The bottom-quark mass from non-relativistic sum rules at NNNLO

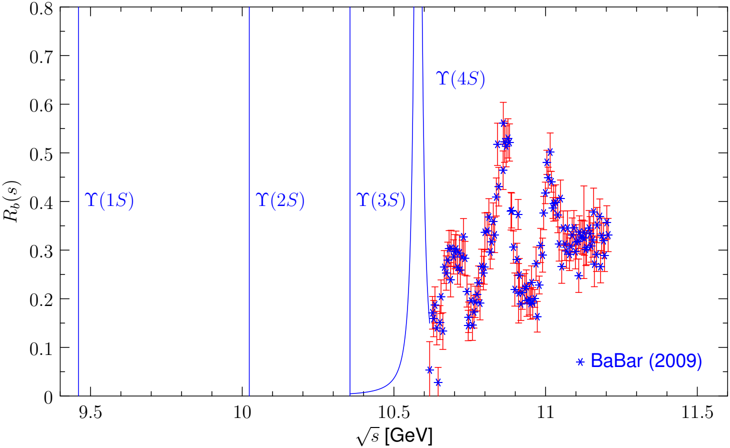
Jan Piclum

based on work in collaboration with

M. Beneke, A. Maier, T. Rauh

Nucl. Phys. B **891** (2015) 42 [arXiv:1411.3132]; arXiv:1601.02949

$e^+e^- \to b\bar{b}$ Near Threshold: Experiment



$$R_b(s) = \frac{\sigma(e^+e^- \to b\bar{b}\,X)}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

Non-relativistic Sum Rules

- consider moments of the cross section

$$\mathcal{M}_n = \int_0^\infty \mathrm{d}s \, \frac{R_b(s)}{s^{n+1}} \, ,$$

- determine bottom-quark mass from $\mathcal{M}_n^{\text{th}} \stackrel{!}{=} \mathcal{M}_n^{\text{exp}}$

Non-relativistic Sum Rules

- consider moments of the cross section

$$\mathcal{M}_n = \int_0^\infty \mathrm{d}s \frac{R_b(s)}{s^{n+1}}, \qquad \frac{1}{s^{n+1}} = \frac{1}{(2m_b + E)^{2(n+1)}} \approx \frac{e^{-(n+1)E/m_b}}{(4m_b^2)^{n+1}}$$

- determine bottom-quark mass from $\mathcal{M}_n^{\text{th}} \overset{!}{=} \mathcal{M}_n^{\text{exp}}$

use large $n \sim 10$:

+ better data
+ more sensitive to m_b
− more sensitive to non-perturbative effects

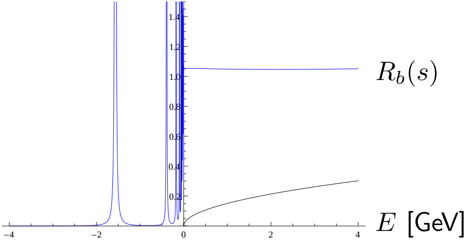
⇝ requires accurate description of non-relativistic dynamics near threshold, including bound states

$e^+e^- \to b\bar{b}$ Near Threshold: Theory

relative velocity of quark-antiquark pair is small:

$$\sqrt{s} = E + 2m_b \approx 2m_b \quad \Rightarrow \quad v = \sqrt{\frac{E}{m_b}} \sim 1/\sqrt{n}$$

- multi-scale problem: mass m_b, momentum $m_b v$, energy $m_b v^2$
- perturbation theory breaks down due to terms proportional to $\frac{\alpha_s}{v}$
  $\rightsquigarrow$ Coulomb resummation
- formation of bound states below threshold



Effective Theory Setup

[Thacker, Lepage; Lepage, Magnea, Nakhleh, Magnea, Hornbostel; Bodwin, Braaten, Lepage]
[Pineda, Soto; Beneke, Signer, Smirnov; Brambilla, Pineda, Soto, Vairo]

scale hierarchy: $m_b \gg m_b v \gg m_b v^2 \gg \Lambda_{\mathrm{QCD}}$, $v \sim 1/\sqrt{n}$

