow \chi_c$
- mass exchange: $m_1 \leftrightarrow m_2$

The potential in momentum space

$$\tilde{V}_s \equiv \langle \mathbf{p}' | V_s | \mathbf{p} \rangle$$

- **Static potential and $1/m$ potentials:**

$$\tilde{V}^{(0)} = -\frac{1}{\mathbf{k}^2}\tilde{D}^{(0)}(k), \qquad \tilde{V}^{(1,0)} \equiv \frac{1}{k}\tilde{D}^{(1,0)}(k)$$

- **$1/m^2$ (spin-independent) potential:**

$$\tilde{V}_{SI}^{(2,0)} = \frac{\mathbf{p}^2+\mathbf{p}'^2}{2\mathbf{k}^2}\tilde{D}_{\mathbf{p}^2}^{(2,0)}(k) + \tilde{D}_r^{(2,0)}(k) + \frac{(\mathbf{p}'^2-\mathbf{p}^2)^2}{\mathbf{k}^4}\tilde{D}_{\rm off}^{(2,0)}(k)$$

The $\tilde{D}$-coefficients have $D = d + 1 = 4 + 2\epsilon$ dimensions

It is the $\tilde{D}_{\rm off}$-coefficients that are *scheme-dependent*

Momentum vs. position space

We can relate both bases in terms of the $\tilde{D}$ coefficients.

Example

$$V^{(0)} = \int \frac{d^d q}{(2\pi)^d} e^{-i\mathbf{q}\cdot\mathbf{r}} \tilde{V}^{(0)}(\mathbf{q}) = -C_F \sum_{n=0}^{\infty} \frac{g_B^{2n+2}}{(4\pi)^{2n}} \mathcal{F}_{2-2n\epsilon}(r) \tilde{D}^{(0)}_{n+1}(\epsilon)$$

The $\mathbf{L}^2$ operator in d-dimensions

$$\frac{\mathbf{L}^2}{\mathbf{r}^2} \equiv p^i (\delta^{ij} - \frac{r^i r^j}{\mathbf{r}^2}) p^j$$

which is the usual L^2 in 3-dimensions

The transform of the off-shell potential

it contributes to the **three** position space structures

$$V^{(2,0)}_{\text{off}} = 4 \left(\frac{d^2 g^{(2,0)}_{\text{off}}}{dr^2} - \frac{1}{r} \frac{d g^{(2,0)}_{\text{off}}}{dr} \right) \frac{\mathbf{L}^2}{r^2} - 2 \left\{ \frac{d^2 g^{(2,0)}_{\text{off}}}{dr^2}, \mathbf{p}^2 \right\} + 2[p^i, [p^i, \frac{d^2 g^{(2,0)}_{\text{off}}}{dr^2}]] + h_{\text{off}}(r)$$

Field redefinition

$\tilde{V}_1$ and $\tilde{V}_{\text{off}}$ are related by a field redefinition: mixing

$$h_s = \frac{\mathbf{p}^2}{2m_r} + V^{(0)}(r) + \frac{\delta V_1(r)}{m_r} + \cdots$$

$\Downarrow$ unitary transformation

$$h_s' = \frac{\mathbf{p}^2}{2m_r} + V^{(0)} + \delta\tilde{V}_{\rm FR} + \cdots$$

$$\delta\tilde{V}_{\rm FR} = \langle \mathbf{p}'|\delta V_{\rm FR}|\mathbf{p}\rangle = \frac{1}{2m_r^2}\frac{(\mathbf{p}'^2 - \mathbf{p}^2)^2}{\mathbf{k}^4}\tilde{g}(k)\,,$$

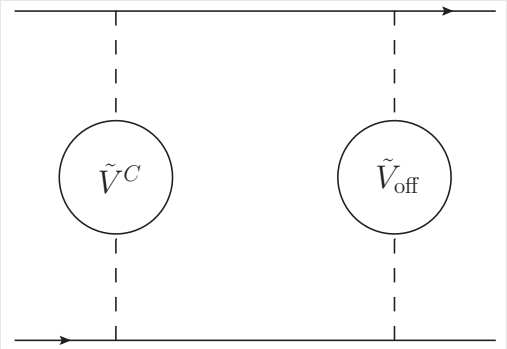
where $\tilde{g}(k) \sim \tilde{g}(k, V_0, \delta V_1)$

we can exchange $1/m_r$ terms by $1/m_r^2$ off-shell contributions

Matching with on-shell Green functions

- on-shell Green functions $\equiv$ S-matrix elements
- asymptotic quarks fulfilling the free EOM order by order

