XIIth International Conference on Quark Confinement and the Hadron Spectrum
Thessaloniki, Greece, 2 September 2016

A gauge-independent Higgs mechanism and the implications for quark confinement

Kei-Ichi Kondo (Chiba Univ., Japan)

Based on e-Print: arXiv:1606.06194 [hep-th]
Higgs mechanism without spontaneous symmetry breaking and quark confinement

Short abstract:

• We give a manifestly gauge-invariant description of the Higgs mechanism in the operator level, without relying on spontaneous symmetry breaking which is signaled by a non-vanishing vacuum expectation value $\langle \phi \rangle$ of the scalar field ϕ.

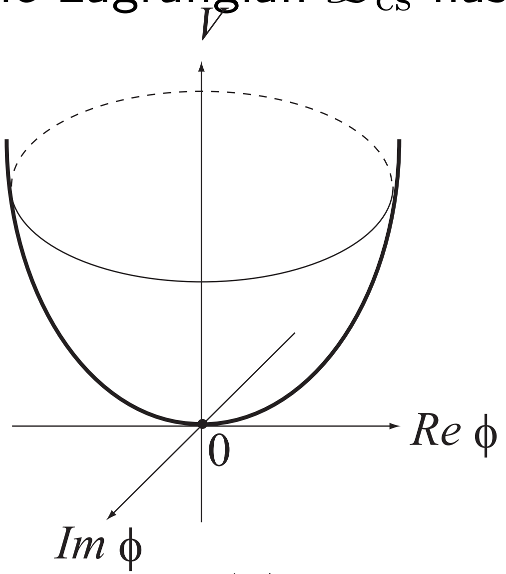
§ Introduction: SSB, NG boson and Higgs mechanism

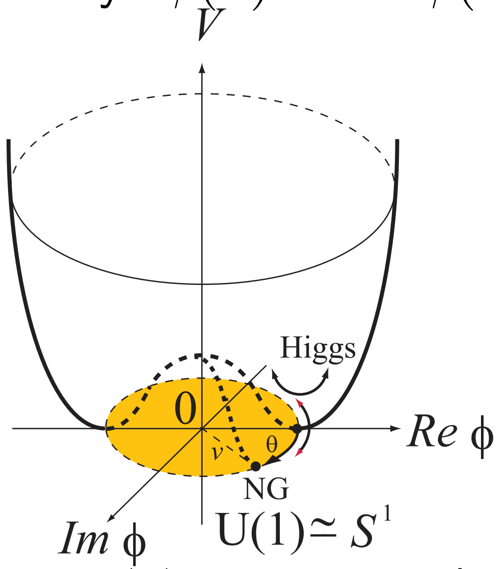
Spontaneous symmetry breaking (SSB) is an important concept in physics.
SSB occurs when the lowest energy state or the vacuum is degenerate.

⊙ SSB of global continuous symmetry $G = U(1)$ in the complex scalar field theory

$$\mathscr{L}_{\rm cs} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi^*\phi), \quad V(\phi^*\phi) = \frac{\lambda}{2}\left(\phi^*\phi - \frac{\mu^2}{\lambda}\right)^2, \ \phi \in \mathbb{C}, \ \lambda > 0. \qquad (1)$$

The Lagrangian $\mathscr{L}_{\rm cs}$ has the global $U(1)$ symmetry: $\phi(x) \to e^{i\theta}\phi(x)$.





unbroken $U(1)$ symmetry ($\mu^2 \leq 0$): $\Longrightarrow$ SSB of $U(1)$ symmetry ($\mu^2 > 0$):

$\langle \phi(x) \rangle = 0$ $\qquad\qquad$ $\langle \phi(x) \rangle = v \neq 0$

For $\phi(x) = |\phi(x)| e^{i\pi(x)/v} \in \mathbb{C}$

- flat direction: $\pi(x) \to$ massless Nambu-Goldstone particle
- curved direction: $|\phi(x)| \to$ massive Higgs particle,

⊙ What happens in the gauge field theory with the local continuous symmetry?

