Superconformal Algebraic Approach to Hadron Structure:
The Perturbative-Nonperturbative Interface in QCD

Guy F. de Téramond

Universidad de Costa Rica

In collaboration with Stan Brodsky, Hans G. Dosch, Alexandre Deur and Cédric Lorce
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1 Preamble: The complexity of QCD

- The QCD Lagrangian in the limit of massless quarks has no scale: still confinement and a mass scale should emerge from the quantum theory built upon the classical QCD conformal theory.

- Increase of the QCD coupling at low energies implies that an infinite number of quark and gluons are strongly coupled.

- Description of the dynamics is vastly complex and understanding the mechanism of confinement is an unsolved problem.

- We require a semiclassical approximation which captures essential aspects of the nonperturbative confinement dynamics, which are not obvious from the QCD Lagrangian.

- Recent progress: nonperturbative QCD dynamics is well captured in a semiclassical effective theory based on superconformal quantum mechanics in the light-front and its holographic embedding on a higher dimensional AdS space (gauge/gravity correspondence).
2 Semiclassical approximation to light front QCD

[GdT and S. J. Brodsky, PRL 102, 081601 (2009)]

- Start with $SU(3)_C$ QCD Lagrangian

\[ \mathcal{L}_{\text{QCD}} = \bar{\psi} (i\gamma^\mu D_\mu - m) \psi - \frac{1}{4} G^{\alpha \mu \nu} G_{\alpha \mu \nu} \]

- Express the hadron 4-momentum generator $P = (P^+, P^-, \mathbf{P}_\perp)$, $P^\pm = P^0 \pm P^3$, in terms of dynamical fields $\psi_+ = \Lambda_+ \psi$ and $\mathbf{A}_\perp$ ($\Lambda_\pm = \gamma^0 \gamma^\pm$) quantized in null plane $x^+ = x^0 + x^3 = 0$

\[ P^- = \frac{1}{2} \int dx^- d^2 \mathbf{x}_\perp \bar{\psi}_+ \gamma^+ (i \nabla_\perp)^2 + m^2 i \partial^+ \psi_+ + \text{interactions} \]

\[ P^+ = \int dx^- d^2 \mathbf{x}_\perp \bar{\psi}_+ \gamma^+ i \partial^+ \psi_+ \]

\[ \mathbf{P}_\perp = \frac{1}{2} \int dx^- d^2 \mathbf{x}_\perp \bar{\psi}_+ \gamma^+ i \nabla_\perp \psi_+ \]

- Construct LF invariant Hamiltonian $P^2 = P_\mu P^\mu = P^- P^+ - \mathbf{P}_\perp^2$ from mass-shell relation

\[ P^2 |\psi(P)\rangle = M^2 |\psi(P)\rangle, \quad |\psi\rangle = \sum_n \psi_n |n\rangle \]

- Simple structure of LF vacuum allows a quantum-mechanical probabilistic interpretation of hadronic states in terms of wave functions: $\psi_n = \langle n | \psi \rangle$
• Compute the hadron matrix element \( \langle \psi(P') | P_\mu P^\mu | \psi(P) \rangle \) for a two-parton bound state

• Factor out the longitudinal \( X(x) \) and orbital kinematical dependence from LFWF \( \psi \)

\[
\psi(x, \zeta, \varphi) = e^{iL\varphi} X(x) \frac{\phi(\zeta)}{\sqrt{2\pi\zeta}}
\]

with invariant transverse impact variable

\[
\zeta^2 = x(1-x)b^2_\perp
\]

• Ultra relativistic limit \( m_q \rightarrow 0 \) longitudinal modes \( X(x) \) decouple \( (L = L^z) \)

\[
M^2 = \int d\zeta \phi^*(\zeta) \sqrt{\zeta} \left( -\frac{d^2}{d\zeta^2} - \frac{1}{\zeta} \frac{d}{d\zeta} + \frac{L^2}{\zeta^2} \right) \frac{\phi(\zeta)}{\sqrt{\zeta}} + \int d\zeta \phi^*(\zeta) U(\zeta) \phi(\zeta)
\]

where effective potential \( U \) includes all interactions, including those from higher Fock states