QCD full theory

[Thacker, Lepage; Lepage, Magnea, Nakhleh, Magnea, Hornbostel; Bodwin, Braaten, Lepage]
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QCD — full theory

↓ integrate out hard modes: $k^0 \sim k^i \sim m_b$
hard subgraphs become point-like vertices
$\rightsquigarrow$ hard matching coefficients

NRQCD

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QCD — full theory

↓

integrate out hard modes: $k^0 \sim k^i \sim m_b$
hard subgraphs become point-like vertices
$\rightsquigarrow$ hard matching coefficients

NRQCD

↓

integrate out soft modes: $k^0 \sim k^i \sim m_b v$
soft subgraphs become instantaneous, non-local vertices
$\rightsquigarrow$ potentials

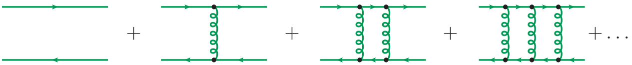
pNRQCD — contains potential quarks and ultrasoft gluons

Potential Non-Relativistic QCD

[Pineda, Soto 1997; Beneke, Signer, Smirnov 1999; Brambilla et al. 1999]

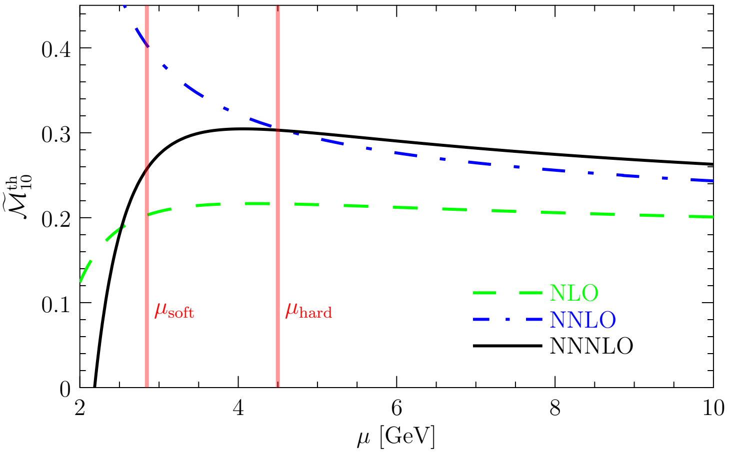
$$\mathcal{L}_{\mathrm{pNRQCD}} = \psi^\dagger \left(i\partial^0 + \frac{\partial^2}{2m_b} + g_s A^0(t,\mathbf{0}) + \ldots \right)\psi$$
$$+ \int \mathrm{d}^3\mathbf{r}\, [\psi^\dagger\psi](x+\mathbf{r}) \left(-\frac{\alpha_s C_F}{r} + \delta V(\mathbf{r}) \right) [\chi^\dagger\chi](x)$$
$$- g_s \psi^\dagger \mathbf{x}\cdot\mathbf{E}(t,\mathbf{0})\psi + \text{antiquark terms} + \ldots$$

- equation of motion is Coulomb Schrödinger equation
- quark-antiquark propagator contains exchange of Coulomb gluons:



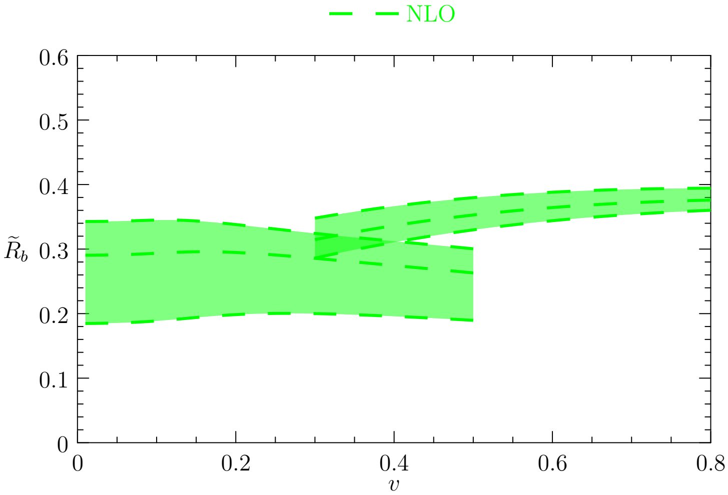
- poles of propagator correspond to bound states