First, we consider the $U(1)$ **gauge-scalar theory**:

$$\mathscr{L}_{\rm AH} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\phi^*\phi),\ V(\phi^*\phi) = \frac{\lambda}{2}\left(\phi^*\phi - \frac{\mu^2}{\lambda}\right)^2,\ \phi \in \mathbb{C},$$
$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad D_\mu = \partial_\mu - iqA_\mu, \tag{2}$$

• The local $U(1)$ gauge symmetry is completely broken spontaneously by choosing a vacuum with the non-vanishing VEV of the scalar field: $\langle\phi(x)\rangle = v/\sqrt{2} \neq 0$.

$$|D_\mu\phi|^2 = |\partial_\mu\phi - iqA_\mu\phi|^2 \to |-iqA_\mu\langle\phi(x)\rangle|^2 + ... = \frac{1}{2}(qv)^2A_\mu A^\mu + ..., \tag{3}$$

• The massless Nambu-Goldstone particle π (boson) must be generated, but it disappears. It is absorbed into the massless gauge boson A_μ to form the massive vector boson as supplying the longitudinal component.
• The Brout-Englert-Higgs (Guralnik-Hagen-Kibble) mechanism or Higgs phenomenon is one of the most well-known mechanisms by which gauge bosons acquire their masses.
• This is a model for superconductor. The superconductivity is understood from the SSB of $U(1)$ gauge symmetry. The Meisser effect is just the Anderson-Schwinger mechanism.

$\odot$ The spontaneous breaking of gauge symmetry is a rather misleading terminology. The lattice gauge theory *à la* Wilson gives a well-defined gauge theory without gauge fixing. The Elitzur theorem tells us that the local continuous gauge symmetry cannot break spontaneously, if no gauge fixing is introduced. The VEV $\langle \phi \rangle$ of ϕ is rigorously zero regardless of the shape of the scalar potential V:

$$\langle \phi \rangle = 0, \tag{4}$$

• Therefore, we are forced to fix the gauge to cause the non-zero VEV. Even after the gauge fixing, however, we still have the problem. Whether SSB occurs or not depends on the gauge choice.
In non-compact $U(1)$ gauge-Higgs model, the SSB occurs $\langle \phi \rangle \neq 0$ only in the Landau gauge $\alpha = 0$, and no SSB occur $\langle \phi \rangle = 0$ in all other covariant gauges with $\alpha \neq 0$, as rigorously shown [KK85,BN86]. In axial gauge, $\langle \phi \rangle = 0$ for compact models [FMS80].

• After imposing the gauge fixing condition for the original local gauge group G, a global subgroup H' remains unbroken. Such a global symmetry H' is called the **remnant global gauge symmetry**. Only a remnant global gauge symmetry H' of the local gauge symmetry G can break spontaneously to cause the Higgs phenomenon.
However, such subgroup H' is not unique and the location of the breaking in the phase diagram depends on the remnant global gauge symmetries in the gauge-Higgs model.

The relevant numerical evidences are given on a lattice for different remnant symmetries allowed for various confinement scenarios.[GOZ04][CG08]
Moreover, the transition occurs in the regions where the Fradkin-Shenker-Osterwalder-Seiler theorem assures us that there is no transition in the phase diagram.

$\Longrightarrow$ The above observations indicate that the Higgs mechanism should be characterized in a gauge-invariant way without breaking the original gauge symmetry. [It is obvious that the non-vanishing VEV of the scalar field is not a gauge-invariant criterion of SSB.]

We show that a gauge boson can acquire the mass in a gauge-invariant way without assuming spontaneous breakdown of gauge symmetry which is signaled by the non-vanishing VEV of the scalar field.

The Higgs phenomenon can be described even without such SSB. The spontaneous symmetry breaking is sufficient but not necessary for the Higgs mechanism to work.

$\Longrightarrow$ Remember that quark confinement is realized in the unbroken gauge symmetry phase with **mass gap**.

Thus, the gauge-invariant description of the Higgs mechanics can shed new light on the complementarity between confinement phase and Higgs phase.