• LF Hamiltonian equation \( P_\mu P^\mu |\psi\rangle = M^2 |\psi\rangle \) is a LF wave equation for \( \phi \)

\[
\left( -\frac{d^2}{d\zeta^2} - \frac{1 - 4L^2}{4\zeta^2} + U(\zeta) \right) \phi(\zeta) = M^2 \phi(\zeta)
\]

• Critical value \( L = 0 \) corresponds to lowest possible stable solution

• Relativistic and frame-independent LF Schrödinger equation: \( U \) is instantaneous in LF time
3 Higher integer-spin wave equations in AdS space

[GdT and S. J. Brodsky, PRL 102, 081601 (2009)]

[GdT, H.G. Dosch and S. J. Brodsky, PRD 87, 075004 (2013)]

• Why is AdS space important? AdS\(_5\) is a 5-dim space of maximal symmetry (conformal), negative curvature and a four-dim boundary: Minkowski space \(ds^2 = \frac{R^2}{z^2} \left( dx_\mu dx^\mu - dz^2 \right)\)

• Isomorphism of \(SO(4, 2)\) group of conformal transformations with generators \(P^\mu, M^{\mu\nu}, K^\mu, D\) with the group of isometries of AdS\(_5\)

• Integer spin-\(J\) in AdS conveniently described by tensor field \(\Phi_{N_1 \ldots N_J}\) with effective action

\[
S_{\text{eff}} = \int d^d x \, dz \, \sqrt{|g|} \, e^{\varphi(z)} \, g^{N_1 N'_1} \ldots g^{N_J N'_J} \left( g^{M M'} D_M \Phi^*_{N_1 \ldots N_J} D_{M'} \Phi_{N'_1 \ldots N'_J} - \mu_{\text{eff}}^2(z) \Phi^*_{N_1 \ldots N_J} \Phi_{N'_1 \ldots N'_J} \right)
\]

Covariant derivative \(D_M\) includes affine connection and dilaton \(\varphi(z)\) effectively breaks conformality

• Effective mass \(\mu_{\text{eff}}(z)\) is determined by precise mapping to light-front physics

• Non-trivial geometry of pure AdS encodes the kinematics and additional deformations of AdS encode the dynamics, including confinement
- Physical hadron has plane-wave and spin components along physical coordinates

\[ \Phi_P(x, z)_{\mu_1 \ldots \mu_J} = e^{iP \cdot x} \Phi(z)_{\mu_1 \ldots \mu_J} \quad \text{with invariant hadronic mass} \quad P_\mu P^\mu = M^2 \]

- Variation of the action gives AdS wave equation for spin-\( J \) field

\[ \Phi(z)_{\mu_1 \ldots \mu_J} = \Phi_J(z) e_{\mu_1 \ldots \mu_J}(P) \]

\[
\left[ -z^{d-1-2J} \frac{e^{\varphi(z)}}{e^{\varphi(z)}} \frac{\partial_z}{z^{d-1-2J}} \left( \frac{e^{\varphi(z)}}{z^{d-1-2J}} \partial_z \right) + \left( \frac{\mu R}{z} \right)^2 \right] \Phi_J = M^2 \Phi_J
\]

- Upon substitution \( \Phi_J(z) \sim z^{(d-1)/2-J} e^{-\varphi(z)/2} \phi_J(z) \) and \( z \to \zeta \) we find LFWE (slide 5)

\[
\left( -\frac{d^2}{d\zeta^2} - \frac{1 - 4L^2}{4\zeta^2} + U(\zeta) \right) \phi_J(\zeta) = M^2 \phi_J(\zeta)
\]

for \( (d = 4) \) where

\[ U(\zeta) = \frac{1}{2} \varphi''(\zeta) + \frac{1}{4} \varphi'(\zeta)^2 + \frac{2J - 3}{2z} \varphi'(\zeta) \]

and \( (\mu R)^2 = -(2 - J)^2 + L^2 \)

- Unmodified AdS equations correspond to the kinetic energy terms for the partons, and effective LF potential \( U(\zeta) \) corresponds to the IR modification of AdS space