Imperfect cancellation between NRQCD and pNRQCD potential loops

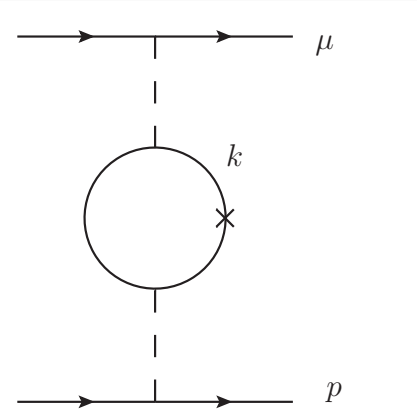


$\Rightarrow$Nontrivial mass dependence in the $1/m$ potential: $\sim \frac{1}{m_1+m_2} = \frac{m_r}{m_1 m_2}$

- As expected: $\tilde{D}^{(a,b)}_{\rm off}(k) = 0$

Matching with off-shell Green functions

- Gauge dependent: Coulomb and Feynman gauges
- Freedom treating energy dependence:
  different choices affect the $1/m$ and $1/m^2$ potentials

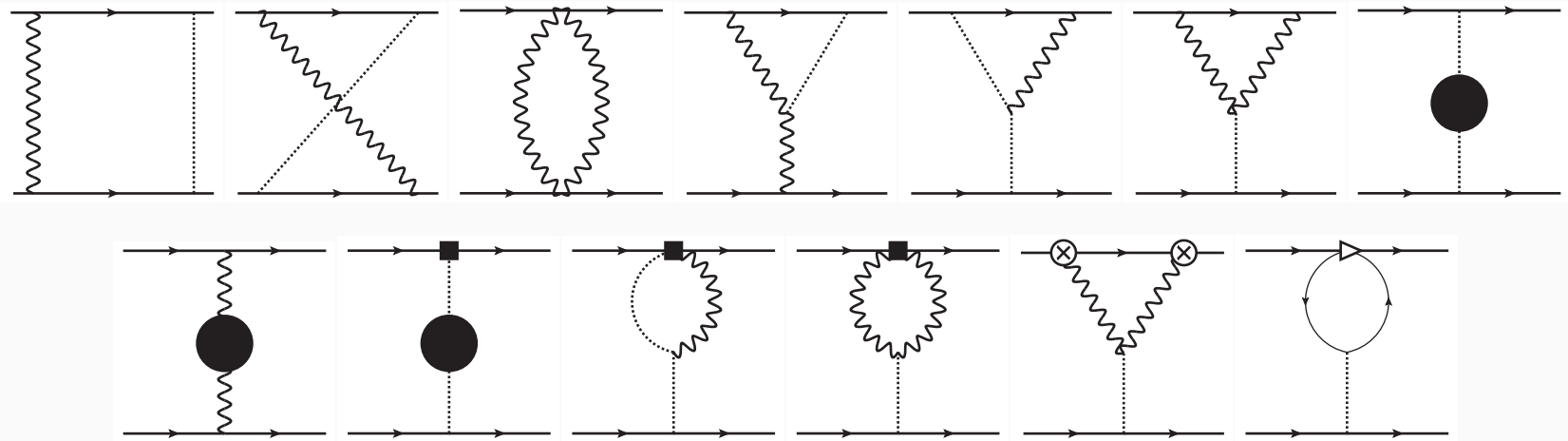


Our choice: the one that exhibits the divergence structure of the on-shell potential most "naturally"

Matching with off-shell Green functions

The $1/m^2$ potential in the Coulomb gauge:

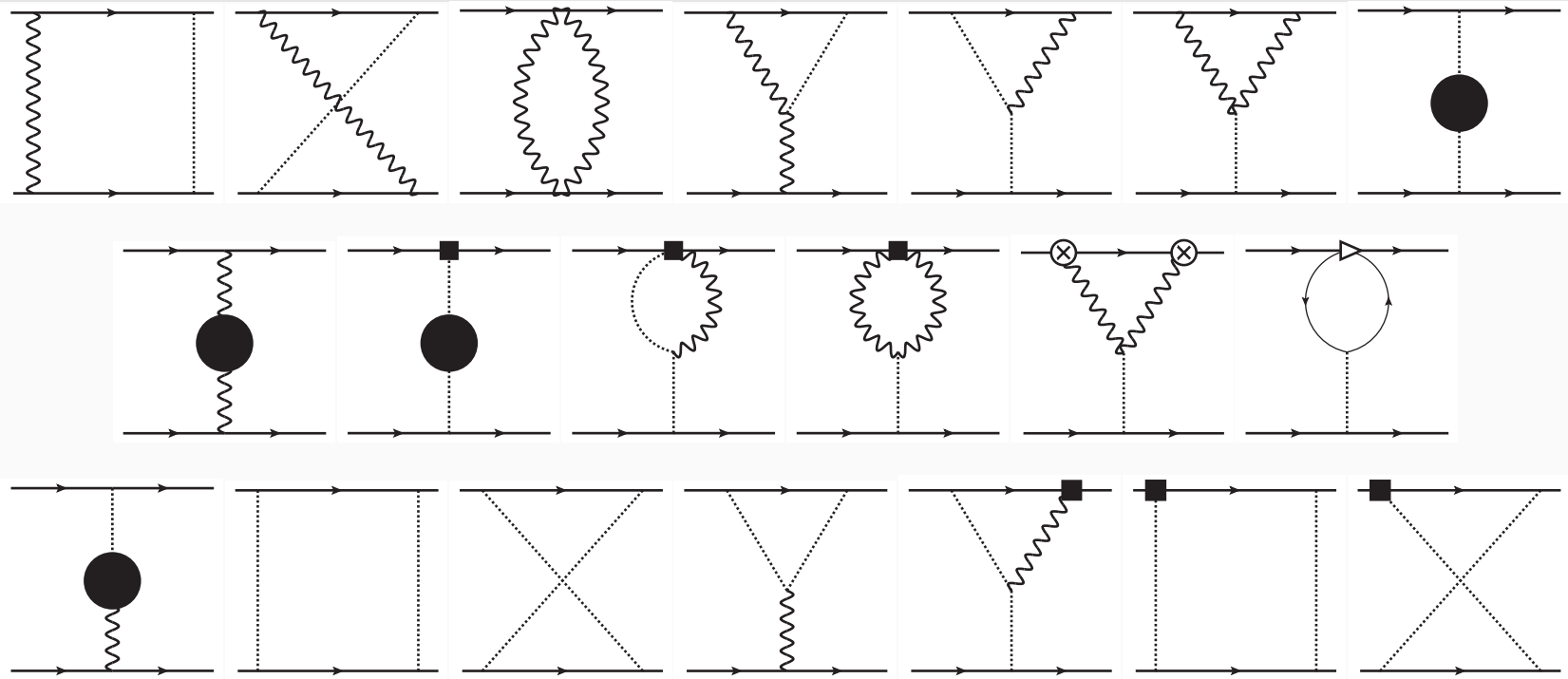
- Less diagrams
- Efficient way to perform the computation
- Natural choice of treating energies



Matching with off-shell Green functions

The $1/m^2$ potential in the Feynman gauge:

- More diagrams
- Heavy dependence on the energies
- Computations are easily automated



Matching with Wilson loops

$$W_\square \equiv \mathrm{P}\exp\left\{-ig\oint_{r\times T_W} dz^\mu A_\mu(z)\right\}$$

- Green functions in **position space** and setting the **time** of the quark and anti-quark **equal**
- Gauge independent

The "quasi-static" energy E_s

$$\frac{\mathbf{p}^2}{2\,m_{\rm r}}+\frac{\mathbf{P}_{\mathbf{R}}^2}{2\,M}+E^{(0)}+\frac{E^{(1,0)}}{m_1}+\frac{E^{(0,1)}}{m_2}+\frac{E^{(2,0)}}{m_1^2}+\frac{E^{(0,2)}}{m_2^2}+\frac{E^{(1,1)}}{m_1\,m_2}+\cdots$$

Example

$$E^{(1,1)}_{\mathbf{L}^2}(r)=\frac{i}{d-1}\left(\delta^{ij}-d\frac{r^i r^j}{r^2}\right)\lim_{T\to\infty}\int_0^T dt\,t^2\langle\langle g\mathbf{E}_1^i(t)g\mathbf{E}_2^j(0)\rangle\rangle_c$$

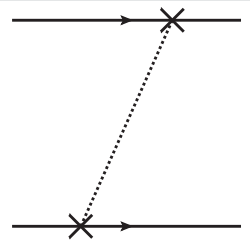
where $\langle\langle\ldots\rangle\rangle\equiv\langle\ldots W_\square\rangle/\langle W_\square\rangle$