§ Conventional Higgs mechanism due to SSB

We consider $G = SU(N)$ Yang-Mills-Higgs theory with the gauge-invariant Lagrangian:

$$\mathscr{L}_{\rm YMH} = -\frac{1}{4}\mathscr{F}^{\mu\nu}(x) \cdot \mathscr{F}_{\mu\nu}(x) + \frac{1}{2}(\mathscr{D}^{\mu}[\mathscr{A}]\boldsymbol{\phi}(x)) \cdot (\mathscr{D}_{\mu}[\mathscr{A}]\boldsymbol{\phi}(x)) - V(\boldsymbol{\phi}(x) \cdot \boldsymbol{\phi}(x)). \tag{1}$$

We assume that the adjoint scalar field $\boldsymbol{\phi}(x) = \phi^A(x)T_A$ has the fixed radial length,

$$\boldsymbol{\phi}(x) \cdot \boldsymbol{\phi}(x) \equiv \boldsymbol{\phi}^A(x)\boldsymbol{\phi}^A(x) = v^2. \tag{2}$$

The Yang-Mills field $\mathscr{A}_\mu(x) = \mathscr{A}_\mu^A(x)T_A$ and $\boldsymbol{\phi}(x)$ obey the gauge transformation:

$$\begin{aligned} \mathscr{A}_\mu(x) &\to U(x)\mathscr{A}_\mu(x)U^{-1}(x) + ig^{-1}U(x)\partial_\mu U^{-1}(x), \\ \boldsymbol{\phi}(x) &\to U(x)\boldsymbol{\phi}(x)U^{-1}(x), \quad U(x) \in G = SU(N). \end{aligned} \tag{3}$$

Notice that $\boldsymbol{\phi}(x) \cdot \boldsymbol{\phi}(x)$ is a gauge-invariant combination. The covariant derivative $\mathscr{D}_\mu[\mathscr{A}] := \partial_\mu - ig[\mathscr{A}_\mu, \cdot]$ transforms $\mathscr{D}_\mu[\mathscr{A}] \to U(x)\mathscr{D}_\mu[\mathscr{A}]U^{-1}(x)$.
Notation: For the Lie-algebra valued quantities $\mathscr{A} = \mathscr{A}^A T_A$ and $\mathscr{B} = \mathscr{B}^A T_A$

$$\mathscr{A} \cdot \mathscr{B} := 2\mathrm{tr}(\mathscr{A}\mathscr{B}) = \mathscr{A}^A\mathscr{B}^B 2\mathrm{tr}(T_A T_B) = \mathscr{A}^A\mathscr{B}^A \quad (A = 1, ..., N^2 - 1). \tag{4}$$

First, we recall the conventional description for the Higgs mechanism.
If $\phi(x)$ acquires a non-vanishing VEV $\langle\phi(x)\rangle = \langle\phi\rangle = \langle\phi^A\rangle T_A$, then

$$\mathscr{D}_\mu[\mathscr{A}]\phi(x) := \partial_\mu \phi(x) - ig[\mathscr{A}_\mu(x), \phi(x)] \to -ig[\mathscr{A}_\mu(x), \langle\phi\rangle] + ..., \tag{5}$$

and

$$\begin{aligned}\mathscr{L}_{\mathrm{YMH}} \to & -\frac{1}{2}\mathrm{tr}_G\{\mathscr{F}^{\mu\nu}(x)\mathscr{F}_{\mu\nu}(x)\} - g^2 \mathrm{tr}_G\{[\mathscr{A}^\mu(x), \langle\phi\rangle][\mathscr{A}_\mu(x), \langle\phi\rangle]\} + ... \\ = & -\frac{1}{2}\mathrm{tr}_G\{\mathscr{F}^{\mu\nu}(x)\mathscr{F}_{\mu\nu}(x)\} - g^2 \mathrm{tr}_G\{[T_A, \langle\phi\rangle][T_B, \langle\phi\rangle]\}\mathscr{A}^{\mu A}(x)\mathscr{A}_\mu^B(x).\end{aligned} \tag{6}$$

To break spontaneously the local continuous gauge symmetry G by realizing the non-vanishing VEV $\langle\phi\rangle$ of the scalar field ϕ, we choose the unitary gauge in which $\phi(x)$ is pointed to a specific direction $\phi(x) \to \phi_\infty$ uniformly over the spacetime.

