Dyson–Schwinger approach to Hamiltonian Quantum Chromodynamics

D. Campagnari  H. Reinhardt  J. Heffner  P. Vastag  E. Ebadati
Institut für Theoretische Physik, Eberhard-Karls-Universität Tübingen

M. Huber
Institut für Physik, Karl-Franzens-Universität Graz

Quark Confinement & Hadron Spectrum XII
Thessaloniki, 29th August 2016
Outline

1 Motivation

2 Canonical Dyson–Schwinger Equations
   - Hamiltonian approach to QCD
   - Vacuum wave functional and cDSEs

3 Variational solution for the Yang–Mills sector
   - Gluon Propagator
   - Ghost-Gluon Vertex
   - Three-Gluon Vertex

4 Fermions

5 Conclusions
### Motivation

#### Yang–Mills sector
Results hitherto achieved in Coulomb gauge
- IR behaviour of propagators
- linearly rising potential between static charges
- Polyakov loop potential and deconfinement phase transition (see talk by H. Reinhardt)

#### Fermion Sector
Inclusion of quarks should explain
- constituent quark mass
- chiral condensate
- QCD phase diagram
(Hamiltonian Approach to) Coulomb gauge

The good...

- well-defined (lattice) Hamiltonian
- analogy to standard quantum mechanics
- appeals to physical intuition

... the bad...

- non-covariant
- non-perturbative renormalization not clear

... and the ugly

- calculations can become very involved
Hamilton operator of QCD in Coulomb gauge

Steps
- start from canonically quantized theory in temporal gauge $A_0 = 0$
- eliminate longitudinal degrees of freedom by means of Gauss’s law

$$H = \frac{1}{2} \left[ -\mathcal{J}_A^{-1} \frac{\delta}{\delta A} \mathcal{J}_A \frac{\delta}{\delta A} + B^2 \right] + \psi^\dagger (-i \alpha \cdot \nabla + \beta m) \psi$$
- $B$ is the non-abelian magnetic field
- $\rho^a = \psi^\dagger t^a \psi - i f^{abc} A^b \frac{\delta}{\delta A^c}$ is the colour charge density
- $F_A = (-\nabla \cdot D)^{-1} (-\nabla^2) (-\nabla \cdot D)^{-1}$ is the Coulomb kernel
Static Green’s functions

\[ \langle K \rangle = \int \mathcal{D}A \mathcal{J}_A \mathcal{D}\xi \mathcal{D}\xi^\dagger \Psi^*[A, \xi, \xi^\dagger] K \Psi[A, \xi, \xi^\dagger] \]

- \( \mathcal{J}_A = \text{Det}(\mathcal{D} - \partial \cdot \mathcal{D}) \) \((\text{with } \mathcal{D} = \partial + A)\) is the Faddeev–Popov determinant of Coulomb gauge
- integration over transverse field configurations
- \( \xi \) and \( \xi^\dagger \) are Grassmann fields
- \( \Psi \) is the vacuum wave functional

The expectation values of products of fields

\[ \langle AA \rangle, \quad \langle \xi \xi^\dagger \rangle, \quad \langle \xi \xi^\dagger A \rangle, \ldots \]

are the static (equal-time) Green functions.
Vacuum wave functional

Formal equivalence to Lagrangian approach

Writing the vacuum wave functional as

$$|\Psi[A, \xi, \xi^\dagger]|^2 =: \exp\left\{-S[A, \xi, \xi^\dagger]\right\}$$

we have an Euclidean QFT defined by an “action” $S[A, \xi, \xi^\dagger]$.

Expansion of the vacuum wave functional

$$S[A, \xi, \xi^\dagger] = \frac{1}{2} \gamma_2 A^2 + \frac{1}{3!} \gamma_3 A^3 + \frac{1}{4!} \gamma_4 A^4 + \xi^\dagger (\bar{\gamma} + \bar{\Gamma}_0 A) \xi + \ldots$$
Canonical Dyson–Schwinger equations

Gluon and quark cDSEs are derived from the identity

$$0 = \int \mathcal{D}A \mathcal{D}\xi \mathcal{D}\xi^\dagger \frac{\delta}{\delta \phi} \left\{ \mathcal{J}_A \ e^{-S[A,\xi,\xi^\dagger]} \ K[A,\xi,\xi^\dagger] \right\}$$

where $\phi \in \{A,\xi,\xi^\dagger\}$.