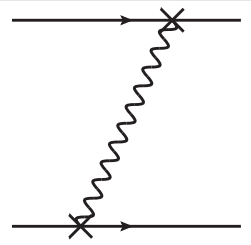
The $1/m^2$ potential with Wilson loops

- We obtain: $V_{s,W}(r) = E_s(r)|_{\rm soft}$

- Feynman rules for chromoelectric insertions

Example:



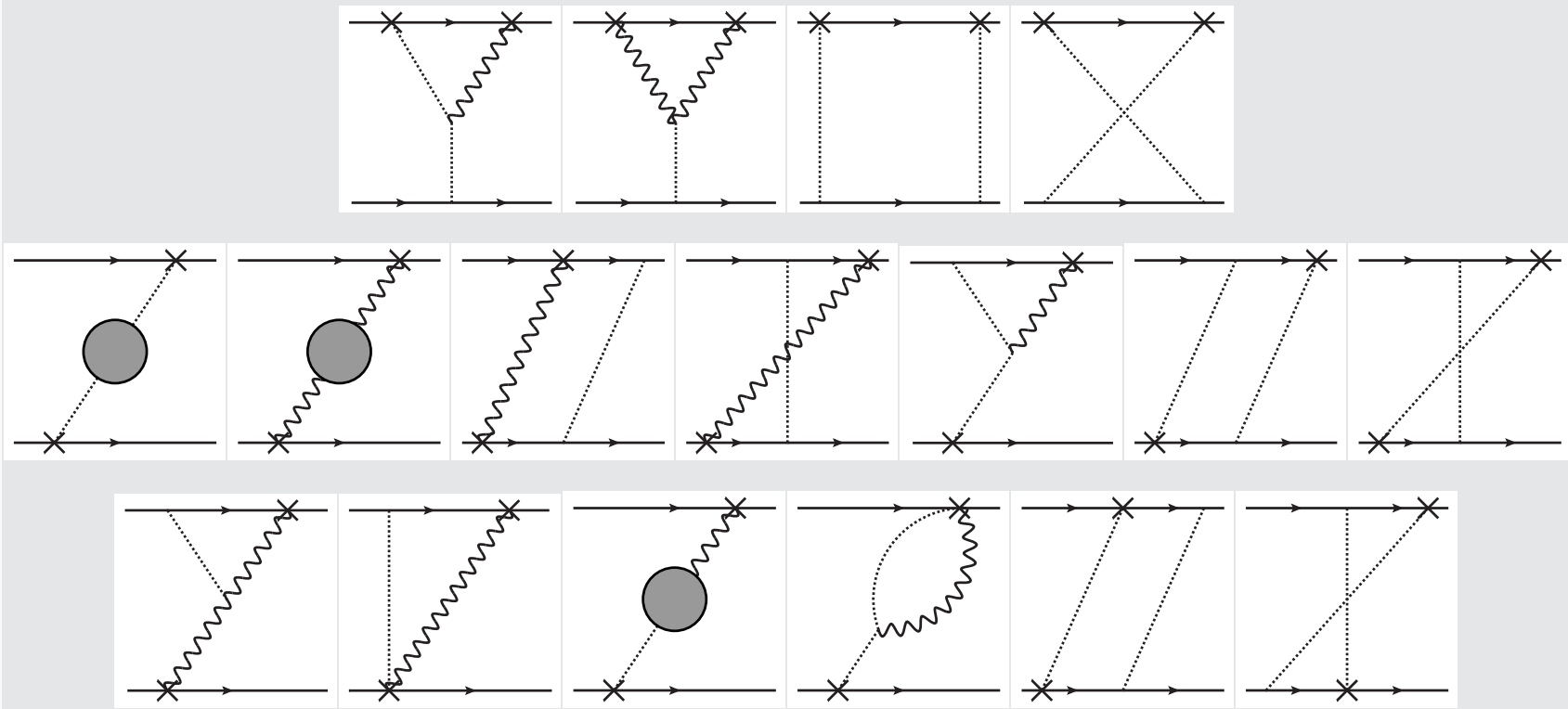


$$V^{(1,1)}_{\mathbf{L}^2,W}(r) = \frac{i g_B^2 C_F}{(d-1)} \left(\delta^{ij} - d\frac{r^i r^j}{r^2}\right) \lim_{T\to\infty} \int_0^T dt\ t^2 \int \frac{d^D k}{(2\pi)^D} e^{i(k_0 t - \mathbf{k}\mathbf{r})} \frac{i k_0^2}{k^2 + i0} P_{ij}(\mathbf{k})$$