By this procedure the original gauge symmetry G is not completely broken. Indeed, there may exist a subgroup H (of G) which does not change ϕ_∞.
This is the partial SSB $G \to H$: the mass is provided for the coset G/H (broken parts), while the mass is not supplied for the H-commutative part of $\mathscr{A}_\mu$:

$$\mathscr{L}_{\mathrm{YMH}} \to -\frac{1}{2}\mathrm{tr}_G\{\mathscr{F}^{\mu\nu}(x)\mathscr{F}_{\mu\nu}(x)\} - (gv)^2 \mathrm{tr}_{G/H}\{\mathscr{A}^\mu(x)\mathscr{A}_\mu(x)\}. \tag{7}$$

Thus the theory reduces to a gauge theory with the **residual gauge group** H.

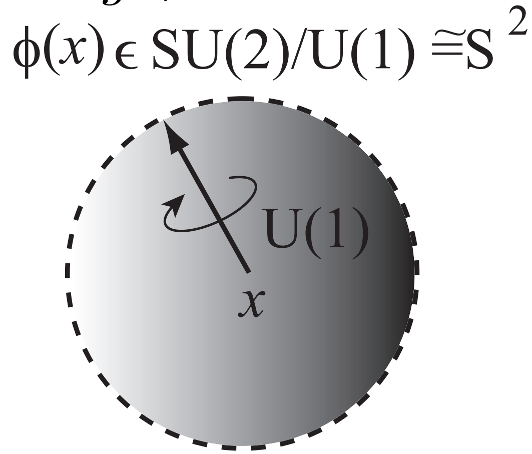
For $G = SU(2)$, by taking the usual **unitary gauge**

$$\langle \boldsymbol{\phi}_\infty \rangle = vT_3, \quad \text{or} \quad \langle \phi_\infty^A \rangle = v\delta^{A3}, \tag{8}$$

the kinetic term generates the mass term,

$$-g^2 v^2 \mathrm{tr}_G\{[T_A, T_3][T_B, T_3]\}\mathscr{A}^{\mu A}\mathscr{A}_\mu^B = \frac{1}{2}(gv)^2(\mathscr{A}^{\mu 1}\mathscr{A}_\mu^1 + \mathscr{A}^{\mu 2}\mathscr{A}_\mu^2). \tag{9}$$

- The off-diagonal gluons $\mathscr{A}_\mu^1, \mathscr{A}_\mu^2$ acquire the same mass $M_W := gv$,
- The diagonal gluon $\mathscr{A}_\mu^3$ remains massless.



Even after taking the unitary gauge (8),
$U(1)$ gauge symmetry described by $\mathscr{A}_\mu^3$ still remains
as the residual local gauge symmetry $H = U(1)$,
which leaves $\boldsymbol{\phi}_\infty$ invariant
(the local rotation around the axis of the scalar field $\boldsymbol{\phi}_\infty$).

Thus, the SSB is sufficient for the Higgs mechanism to take place.
But, it is not clear whether the SSB is necessary or not for the Higgs mechanism to work. This description depends on the specific gauge.

§ Gauge-invariant Higgs mechanism: SU(2) case

Next, we give a gauge-invariant description for the Higgs mechanism.

• We construct a composite vector field $\mathscr{W}_\mu(x)$ from $\mathscr{A}_\mu(x)$ and $\boldsymbol{\phi}(x)$ by

$$\mathscr{W}_\mu(x) := -ig^{-1}[\hat{\boldsymbol{\phi}}(x), \mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x)], \quad \hat{\boldsymbol{\phi}}(x) := \boldsymbol{\phi}(x)/v. \tag{1}$$

• We find that the kinetic term of the Yang-Mills-Higgs model is identical to the "mass term" of the vector field $\mathscr{W}_\mu(x)$:

$$\frac{1}{2}\mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi}(x) \cdot \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}(x) = \frac{1}{2}M_W^2 \mathscr{W}^\mu(x) \cdot \mathscr{W}_\mu(x), \quad M_W := gv, \tag{2}$$

as far as the constraint ($\boldsymbol{\phi} \cdot \boldsymbol{\phi} = 1$) is satisfied. This fact is shown explicitly $G = SU(2)$,