- AdS Breitenlohner-Freedman bound \( (\mu R)^2 \geq -4 \) equivalent to LF QM stability condition \( L^2 \geq 0 \)
4 Higher half-integer-spin wave equations in AdS space

[J. Polchinski and M. J. Strassler, JHEP 0305, 012 (2003)]
[GdT and S. J. Brodsky, PRL 94, 201601 (2005)]
[GdT, H.G. Dosch and S. J. Brodsky, PRD 87, 075004 (2013)]

• Extension of holographic ideas to spin-\(\frac{1}{2}\) (and higher half-integral \(J\)) hadrons by considering wave equations for Rarita-Schwinger spinor fields in AdS space and their mapping to light-front physics

• LF clustering decomposition of invariant variable \(\zeta\): same multiplicity of states for mesons and baryons

• The invariant variable \(\zeta\) is the light-front weighted distribution of the spectator diquark cluster relative to the active quark

\[
\zeta = \sqrt{\frac{x}{1-x}} \left| \sum_{j=1}^{N-1} x_j b_{\perp j} \right|
\]

• LF cluster decomposition follows from mapping of EM form factor in AdS to the light front [S. J. Brodsky and GdT, PRL 96, 201601 (2006)]
- Half-integer spin $J = T + \frac{1}{2}$ represented by RS spinor $[\Psi_{N_1\cdots N_T}]_{\alpha}$ with effective AdS action

$$S_{\text{eff}} = \frac{1}{2} \int d^d x \, dz \, \sqrt{|g|} g^{N_1 N'_1} \cdots g^{N_T N'_T}$$

$$\left[ \overline{\Psi}_{N_1\cdots N_T} \left( i \Gamma^A e^M_A D_M - \mu - \rho(z) \right) \Psi_{N'_1\cdots N'_T} + \text{h.c.} \right]$$

where the covariant derivative $D_M$ includes the affine connection and the spin connection

- $e^A_M$ is the vielbein and $\Gamma^A$ tangent space Dirac matrices $\{\Gamma^A, \Gamma^B\} = \eta^{AB}$

- Variation of the action gives linear equations

$$- \frac{d}{dz} \psi_- - \frac{L + \frac{1}{2}}{z} \psi_- - V(z) \psi_- = M \psi_+$$

$$\frac{d}{dz} \psi_+ - \frac{L + \frac{1}{2}}{z} \psi_+ - V(z) \psi_+ = M \psi_-$$

where $L = \mu R - \frac{1}{2}$, $\psi \equiv \Psi_T$ and

$$V(z) = \frac{R}{z} \rho(z)$$

a $J$-independent potential – No spin-orbit coupling along a given trajectory!

[See also: T. Gutsche, V. E. Lyubovitskij, I. Schmidt and A. Vega, Phys. Rev. D 85, 076003 (2012)]
• Mapping to the light front, $z \to \zeta$ system of linear eqs. is equivalent to the second order equations:

\[
\left(-\frac{d^2}{d\zeta^2} - \frac{1 - 4L^2}{4\zeta^2} + U^+(\zeta)\right)\psi_+ = M^2\psi_+ \\
\left(-\frac{d^2}{d\zeta^2} - \frac{1 - 4(L + 1)^2}{4\zeta^2} + U^-(-\zeta)\right)\psi_- = M^2\psi_-
\]

where

\[
U^\pm(\zeta) = V^2(\zeta) \pm V'(\zeta) + \frac{1 + 2L}{\zeta}V(\zeta)
\]

and

\[
V(\zeta) = \frac{R}{\zeta}\rho(\zeta)
\]

• The plus and minus component equations correspond to LF orbital angular momentum $L$ and $L + 1$
5 Superconformal quantum mechanics and nucleons

[S. Fubini and E. Rabinovici, NPB 245, 17 (1984)]


- SUSY QM contains two fermionic generators $Q$ and $Q^\dagger$, and a bosonic generator, the Hamiltonian $H$
  [E. Witten, NPB 188, 513 (1981)]