Ghost cDSEs follow from the operator identity

$$G_A = G_0 - G_0 \tilde{\Gamma}_0 A G_A$$

where $G_A^{-1} = -\partial \cdot D$, and $\tilde{\Gamma}_0$ is the bare ghost-gluon vertex.

Not quite equations of motion, rather relations between the Green functions and the — so far undetermined — variational kernels.
Kernels of the vacuum wave functional

\[ S[A, \xi, \xi^\dagger] = \frac{1}{2} \gamma_2 A^2 + \frac{1}{3!} \gamma_3 A^3 + \frac{1}{4!} \gamma_4 A^4 + \xi^\dagger (\bar{\gamma} + \bar{\Gamma}_0 A) \xi + \ldots \]

“Bare” vertices?

The **coefficients** in the vacuum wave functional play the role of the bare vertices, but

- are non-local functions
- have a non-trivial expansion in powers of the coupling
- will be represented diagrammatically by small empty boxes

\[ \gamma_2 = \ldots , \quad \bar{\gamma} = \ldots , \quad \bar{\Gamma}_0 = \ldots , \quad \ldots \]

- are not known... \( \Rightarrow \) variational kernels
Kernels of the vacuum wave functional

\[ S[A, \xi, \xi^\dagger] = \frac{1}{2} \gamma_2 A^2 + \frac{1}{3!} \gamma_3 A^3 + \frac{1}{4!} \gamma_4 A^4 + \xi^\dagger (\bar{\gamma} + \bar{\Gamma}_0 A) \xi + \ldots \]

“Bare” vertices?

The coefficients in the vacuum wave functional play the role of the bare vertices, but

- are non-local functions
- have a non-trivial expansion in powers of the coupling
- will be represented diagrammatically by small empty boxes

\[ \gamma_2 = \begin{array}{c} \text{empty square} \end{array}, \quad \bar{\gamma} = \begin{array}{c} \text{empty square} \end{array}, \quad \bar{\Gamma}_0 = \begin{array}{c} \text{empty square} \end{array}, \quad \ldots \]

- are not known... \Rightarrow \text{variational kernels}
Variational method

- evaluate the energy in the state defined by the chosen Ansatz
- use the cDSEs to express the energy density as a function of the variational kernels
- minimize the energy by taking functional derivatives w.r.t. the variational kernels

This gives a set of gap equations, which can be combined with the cDSEs.
1 Motivation

2Canonical Dyson–Schwinger Equations
   - Hamiltonian approach to QCD
   - Vacuum wave functional and cDSEs

3Variational solution for the Yang–Mills sector
   - Gluon Propagator
   - Ghost-Gluon Vertex
   - Three-Gluon Vertex

4Fermions

5Conclusions
Yang–Mills sector

Gluon propagator with non-Gaussian functional
Yang–Mills sector

Gluon propagator with non-Gaussian functional
**Yang–Mills sector**

**Ghost-gluon vertex**

### Truncated cDSE

\[ \Sigma_{\text{Ab}} \]  
\[ \Sigma_{\text{non-Ab}} \]

\[ \Sigma_{\text{Ab}} = \Sigma_{\text{full}} + \Sigma_{\text{approx.}} \]
\[ \Sigma_{\text{non-Ab}} = \Sigma_{\text{full}} + \Sigma_{\text{approx.}} \]

---

**Graphical Representation**

- **Left Graph**: Plot of \( \Sigma^{(\text{non-Ab})}(p,p,2\pi/3) \) showing curves for full and approximate \( \Sigma_{\text{Ab}}, \Sigma_{\text{non-Ab}} \).
- **Right Graph**: Plot of \( D^a_{\bar{c}c}(p,p,2\pi/3) \) showing full and approximate curves.
Yang–Mills sector