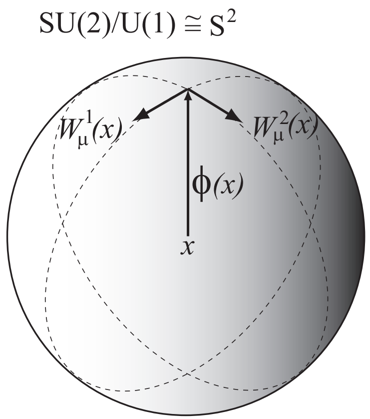
$$\begin{aligned} g^2 v^2 \mathscr{W}^\mu \cdot \mathscr{W}_\mu =& v^{-2} 2\mathrm{tr}([\boldsymbol{\phi}, \mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi}][\boldsymbol{\phi}, \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}]) \\ =& v^{-2}\{(\boldsymbol{\phi} \cdot \boldsymbol{\phi})(\mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi} \cdot \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}) - (\boldsymbol{\phi} \cdot \mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi})(\boldsymbol{\phi} \cdot \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi})\} \\ =& (\mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi}) \cdot (\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}), \end{aligned} \tag{3}$$

where we have used the constraint ($\boldsymbol{\phi} \cdot \boldsymbol{\phi} \equiv v^2$) and $\boldsymbol{\phi} \cdot \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi} = \boldsymbol{\phi} \cdot \partial_\mu \boldsymbol{\phi} + \boldsymbol{\phi} \cdot (g\mathscr{A}_\mu \times \boldsymbol{\phi}) = g\mathscr{A}_\mu \cdot (\boldsymbol{\phi} \times \boldsymbol{\phi}) = 0$, with $\boldsymbol{\phi} \cdot \partial_\mu \boldsymbol{\phi} = 0$ following from differentiating the constraint.

• Remarkably, the above "mass term" of $\mathscr{W}_\mu$ is gauge invariant, since $\mathscr{W}_\mu$ obeys the adjoint gauge transformation:

$$\mathscr{W}_\mu(x) \to U(x)\mathscr{W}_\mu(x)U^{-1}(x). \tag{4}$$

Therefore, $\mathscr{W}_\mu$ becomes massive without breaking the original gauge symmetry.
The above $\mathscr{W}_\mu$ gives a gauge-independent definition of the massive gluon mode in the operator level.
The SSB of gauge symmetry is not necessary for generating the mass of $\mathscr{W}_\mu$.
(We do not need to choose a specific vacuum from all possible degenerate ground states distinguished by the direction of $\boldsymbol{\phi}$.)



How is this description related to the conventional one?

• Despite its appearance of $\mathscr{W}_\mu$,
the independent internal degrees of freedom
of $\mathscr{W}_\mu = (\mathscr{W}_\mu^A)$ $(A = 1, 2, 3)$ is equal to $\dim(G/H) = 2$, since

$$\mathscr{W}_\mu(x) \cdot \boldsymbol{\phi}(x) = 0. \tag{5}$$

Notice that this is a gauge-invariant statement. Thus, $\mathscr{W}_\mu(x)$ represent the massive modes corresponding to the coset space G/H components as expected. [We understand the **residual gauge symmetry** left in the partial SSB: $G = SU(2) \to H = U(1)$.]

In fact, by taking the unitary gauge $\boldsymbol{\phi}(x) \to \boldsymbol{\phi}_\infty = v\hat{\boldsymbol{\phi}}_\infty$, $\mathscr{W}_\mu$ reduces to

$$\begin{aligned}\mathscr{W}_\mu(x) \to -ig^{-1}[\hat{\boldsymbol{\phi}}_\infty, \mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}_\infty] =&[\hat{\boldsymbol{\phi}}_\infty, [\hat{\boldsymbol{\phi}}_\infty, \mathscr{A}_\mu(x)]] \\ =&\mathscr{A}_\mu(x) - (\mathscr{A}_\mu(x) \cdot \hat{\boldsymbol{\phi}}_\infty)\hat{\boldsymbol{\phi}}_\infty. \end{aligned} \tag{6}$$