- Closure under the graded algebra $sl(1/1)$:
  \[
  \frac{1}{2} \{ Q, Q^\dagger \} = H \\
  \{ Q, Q \} = \{ Q^\dagger, Q^\dagger \} = 0 \\
  [Q, H] = [Q^\dagger, H] = 0
  \]

- Since $[Q^\dagger, H] = 0$, the states $|E\rangle$ and $Q^\dagger |E\rangle$ have identical eigenvalues $E$, but for a zero eigenvalue we can have the trivial solution $|E = 0\rangle = 0$

- For a conformal theory
  \[
  Q = \psi^\dagger \left( -\frac{d}{dx} + \frac{f}{x} \right), \quad Q^\dagger = \psi \left( \frac{d}{dx} + \frac{f}{x} \right)
  \]
  where $\psi$ and $\psi^\dagger$ are spinor operators with $\{ \psi, \psi^\dagger \} = 1$ and $f$ is dimensionless
**Conformal graded-Lie algebra** has in addition to the Hamiltonian $H$ and supercharges $Q$ and $Q^\dagger$, a new operator $S$ related to the generator of conformal transformations $K$

$$S = \chi x, \quad S^\dagger = \chi^\dagger x$$

- **Find enlarged algebra** (Superconformal algebra of Haag, Lopuszanski and Sohnius (1974))

$$\frac{1}{2} \{Q, Q^\dagger\} = H, \quad \frac{1}{2} \{S, S^\dagger\} = K,$$

$$\{Q, S^\dagger\} = f - B + 2iD, \quad \{Q^\dagger, S\} = f - B - 2iD$$

where $B = \frac{1}{2} [\psi^\dagger, \psi]$, and the generators of translation, dilatation and the special conformal transformation $H$, $D$ and $K$

$$H = \frac{1}{2} \left( -\frac{d^2}{dx^2} + \frac{f^2 + 2Bf}{x^2} \right)$$

$$D = \frac{i}{4} \left( \frac{d}{dx} x + x \frac{d}{dx} \right)$$

$$K = \frac{1}{2} x^2$$

satisfy the conformal algebra

$$[H, D] = iH, \quad [H, K] = 2iD, \quad [K, D] = -iK$$
• Following F&R define a fermionic generator $R$, a linear combination of the generators $Q$ and $S$

$$ R_\lambda = Q + \lambda S $$

which generates a new Hamiltonian

$$ G_\lambda = \{ R_\lambda, R_\lambda^\dagger \} $$

where by construction

$$ \{ R_\lambda, R_\lambda \} = \{ R_\lambda^\dagger, R_\lambda^\dagger \} = 0, \quad [ R_\lambda, G_\lambda ] = [ R_\lambda^\dagger, G_\lambda ] = 0 $$

which also closes under the graded algebra $sl(1/1)$

• In a Pauli matrix representation $G_\lambda$ is given by

$$ G_\lambda = 2H + 2\lambda^2 K + 2\lambda ( f - \sigma_3 ) $$

and leads to the eigenvalue equations

$$ \left( - \frac{d^2}{dx^2} + \lambda^2 x^2 + 2\lambda f - \lambda + \frac{4(f + \frac{1}{2})^2 - 1}{4x^2} \right) \phi_1 = E \phi_1 $$

$$ \left( - \frac{d^2}{dx^2} + \lambda^2 x^2 + 2\lambda f + \lambda + \frac{4(f - \frac{1}{2})^2 - 1}{4x^2} \right) \phi_2 = E \phi_2 $$
Light-front mapping and nucleon spectrum

- Upon the substitutions $x \mapsto \zeta$, $E \mapsto M^2$ and $f \mapsto L + \frac{1}{2}$ in the AdS wave equations (slide 10) we find $U^+ = \lambda^2 \zeta^2 + 2\lambda(L + 1)$ and $U^- = \lambda^2 \zeta^2 + 2\lambda L$, thus the effective potential $V = \lambda \zeta$ and the bound state equations:

\[
\left( -\frac{d^2}{d\zeta^2} - \frac{1 - 4L^2}{4\zeta^2} + \lambda^2 \zeta^2 + 2\lambda(L + 1) \right) \psi_+ = M^2 \psi_+ \\
\left( -\frac{d^2}{d\zeta^2} - \frac{1 - 4(L + 1)^2}{4\zeta^2} + \lambda^2 \zeta^2 + 2\lambda L \right) \psi_- = M^2 \psi_-, 
\]