Choice of Scale

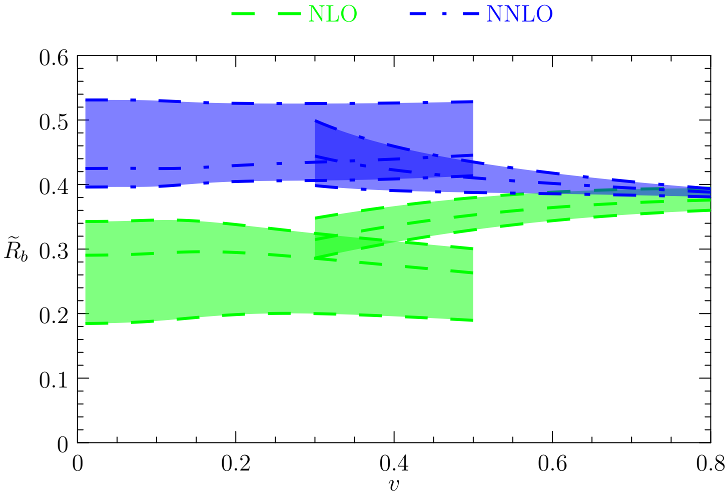


use $\mu = m_b$ as central value and vary scale between 3 and 10 GeV

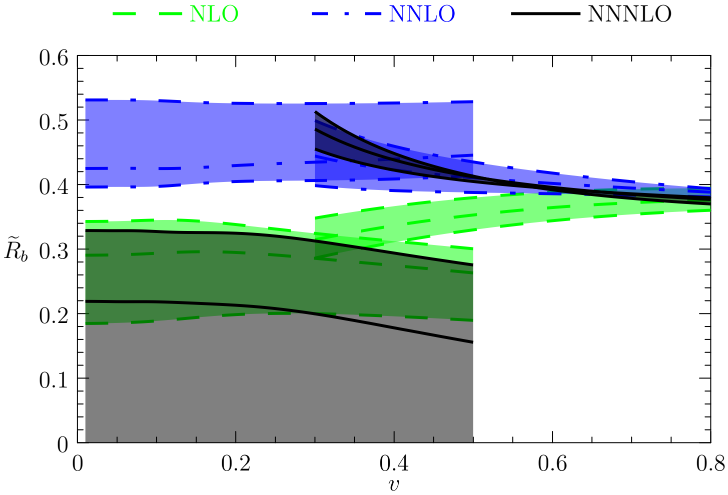
Continuum Contribution



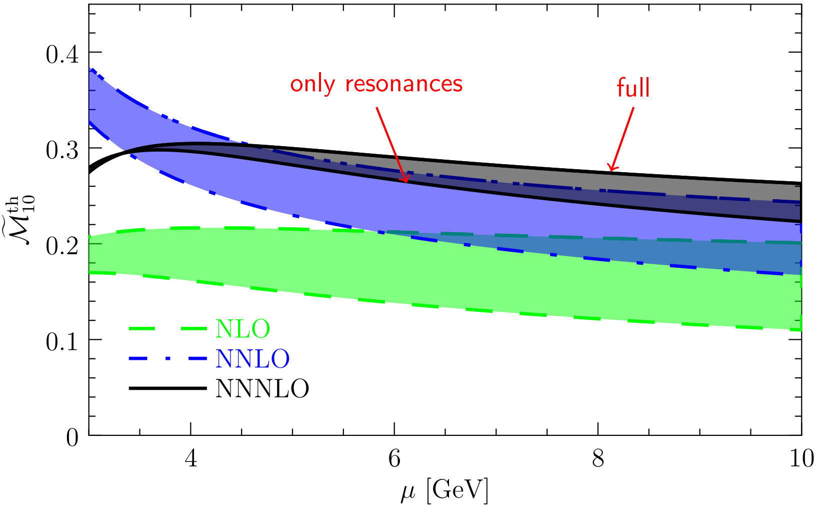
Continuum Contribution



Continuum Contribution

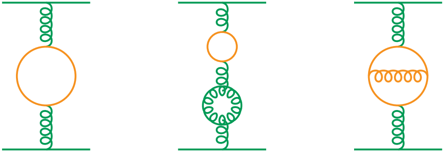


Continuum Contribution — Impact on $\mathcal{M}_{10}^{\text{th}}$



Charm Mass Contribution

- charm-quark mass is soft: $m_c \sim m_b v \rightsquigarrow$ correction to potential