Then $\mathscr{W}_\mu$ agrees with the off-diagonal components for the specific choice $\hat{\phi}_\infty^A = \delta^{A3}$:

$$\mathscr{W}_\mu^A(x) \to \begin{cases} \mathscr{A}_\mu^a(x) & (A = a = 1, 2) \\ 0 & (A = 3) \end{cases}. \tag{7}$$

The constraint $\boldsymbol{\phi} \cdot \boldsymbol{\phi} = v^2$ represents the vacuum manifold in the target space of the scalar field $\boldsymbol{\phi}$. The scalar field $\boldsymbol{\phi}$ subject to the constraint $\boldsymbol{\phi} \cdot \boldsymbol{\phi} = v^2$ is regarded as the **Nambu-Goldstone modes** living in the **flat direction** at the bottom of the potential $V(\phi)$, giving the **degenerate lowest energy states**.
Therefore, the massive field $\mathscr{W}_\mu$ is formed by combining the **massless (would-be) Nambu-Goldstone modes** with the original massless Yang-Mills field $\mathscr{A}_\mu$.
This corresponds to the conventional explanation that the gauge boson acquires the mass by absorbing the Nambu-Goldstone boson appeared in association with the SSB.

This suggests that the original gauge field $\mathscr{A}_\mu$ is separated into two pieces:

$$\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{W}_\mu(x). \tag{8}$$

By definition, $\mathscr{V}_\mu(x)$ transforms under the gauge transformation just like $\mathscr{A}_\mu(x)$:

$$\mathscr{V}_\mu(x) \to U(x)\mathscr{V}_\mu(x)U^{-1}(x) + ig^{-1}U(x)\partial_\mu U^{-1}(x). \tag{9}$$

According to the definition of $\mathscr{W}_\mu(x)$, it is shown that $\mathscr{W}_\mu(x) = 0$ is equivalent to

$$\mathscr{D}_\mu[\mathscr{V}]\hat{\boldsymbol{\phi}}(x) = 0. \tag{10}$$

Using (5) and (10), we find that $\mathscr{V}_\mu$ is constructed from $\mathscr{A}_\mu$ and $\boldsymbol{\phi}$ as [Manton (1977)]

$$\mathscr{V}_\mu(x) = c_\mu(x)\hat{\boldsymbol{\phi}}(x) + ig^{-1}[\hat{\boldsymbol{\phi}}(x), \partial_\mu\hat{\boldsymbol{\phi}}(x)], \quad c_\mu(x) := \mathscr{A}_\mu(x) \cdot \hat{\boldsymbol{\phi}}(x). \tag{11}$$

In the unitary gauge $\boldsymbol{\phi}(x) \to \boldsymbol{\phi}_\infty = v\hat{\boldsymbol{\phi}}_\infty$, $\hat{\boldsymbol{\phi}}_\infty^A = \delta^{A3}$, $\mathscr{V}_\mu$ agrees with the diagonal component

$$\mathscr{V}_\mu(x) \to (\mathscr{A}_\mu(x) \cdot \hat{\boldsymbol{\phi}}_\infty)\hat{\boldsymbol{\phi}}_\infty \to \begin{cases} 0 & (A = a = 1, 2) \\ \mathscr{A}_\mu^3(x) & (A = 3) \end{cases}. \tag{12}$$

Thus, the above arguments go well in the topologically trivial sector.
The topologically non-trivial sector is discussed later.

• First, we introduce $\mathscr{V}_\mu(x)$ and $\mathscr{W}_\mu(x)$ as composite field operators of $\mathscr{A}_\mu(x)$ and $\hat{\boldsymbol{\phi}}(x)$.
• Then we regard a set of field variables $\{c_\mu(x), \mathscr{W}_\mu(x), \hat{\boldsymbol{\phi}}(x)\}$ as obtained from $\{\mathscr{A}_\mu(x), \hat{\boldsymbol{\phi}}(x)\}$ based on **change of variables**:

$$\{c_\mu(x), \mathscr{W}_\mu(x), \hat{\boldsymbol{\phi}}(x)\} \leftarrow \{\mathscr{A}_\mu(x), \hat{\boldsymbol{\phi}}(x)\}. \tag{13}$$

• Finally, we identify $c_\mu(x)$, $\mathscr{W}_\mu(x)$ and $\hat{\boldsymbol{\phi}}(x)$ with the fundamental field variables (independent up to the constraint (2)) for describing the massive Yang-Mills theory anew.
(Here fundamental means that the quantization should be performed with respect to those variables $\{c_\mu(x), \mathscr{W}_\mu(x), \hat{\boldsymbol{\phi}}(x)\}$ which appear in the path-integral measure.)