Ghost-gluon vertex

Two different cDSEs

\[ \begin{align*}
\text{Diagram 1} & \quad = \\
\text{Diagram 2} & \quad = \\
\end{align*} \]
Yang–Mills sector

Ghost-gluon vertex

\[ D^A \bar{c}(p,0,\pi/2) \]

- Red: \( \bar{c} \) bare vertex
- Green: A bare vertex
Yang–Mills sector

The three-gluon vertex

(Truncated) Three-gluon vertex cDSE

Dashed line: Ghost triangle only
Full line: Full cDSE

Full red line: Bare ghost-gluon vertex
Dashed green line: Full ghost-gluon vertex
1 Motivation

2 Canonical Dyson–Schwinger Equations
   - Hamiltonian approach to QCD
   - Vacuum wave functional and cDSEs

3 Variational solution for the Yang–Mills sector
   - Gluon Propagator
   - Ghost-Gluon Vertex
   - Three-Gluon Vertex

4 Fermions

5 Conclusions
Fermion Sector

The two-quark and quark-gluon kernel

In the exponent of the wave functional

\[ S[A, \xi, \xi^\dagger] = \frac{1}{2} \gamma_2 A^2 + \frac{1}{3!} \gamma_3 A^3 + \frac{1}{4!} \gamma_4 A^4 + \xi^\dagger (\bar{\gamma} + \bar{\Gamma}_0 A) \xi \]

take the most simple Dirac and colour structure

\[ \bar{\gamma} \sim \beta s(p), \quad \bar{\Gamma}_0 \sim \alpha_i t^a \nu(p, q) \]

with \( s \) and \( \nu \) being scalar variational kernels.
Fermion Sector

Gluon propagator with quark loop

\[ D(p) \]
Fermion Sector

The quark gap equation

\[
M(p) = \frac{g^2 C_F}{2} \int \frac{d^3 q}{(2\pi)^3} \frac{F(p-q)}{\sqrt{q^2 + M^2(q)}} \left[ M(q) - \frac{p \cdot q}{p^2} M(p) \right] \\
+ \frac{g^2 C_F}{2} \int \frac{d^3 q}{(2\pi)^3} \frac{\nu^2(p, q)}{\Omega(p+q) \sqrt{q^2 + M^2(q)}} \times \{ \text{stuff arising from the Dirac trace} \}
\]

Looks harmless (maybe even nice), but

- bad, bad linear divergences
- chiral condensate and constituent quark mass way too small

Are we missing something?
Fermion Sector

The quark gap equation

\[
M(p) = \frac{g^2 C_F}{2} \int \frac{d^3q}{(2\pi)^3} \frac{F(p-q)}{\sqrt{q^2 + M^2(q)}} \left[ M(q) - \frac{p \cdot q}{p^2} M(p) \right] + \frac{g^2 C_F}{2} \int \frac{d^3q}{(2\pi)^3} \frac{\nu^2(p,q)}{\Omega(p+q)\sqrt{q^2 + M^2(q)}} \times \{ \text{stuff arising from the Dirac trace} \}
\]

Looks harmless (maybe even nice), but

- bad, bad linear divergences
- chiral condensate and constituent quark mass way too small

Are we missing something?
**Fermion Sector**

The quark gap equation

\[
M(p) = \frac{g^2 C_F}{2} \int \frac{d^3q}{(2\pi)^3} \frac{F(p-q)}{\sqrt{q^2 + M^2(q)}} \left[ M(q) - \frac{p \cdot q}{p^2} M(p) \right]
+ \frac{g^2 C_F}{2} \int \frac{d^3q}{(2\pi)^3} \frac{\nu^2(p,q)}{\Omega(p+q)\sqrt{q^2 + M^2(q)}} \times \{\text{stuff arising from the Dirac trace}\}
\]