- known to NNLO [Melles]
- expect significant impact on m_b: $\delta m_b \sim -30$ MeV [Hoang]

we find: $\delta m_b \sim -3$ MeV at NNLO

QQbar_threshold

energy levels, wave function, and cross section can be obtained with the public program QQbar_threshold

https://qqbarthreshold.hepforge.org/

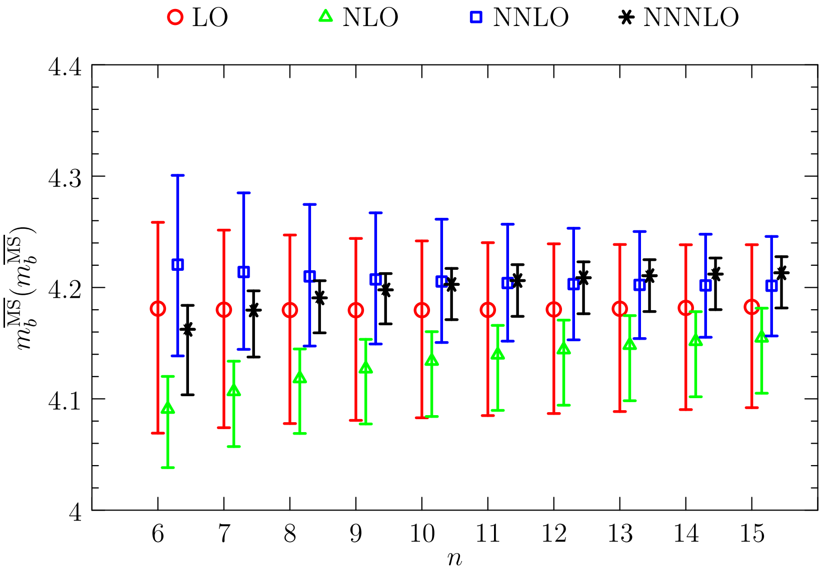
- C++ library
- Mathematica interface
- numerous options for detailed analyses

```
Needs["QQbarThreshold`"];

With[
   {mbPS = 4.5, scale = 4.5},
   E1 = BBbarEnergyLevel[ 1, scale, mbPS, "LO" ];
];

Print[E1];
```

```
#include <iostream>
#include "QQbar_threshold/energy_levels.hpp"
using namespace QQbar_threshold;
int main(){
  const double mb_PS = 4.5;
  double E_1 = bbbar_energy_level( 1, mb_PS, mb_PS, LO );
  std::cout << E_1 << '\n';
}
```

[Beneke, Kiyo, Maier, JP]