The $1/m^2$ potential with Wilson loops

- We obtain: $V_{s,W}(r) = E_s(r)|_{\rm soft}$

Only need compute off-shell potential



The $1/m$ potential

- Profit from previous computation: on-shell, for equal masses and to $\mathcal{O}(\epsilon)$ (Kniehl et al.)
- The $\mathcal{O}(\frac{1}{m^2})$ scheme difference is off-shell: $\tilde{V}_{s,X} = \tilde{V}_{s,\mathrm{on-shell}} + \delta \tilde{V}_X^{(2)}$

We exploit the relation:

$$\frac{\tilde{V}_X^{(1,0)}}{m} + \frac{\tilde{V}_X^{(0,1)}}{m} + \delta \tilde{V}_X^{(1)}\Big|_{m=m_1=m_2} = \left[\frac{\tilde{V}_{\mathrm{on-shell}}^{(1,0)}}{m} + \frac{\tilde{V}_{\mathrm{on-shell}}^{(0,1)}}{m}\right]_{m=m_1=m_2}$$

- We compute the $\mathcal{O}(\alpha_s^3/m)$ potential for different masses in **all previous schemes** to $\mathcal{O}(\epsilon)$

The renormalised potential

The (singlet) heavy quarkonium self-energy

$$\Sigma_B(1-\text{loop}) = -g_B^2 C_F V_A^2 (1+\epsilon) \frac{\Gamma(2+\epsilon)\Gamma(-3-2\epsilon)}{\pi^{2+\epsilon}} \mathbf{r}\,(h_s - E + \Delta V)^{3+2\epsilon}\mathbf{r}$$

where $\Delta V \equiv V_o^{(0)} - V^{(0)}$

- Its UV divergences (δV_s) **cancel** the US divegrences of the soft potential:

$$V_s^{\overline{\text{MS}}} + \delta V_s = V_s$$

- $V_s^{\overline{\text{MS}}}$ produces finite physical results
- δV_s is ambiguous: we choose it so that the 4-dimensional potentials are finite

Poincaré invariance

Poincaré invariance constrains

for potentials with an exact expansion on the masses:

$$2V^{(2,0)}_{\mathbf{L}^2} - V^{(1,1)}_{\mathbf{L}^2} + \frac{r}{2}\frac{dV^{(0)}(r)}{dr} = 0$$

$$-4V^{(2,0)}_{\mathbf{p}^2} + 2V^{(1,1)}_{\mathbf{p}^2} - V^{(0)}(r) + r\frac{dV^{(0)}(r)}{dr} = 0$$

- Our bare and renormalized potentials fulfil them
- They are **not affected by field redefinitions**: they produce $\delta V^{(1,1)}_{\mathbf{L}^2} = 2\delta V^{(2,0)}_{\mathbf{L}^2}$, $\delta V^{(1,1)}_{\mathbf{p}^2} = 2\delta V^{(2,0)}_{\mathbf{p}^2}$

Example of potential result

The $1/m_1^2$ renormalized potential with Wilson loops :

$$
\begin{aligned}
V_{r,W}^{(2,0),\overline{\mathrm{MS}}}(r) = \frac{C_F\alpha_s}{8}\Bigg(c_D^{(1)} + \frac{\alpha_s}{\pi}\Bigg\{-\frac{5}{9}\left(c_D^{(1)} + c_1^{hl\,(1)}\right)T_F n_f + \left(\frac{13}{36}c_F^{(1)\,2} + \frac{8}{3}\right)C_A \\
+ \left(\left(\frac{4}{3} + \frac{5}{6}c_F^{(1)\,2}\right)C_A - \frac{2}{3}\left(c_D^{(1)} + c_1^{hl\,(1)}\right)T_F n_f\right)\ln(\nu)\Bigg\}\Bigg)4\pi\delta^{(3)}(\mathbf{r}) \\
+ \frac{C_F\alpha_s^2}{8\pi}\left\{\left(\frac{4}{3} + \frac{5}{6}c_F^{(1)\,2}\right)C_A - \frac{2}{3}\left(c_D^{(1)} + c_1^{hl\,(1)}\right)T_F n_f\right\}\mathrm{reg}\frac{1}{r^3}, \\
V_{\mathbf{L}^2,W}^{(2,0),\overline{\mathrm{MS}}}(r) = \frac{C_A C_F\alpha_s^2}{4\pi r}\left(\frac{11}{3} - \frac{8}{3}\ln\left(r\nu e^{\gamma_E}\right)\right), \\
V_{\mathbf{p}^2,W}^{(2,0),\overline{\mathrm{MS}}}(r) = -\frac{C_A C_F\alpha_s^2}{\pi r}\left(\frac{2}{3} + \frac{1}{3}\ln\left(r\nu e^{\gamma_E}\right)\right)
\end{aligned}
$$