- Eigenfunctions

$\psi_+(\zeta) \sim \zeta^{\frac{1}{2} + L} e^{-\lambda \zeta^2 / 2} L_n^L(\lambda \zeta^2)$

$\psi_-(\zeta) \sim \zeta^{\frac{3}{2} + L} e^{-\lambda \zeta^2 / 2} L_n^{L+1}(\lambda \zeta^2)$

- Eigenvalues

$M^2 = 4\lambda(n + L + 1)$

- $L$ is the relative LF angular momentum between the active quark and spectator cluster
6 Superconformal meson-baryon symmetry

[H.G. Dosch, GdT, and S. J. Brodsky, PRD 91, 085016 (2015)]

- Extend superconformal QM to relate bound-state equations for mesons and baryons
- Upon the substitutions (slide 13) $x \mapsto \zeta$, $E \mapsto M^2$, $\lambda \mapsto \lambda_B = \lambda_M$ and $f \mapsto L_M - \frac{1}{2} = L_B + \frac{1}{2}$ find the bound-state equations

\[
\begin{align*}
\left(- \frac{d^2}{d\zeta^2} + \frac{4L_M^2 - 1}{4\zeta^2} + \lambda_M^2 \zeta^2 + 2\lambda_M (L_M - 1)\right) \phi_{Meson} &= M^2 \phi_{Meson} \\
\left(- \frac{d^2}{d\zeta^2} + \frac{4L_N^2 - 1}{4\zeta^2} + \lambda_N^2 \zeta^2 + 2\lambda_N (L_N + 1)\right) \phi_{Nucleon} &= M^2 \phi_{Nucleon}
\end{align*}
\]

- Superconformal QM imposes the condition $\lambda = \lambda_M = \lambda_B$ (equality of Regge slopes) and the remarkable relation $\Rightarrow L_M = L_B + 1$
- Also $R^\dagger |M, L\rangle = |B, L - 1\rangle$, $R^\dagger |M, L = 0\rangle = 0$
  Special role of the pion as a unique state of zero energy
- Emerging dynamical SUSY from SU(3) color
  (Hadronic SUSY introduced by H. Miyazawa (1966))

\[
\begin{align*}
\bar{3} \rightarrow 3 \times 3 \\
L &\rightarrow L - 1
\end{align*}
\]
Superconformal Algebraic Approach to Hadron Structure

- Superconformal spin-dependent Hamiltonian to describe mesons and baryons (chiral limit)

\[ G = \{ R^\dagger_\lambda, R_\lambda \} + 2\lambda S \]

\[ S = 0, 1 \]

Mesons: \( M^2 = 4\lambda (n + L_M) + 2\lambda S \),  
Baryons: \( M^2 = 4\lambda (n + L_B + 1) + 2\lambda S \)

Superconformal meson-nucleon partners: solid line corresponds to \( \sqrt{\lambda} = 0.53 \text{ GeV} \)
How universal is the semiclassical approximation based on superconformal QM and its LF holographic embedding?


\[ \lambda = 0.523 \pm 0.024 \text{ GeV} \]

Best fit for hadronic scale \( \sqrt{\lambda} \) from the different sectors including radial and orbital excitations
7 Supersymmetry across the heavy-light hadronic spectrum


[H.G. Dosch, GdT, and S. J. Brodsky IN PREPARATION]