Bottom Quark Mass



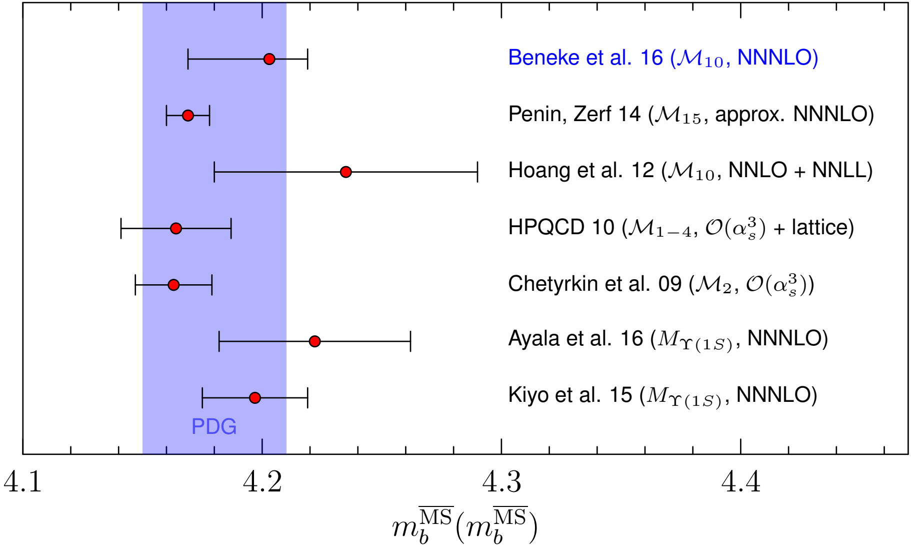
Uncertainty

$$m_b^{\mathsf{PS}}(2\,\mathsf{GeV}) = \Big[4.532^{+0.002}_{-0.035}(\mu) \pm 0.010(\alpha_s)^{+0.003}_{-0}(\mathsf{res}) \pm 0.001(\mathsf{conv})$$
$$\pm 0.002(\mathsf{charm})^{+0.007}_{-0.013}(n) \pm 0.003(\mathsf{exp})\Big]\ \mathsf{GeV}$$
$$= 4.532^{+0.013}_{-0.039}\ \mathsf{GeV}$$

uncertainty estimate includes:

- scale variation: vary $\mu \in [3, 10]$ GeV
- strong coupling: $\alpha_s(M_Z) = 0.1184 \pm 0.0010$
- number of resonances: difference between 6 and 4 resonances
- dependence on $\mu_f = 2$ GeV: vary $\mu_f \in [1, 3]$ GeV
- charm-mass contribution: 100% uncertainty
- choice of $n = 10$: vary $n \in [8, 12]$
- experimental uncertainty

Comparison


Beneke et al. 16 ($\mathcal{M}_{10}$, NNNLO)
Penin, Zerf 14 ($\mathcal{M}_{15}$, approx. NNNLO)
Hoang et al. 12 ($\mathcal{M}_{10}$, NNLO + NNLL)
HPQCD 10 ($\mathcal{M}_{1-4}$, $\mathcal{O}(\alpha_s^3)$ + lattice)
Chetyrkin et al. 09 ($\mathcal{M}_2$, $\mathcal{O}(\alpha_s^3)$)
Ayala et al. 16 ($M_{\Upsilon(1S)}$, NNNLO)
Kiyo et al. 15 ($M_{\Upsilon(1S)}$, NNNLO)
PDG
4.1
4.2
4.3
4.4
$m_b^{\overline{\mathrm{MS}}}(m_b^{\overline{\mathrm{MS}}})$

Summary

- precise bottom-quark mass from non-relativistic sum rules at NNNLO

$$m_b^{\mathsf{PS}}(2\ \mathrm{GeV}) = 4.532^{+0.013}_{-0.039}\ \mathrm{GeV}$$

$$m_b^{\overline{\mathsf{MS}}}(m_b^{\overline{\mathsf{MS}}}) = 4.203^{+0.016}_{-0.034}\ \mathrm{GeV}$$

- uncertainty dominated by theory

further investigation needed for:

- charm-mass correction
- behaviour of continuum

Moments

- experimental:

$$\mathcal{M}_n^{\text{exp}} = 9\pi \sum_{N=1}^{4} \frac{1}{\alpha(M_{\Upsilon(NS)})^2} \frac{\Gamma_{\Upsilon(NS)\to l^+l^-}}{M_{\Upsilon(NS)}^{2n+1}} + \int_{s_{\text{cont}}}^{\infty} ds\, \frac{R_b(s)}{s^{n+1}}$$

- theoretical:

$$\mathcal{M}_n^{\text{th}} = \frac{12\pi^2 N_c e_b^2}{m_b^2} \sum_{N=1}^{\infty} \frac{Z_N}{(2m_b + E_N)^{2n+1}} + \int_{4m_b^2}^{\infty} ds\, \frac{R_b(s)}{s^{n+1}}$$

Low vs. High Moments