Heavy quarkonium with different masses

The N^3LO spectrum

The B_c spectrum

US energy correction

$$\delta E_{nl}^{US} = -E_n^C \frac{\alpha_s^3}{\pi}\left[\frac{2}{3}C_F^3 L_{nl}^E + \frac{1}{3}C_A\left(L_\nu - L_{US} + \frac{5}{6}\right)\left(\frac{C_A^2}{2} + \frac{4C_A C_F}{(2l+1)n} + 2C_F^2\left(\frac{8}{(2l+1)n} - \frac{1}{n^2}\right)\right) + \frac{8\delta_{l0}}{3n}C_F^2\left(C_F - \frac{C_A}{2}\right)\left(L_\nu - L_{US} + \frac{5}{6}\right)\right],$$

where L_n^E is the Bethe logarithm

Energy correction associated to the static potential

$$\delta E(n,l,s,j)\Big|_{V^{(0)}} = E_n^C\left(1 + \frac{\alpha_s}{\pi}P_1(L_\nu) + \left(\frac{\alpha_s}{\pi}\right)^2 P_2^c(L_\nu) + \left(\frac{\alpha_s}{\pi}\right)^3 P_3^c(L_\nu)\right),$$

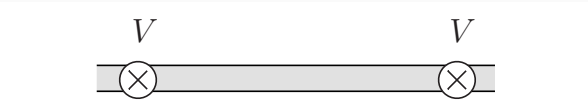
(Kiyo, Sumino)

The B_c spectrum

Energy correction associated to the relativistic potentials

$$\delta E(n,l,s,j) = E_n^C \left[\left(\frac{\alpha_s}{\pi}\right)^2 c_2^{\rm nc} + \left(\frac{\alpha_s}{\pi}\right)^3 c_3^{\rm nc} \right]$$

- $c_3^{\rm nc}$ involves the use of perturbation theory



$$\delta E_{nlj}^{V\times V} = \langle \psi_{nlj} | V \frac{1}{(E_n^C - h)'} V | \psi_{nlj} \rangle$$
$$= \int \mathrm{d}\mathbf{r_2} \mathrm{d}\mathbf{r_1} \psi_{nlj}^*(\mathbf{r_2}) V(\mathbf{r_2}) G'_{nl}(\mathbf{r_1},\mathbf{r_2}) V(\mathbf{r_1}) \psi_{nlj}(\mathbf{r_1})$$

The B_c spectrum

Complete N^{3}LO spectrum for the B_c

$$E(n,l,s,j) = E_n^C \left(1 + \frac{\alpha_s}{\pi} P_1(L_\nu) + \left(\frac{\alpha_s}{\pi}\right)^2 P_2(L_\nu) + \left(\frac{\alpha_s}{\pi}\right)^3 P_3(L_\nu)\right),$$

$$P_1(L_\nu) = \beta_0 L_\nu + \frac{a_1}{2},$$

$$P_2(L_\nu) = \frac{3}{4}\beta_0^2 L_\nu^2 + \left(-\frac{\beta_0^2}{2} + \frac{\beta_1}{4} + \frac{3\beta_0 a_1}{4}\right) L_\nu + c_2,$$

$$P_3(L_\nu) = \frac{1}{2}\beta_0^3 L_\nu^3 + \left(-\frac{7\beta_0^3}{8} + \frac{7\beta_0\beta_1}{16} + \frac{3}{4}\beta_0^2 a_1\right) L_\nu^2 + \left(\frac{\beta_0^3}{4} - \frac{\beta_0\beta_1}{4} + \frac{\beta_2}{16} - \frac{3}{8}\beta_0^2 a_1 + 2\beta_0 c_2 + \frac{3\beta_1 a_1}{16}\right) L_\nu + c_3$$

where $c_i = c_i^{\mathrm{c}} + c_i^{\mathrm{nc}}$

Final remarks

Summary of the results

- we develop the **N^{3}LO potential** in pNRQCD for **different masses**
- The potentials obtained are valid for $\mathbf{mv} \gg \mathbf{\Lambda}_{\rm QCD}$