According to the decomposition $\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{W}_\mu(x)$, the field strength $\mathscr{F}_{\mu\nu}(x)$ of the gauge field $\mathscr{A}_\mu(x)$ is decomposed as

$$\begin{aligned}\mathscr{F}_{\mu\nu}[\mathscr{A}] :=& \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu - ig[\mathscr{A}_\mu, \mathscr{A}_\nu] \\ =& \mathscr{F}_{\mu\nu}[\mathscr{V}] + \mathscr{D}_\mu[\mathscr{V}]\mathscr{W}_\nu - \mathscr{D}_\nu[\mathscr{V}]\mathscr{W}_\mu - ig[\mathscr{W}_\mu, \mathscr{W}_\nu]. \end{aligned} \tag{14}$$

By substituting this decomposition into the Yang-Mills-Higgs Lagrangian, we obtain

$$\begin{aligned}\mathscr{L}_{\mathrm{YMH}} =& -\frac{1}{4}\mathscr{F}_{\mu\nu}[\mathscr{V}] \cdot \mathscr{F}^{\mu\nu}[\mathscr{V}] \\ & -\frac{1}{4}(\mathscr{D}_\mu[\mathscr{V}]\mathscr{W}_\nu - \mathscr{D}_\nu[\mathscr{V}]\mathscr{W}_\mu)^2 \\ & +\frac{1}{2}\mathscr{F}_{\mu\nu}[\mathscr{V}] \cdot ig[\mathscr{W}^\mu, \mathscr{W}^\nu] - \frac{1}{4}(ig[\mathscr{W}_\mu, \mathscr{W}_\nu])^2 \\ & +\frac{1}{2}M_W^2 \mathscr{W}^\mu \cdot \mathscr{W}_\mu, \quad \mathscr{D}_\mu[\mathscr{V}] := \partial_\mu - ig[\mathscr{V}_\mu, \cdot]. \end{aligned} \tag{15}$$

The field $\mathscr{W}_\mu$ has the ordinary kinetic term and the mass term. Therefore, there is a massive vector pole in the propagator of $\mathscr{W}_\mu$ (after a certain gauge fixing). Thus, $\mathscr{W}_\mu$ is not an auxiliary field, but is a propagating field with the mass M_W (up to possible quantum corrections).

§ Confined massive phase: SU(2) case

Finally, we discuss the implications for quark confinement.
The field strength $\mathscr{F}_{\mu\nu}[\mathscr{V}](x)$ of $\mathscr{V}_\mu(x)$ is shown to be proportional to $\hat{\boldsymbol{\phi}}(x)$:

$$\begin{aligned}\mathscr{F}_{\mu\nu}[\mathscr{V}](x) =& \hat{\boldsymbol{\phi}}(x)\{\partial_\mu c_\nu(x) - \partial_\nu c_\mu(x) + H_{\mu\nu}(x)\}, \\ & H_{\mu\nu}(x) := ig^{-1}\hat{\boldsymbol{\phi}}(x)\cdot[\partial_\mu\hat{\boldsymbol{\phi}}(x), \partial_\nu\hat{\boldsymbol{\phi}}(x)],\end{aligned} \tag{1}$$

We can introduce the Abelian-like $SU(2)$ gauge-invariant field strength $f_{\mu\nu}$ by

$$f_{\mu\nu}(x) := \hat{\boldsymbol{\phi}}(x)\cdot\mathscr{F}_{\mu\nu}[\mathscr{V}](x) = \partial_\mu c_\nu(x) - \partial_\nu c_\mu(x) + H_{\mu\nu}(x). \tag{2}$$