Looks harmless (maybe even nice), but

- bad, bad linear divergences
- chiral condensate and constituent quark mass way too small

Are we missing something?
Fermion Sector

The quark gap equation

\[
M(p) = \frac{g^2 C_F}{2} \int \frac{d^3 q}{(2\pi)^3} \frac{F(p-q)}{\sqrt{q^2 + M^2(q)}} \left[ M(q) - \frac{p \cdot q}{p^2} M(p) \right] \\
+ \frac{g^2 C_F}{2} \int \frac{d^3 q}{(2\pi)^3} \frac{\nu^2(p, q)}{\Omega(p + q) \sqrt{q^2 + M^2(q)}} \\
\times \{ \text{stuff arising from the Dirac trace} \}
\]

Looks harmless (maybe even nice), but

- bad, bad linear divergences
- chiral condensate and constituent quark mass way too small

Are we missing something?
Fermion Sector

Vector kernel with non-trivial Dirac component

Possible Dirac structures

\[ \alpha_i \quad \beta\alpha_i \quad \gamma_5\alpha_i \quad \ldots \]
# Fermion Sector

Vector kernel with non-trivial Dirac component

Possible Dirac structures

\[ \alpha_i \checkmark \beta \alpha_i \gamma_5 \alpha_i \ldots \]
Fermion Sector

Vector kernel with non-trivial Dirac component

Possible Dirac structures

\[ \alpha_i \quad \beta \alpha_i \quad \gamma_5 \alpha \quad \ldots \]
Fermion Sector

Vector kernel with non-trivial Dirac component

Possible Dirac structures

$$\alpha_i \checkmark \beta \alpha_i \gamma_5 \alpha_i \ldots$$
Fermion Sector

Vector kernel with non-trivial Dirac component

Possible Dirac structures

\[ \alpha_i \quad \beta \alpha_i \quad \gamma_5 \alpha_i \quad \ldots \]

Motivated by “perturbation theory”

\[ |0\rangle^{(0)} \implies |0\rangle^{(1)} \sim \psi^\dagger \alpha_i \psi |0\rangle^{(0)} \]
Fermion Sector

Vector kernel with non-trivial Dirac component

Possible Dirac structures

$$\alpha_i \quad \beta \alpha_i \quad \gamma_5 \alpha \quad \ldots$$

Motivated by “perturbation theory”

$$|\text{BCS}\rangle \quad \Longrightarrow \quad |\text{BCS}\rangle^{(1)} \sim \psi^\dagger \beta \alpha_i \psi |0\rangle^{(0)}$$
Fermion Sector

Vector kernel with non-trivial Dirac component

Possible Dirac structures

\[ \begin{align*}
\alpha_i & \quad \beta \alpha_i \\
\checkmark & \quad \uparrow \\
\gamma_5 \alpha & \quad \ldots
\end{align*} \]

Motivated by “perturbation theory”

\[ |\text{BCS}\rangle \quad \Longrightarrow \quad |\text{BCS}\rangle^{(1)} \sim \psi^\dagger \beta \alpha_i \psi |0\rangle^{(0)} \]

Turns out this is rather relevant!

- completely non-perturbative in nature
- linearly divergent terms cancel
Fermion Sector

Vector kernel with non-trivial Dirac component

Possible Dirac structures

\[ \alpha_i \quad \beta \alpha_i \quad \gamma_5 \alpha \quad \ldots \]

Motivated by “perturbation theory”

\[ |\text{BCS}\rangle \quad \Rightarrow \quad |\text{BCS}\rangle^{(1)} \sim \psi^\dagger \beta \alpha_i \psi |0\rangle^{(0)} \]

Turns out this is rather relevant!

- completely non-perturbative in nature
- linearly divergent terms cancel

Results give

\[ \langle \bar{q}q \rangle \sim (-235 \text{ MeV})^3 \]
Conclusions

- standard DSE techniques can be used to treat arbitrary wave functionals
- coupled quark-gluon system in Hamiltonian approach investigated
- spurious divergences are now under control