The $\mathcal{O}(\alpha_s/m^2)$ potential in different matching schemes

- all schemes are feasible
- they are related by a field redefinition

The $\mathcal{O}(\alpha_s^3/m)$ potential in different matching schemes

- no $C_F^2 T_F n_F$ in off-shell scheme
- no $C_F^2 T_F n_F$ or $C_F^2 C_A$ in Wilson loop scheme

- The US contribution is valid for $\mathbf{mv^2} \gg \mathbf{\Lambda}_{\rm QCD}$

- We computed the full N^{3}LO spectrum for different masses

Future perspectives

- Study the B_c spectrum and decays
  - -obtain the charm mass
- Compute US contribution with Wilson loops
  - -comparison with lattice predictions
- Explicitly compute $1/m$ potential with Wilson loops
  - -few color structures
  - -check on previous computations
- Compute higher order contributions: $\mathcal{O}(m_r \alpha_s^6 \ln(\alpha_s))$

Thank you!

The spin-dependent and the static potentials

The static potential

- gauge independent order-by-order
- already computed to $\mathcal{O}(\alpha^4)$

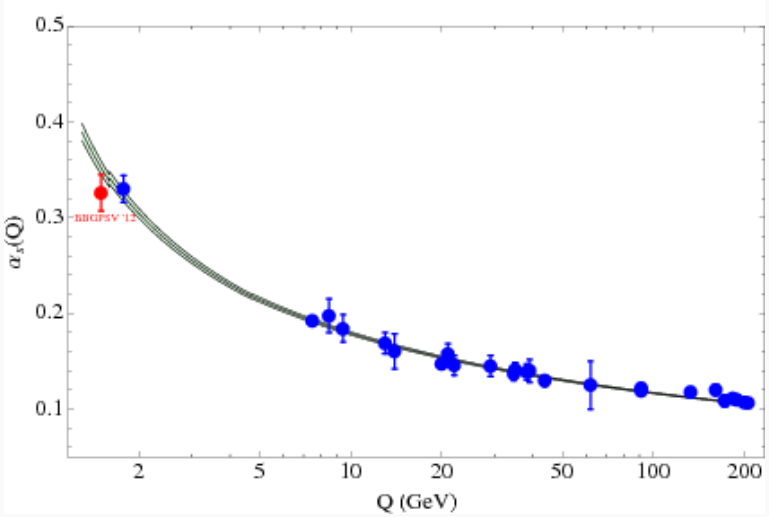
The spin-dependent potential

- renormalized result for different masses computed in 1986 by Pantaleone et al.
- bare potential needs prescription for the D-dimensional Levi-Civita symbol
- the renormalized potential is enough to compute the spectrum

The problematic of QCD

Quantum Chromodynamics describes the interaction of quarks and gluons

- It is asymptotically free: predictions can be made at high energies
- The strong interaction grows at long distances: quarks are confined in hadrons at low energies



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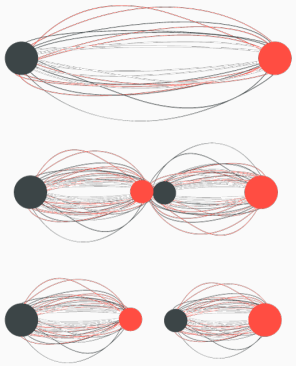
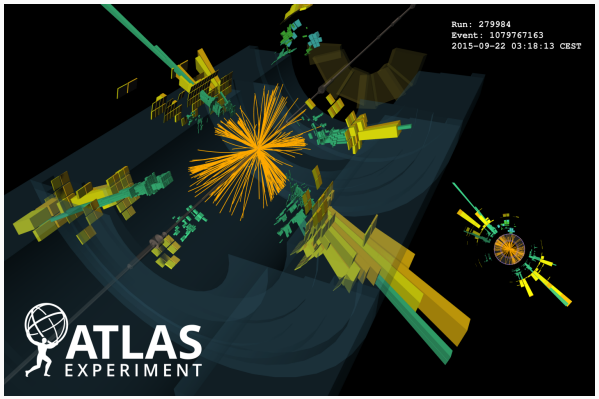




We exploit the EFT tools to overcome our limitation to describe **low energy QCD**