- Introduction of quark masses breaks conformal symmetry without violating supersymmetry
- Embedding of SUSY bound-state equations in AdS to determine the confinement potential
- SUSY potential for a meson can be read-off from slide 10 \( V_{\text{light}} = \lambda \zeta \)
  \[
  U_{\text{susy}} = V^2 - V' + \frac{2L + 1}{\zeta} V
  \]
  where \( V \) is an arbitrary superpotential and \( L = L_M - 1 \)
- Embedding meson bound-state equation in AdS the LF potential is \( \varphi_{\text{light}} = \lambda \zeta^2 \)
  \[
  U_{\text{AdS}} = \frac{1}{4} (\varphi')^2 + \frac{1}{2} \varphi'' + \frac{2L - 1}{2 \zeta} \varphi'
  \]
  where \( \varphi \) is an arbitrary dilaton (slide 7) and \( L = L_M - 1 \) for \( J = L_M \)
• Equating the SUSY and the dilaton potentials $U_{susy} = U_{AdS}$ we find:

$$\varphi'(\zeta) = \int d\zeta \left( \lambda \zeta \sigma(\zeta) - \frac{\lambda^2 \zeta^2 \sigma'(\zeta)}{\lambda^2 \zeta^2 \sigma(\zeta) + 2(L_M - 1)\lambda} \right)$$

$$V(\zeta) = \frac{1}{2} \left( \lambda \zeta \sigma(\zeta) + \frac{\lambda^2 \zeta^2 \sigma'(\zeta)}{\lambda^2 \zeta^2 \sigma(\zeta) + 2(L_M - 1)\lambda} \right)$$

where $\sigma(\zeta)$ is an arbitrary function

• If dilaton profile is independent of angular momentum one has $\sigma'(\zeta) = 0$ and thus $\sigma = \text{const} \equiv 2A$

• Solution:

$$\varphi = \lambda A \zeta^2 + B \quad \text{and} \quad V(\zeta) = \lambda A \zeta$$

• For strongly broken conformal invariance the dilaton is still quadratic, but since $A$ is arbitrary the strength of the potential is not determined
\[ M^2 \text{ vs } L_M = L_B + 1 \]

\[ D/\Lambda_c \text{ trajectory (left) and } D_s/\Xi_c \text{ trajectory (right)} \]
\[ M^2 \text{ vs } L_M = L_B + 1 \]

\[ B/\Lambda_b \text{ trayectory} \]

\[ \sqrt{\lambda_{\text{light}}} \simeq 0.52 \text{ GeV}, \quad \sqrt{\lambda_{\text{charm}}} \simeq 0.7 \text{ GeV}, \quad \sqrt{\lambda_{\text{bottom}}} \simeq 1 \text{ GeV}, \]
8 Infrared behavior of the strong coupling in light-front holographic QCD


- Effective coupling $\alpha_{g_1} = g_1^2/4\pi$ defined from and observable: $g_1$ scheme from Bjorken sum rule

$$\frac{\alpha_{g_1}(Q^2)}{\pi} = 1 - \frac{6}{g_A} \int_0^1 dx g_1^{p-n}(x, Q^2)$$

- Infrared behavior of strong coupling in LF holographic QCD from Fourier transform of the LF transverse coupling:

$$\alpha_{g_1}^{AdS}(Q) = \pi \exp \left(-\frac{Q^2}{4\lambda}\right)$$

- Large $Q$-dependence of $\alpha_s$ is computed from the pQCD $\beta$ series:

$$Q^2 d\alpha_s/dQ^2 = \beta(Q) = -(\beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \beta_2 \alpha_s^4 + \cdots)$$

where coefficients $\beta_i$ are known up to $\beta_4$ in $\overline{MS}$ scheme (five-loops):

- $\alpha_{g_1}^{pQCD}(Q)$ expressed as a perturbative expansion in $\alpha_{\overline{MS}}(Q)$:

$$\alpha_{g_1}^{pQCD}(Q) = \pi \left[ \alpha_{\overline{MS}}/\pi + a_1 \left( \alpha_{\overline{MS}}/\pi \right)^2 + a_2 \left( \alpha_{\overline{MS}}/\pi \right)^3 + \cdots \right]$$

The coefficients $a_i$ are known up to order $a_4$. 
• $\Lambda_{QCD}$ and transition scale $Q_0$ from matching perturbative and nonperturbative regimes:
  $\Lambda_{\overline{MS}} = 0.339 \pm 0.019 \text{ GeV}$  
  (World Average: $\Lambda_{\overline{MS}} = 0.332 \pm 0.019 \text{ GeV}$)
  Transition scale: $Q_0^2 \approx 1 \text{ GeV}^2$

• Renormalization scheme dependence
Thanks!