The spectrum functions

$$\xi^{\rm SD}_{\rm FFF} = \frac{2}{3n}\frac{m_r^2}{m_1 m_2}\left\{\frac{-3(1-\delta_{l0})}{l(l+1)(2l+1)}\left(D_S + X_{\rm LS} + \frac{m_1}{m_2}X_{\rm LS_2} + \frac{m_2}{m_1}X_{\rm LS_1}\right)\right.$$
$$\left. - \; 4\mathcal{S}_{12}\delta_{l0}\left[2 + 3\frac{m_1 m_2}{m_2^2 - m_1^2}\ln\left(\frac{m_1^2}{m_2^2}\right)\right]\right\},$$
$$\xi^{\rm SD}_{\rm FFnf} = \frac{2m_r^2}{9n^2 m_1 m_2}\left\{\frac{1-\delta_{l0}}{l(l+1)(2l+1)}\left[2n(4\mathcal{S}_{12} - D_S)\right.\right.$$
$$+ 6\Big(D_S + \frac{m_2}{m_1}X_{\rm LS_1} + \frac{m_1}{m_2}X_{\rm LS_2} + 2X_{\rm LS}\Big)\left(\frac{3n}{2l+1} + \frac{n}{2l(l+1)(2l+1)} + l + \frac{1}{2}\right.$$
$$\left.\left. + 2n\Big\{S_1(l+n) + S_1(2l-1) - 2S_1(2l+1) - l(\Sigma_1^{(k)} + \Sigma_1^{(m)}) + n\Sigma_b - \Sigma_1^{(m)} + \frac{1}{6}\Big\}\right)\right]$$
$$\left. + 8\delta_{l0}\mathcal{S}_{12}\left[1 + 4n\left(\frac{11}{12} - \frac{1}{n} - S_1(n-1) - S_1(n) + nS_2(n)\right)\right]\right\},$$

Logarithmic functions

$E_n^C = -\frac{C_F^2 \alpha_s^2 m_r}{2n^2}$

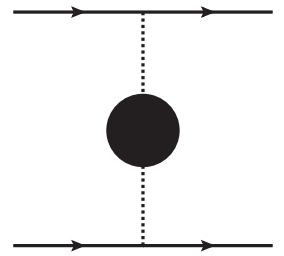
$L_\nu = \ln\left(\frac{n\nu}{2m_r C_F \alpha_s}\right) + S_1(n+l)$ $L_{US} = \ln\left(\frac{C_F \alpha_s n}{2}\right) + S_1(n+l)$

$L_H = \ln\left(\frac{n}{C_F \alpha_s}\right) + S_1(n+l)$

The Bethe logarithm is defined as

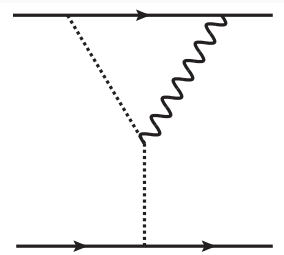
$$L_n^E = \frac{1}{(C_F\alpha_s)^2 E_n^C} \int_0^\infty \frac{d^3k}{(2\pi)^3} |\langle \mathbf{r} \rangle_{\mathbf{k}n}|^2 \left(E_n^C - \frac{k^2}{2m_r}\right)^3 \ln \frac{E_1^C}{E_n^C - \frac{k^2}{2m_r}}$$

Energy dependence in the Coulomb gauge



$$= -\frac{ig_B^4}{3}\frac{k_0^2\mathbf{k}^{2\epsilon-4}\csc(\pi\epsilon)}{2^{4\epsilon+4}\pi^{\epsilon+\frac{1}{2}}\Gamma\left(\epsilon+\frac{5}{2}\right)}\Big[3C_F T_F n_f \epsilon(1+\epsilon)$$

$$-\frac{C_A C_F}{4}\left((\epsilon+1)(\epsilon(56\epsilon+121)+60)\right) - \frac{5\Gamma\left(\epsilon+\frac{3}{2}\right)^2}{\sqrt{\pi}\Gamma\left(2\epsilon+\frac{5}{2}\right)}4^{\epsilon+1}(\epsilon+1)(2\epsilon+3)(4\epsilon+3)\Big]$$



$$= -\frac{ig_B^4}{3m_i}C_A C_F\frac{\mathbf{k}^{2\epsilon-4}(\epsilon+1)(2\epsilon+1)\Gamma(1-\epsilon)\Gamma(2\epsilon)}{4^{2\epsilon+1}\pi^{\epsilon+\frac{3}{2}}\Gamma\left(2\epsilon+\frac{3}{2}\right)}$$

$$\left[E_i\left(\mathbf{k}^2-(\mathbf{p}'^2-\mathbf{p}^2)^2\right)+E_i'\left(\mathbf{k}^2+(\mathbf{p}'^2-\mathbf{p}^2)^2\right)\right]$$