In the low-energy $E \ll M_W$ or the long-distance $r \gg M_W^{-1}$ region, we can neglect $\mathscr{X}_\mu$.
Then the dominant low-energy modes are described by the restricted Yang-Mills theory:

$$\mathscr{L}_{\rm YM}^{\rm rest} = -\frac{1}{4}\mathscr{F}^{\mu\nu}[\mathscr{V}]\cdot\mathscr{F}_{\mu\nu}[\mathscr{V}] = -\frac{1}{4}f^{\mu\nu}f_{\mu\nu}. \tag{3}$$

⊙ In the low-energy $E \ll M_W$ or the long-distance $r \gg M_W^{-1}$ region, the massive components $\mathscr{X}_\mu(x)$ become negligible and the other restricted fields become dominant. This is a phenomenon called the "Abelian" dominance in quark confinement. ['tHooft 81, Ezawa-Iwazaki 82]

The "Abelian" dominance in quark confinement is understood as a consequence of the Higgs mechanism for the relevant (or equivalent) Yang-Mills-Higgs model in the gauge-invariant way.

⊙ In the Yang-Mills-Higgs model, $\mathscr{A}_\mu(x)$ and $\boldsymbol{\phi}(x)$ are independent field variables. However, the Yang-Mills theory should be described by $\mathscr{A}_\mu(x)$ alone and hence $\boldsymbol{\phi}$ must be supplied by the gauge field $\mathscr{A}_\mu(x)$ due to the strong interactions.
[In other words, the scalar field $\boldsymbol{\phi}$ should be given as a (complicated) functional of the gauge field.]

This is achieved by imposing the constraint which we call the **reduction condition**. We choose e.g.,

$$\boldsymbol{\chi}(x) := [\hat{\boldsymbol{\phi}}(x), \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x)] = \mathbf{0} \Longleftrightarrow \mathscr{D}^\mu[\mathscr{V}]\mathscr{W}_\mu(x) = 0. \tag{4}$$

This condition is gauge covariant, $\boldsymbol{\chi}(x) \to U(x)\boldsymbol{\chi}(x)U^{-1}(x)$.
The reduction condition plays the role of eliminating the extra degrees of freedom introduced by the radially fixed scalar field into the Yang-Mills theory, since

$$\boldsymbol{\chi}(x) \cdot \hat{\boldsymbol{\phi}}(x) = 0. \tag{5}$$

See Kondo et al., Phys. Report 579, 1–226 (2015), arXiv:1409.1599 [hep-th].

⊙ Fortunately, the reduction condition is automatically satisfied in the level of field equations. We introduce a Lagrange multiplier field $\lambda(x)$ to incorporate the constraint

$$\mathscr{L}'_{\rm YMH} = \mathscr{L}_{\rm YMH} + \lambda(x)\left(\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x) - v^2\right). \tag{6}$$

Then the field equations are obtained as

$$\frac{\delta S'_{\rm YMH}}{\delta\lambda(x)} = \boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x) - v^2 = 0, \tag{7}$$

$$\frac{\delta S'_{\rm YMH}}{\delta\mathscr{A}^\mu(x)} = \mathscr{D}^\nu[\mathscr{A}]\mathscr{F}_{\nu\mu}(x) - ig[\boldsymbol{\phi}(x), \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}(x)] = 0, \tag{8}$$

$$\frac{\delta S'_{\rm YMH}}{\delta\boldsymbol{\phi}(x)} = -\mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}(x) - 2\boldsymbol{\phi}(x)V'(\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x)) + 2\lambda(x)\boldsymbol{\phi}(x) = 0. \tag{9}$$

The reduction condition is automatically satisfied:
$\mathscr{D}^\mu(8) \Longrightarrow 0 = \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}^\nu[\mathscr{A}]\mathscr{F}_{\nu\mu} = ig\mathscr{D}^\mu[\mathscr{A}][\boldsymbol{\phi}, \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}] = ig[\boldsymbol{\phi}, \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}]$
$[\boldsymbol{\phi}, (9)] \Longrightarrow [\boldsymbol{\phi}, \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}] = [\boldsymbol{\phi}, -2\boldsymbol{\phi}V'(\boldsymbol{\phi}\cdot\boldsymbol{\phi}) + 2\lambda\boldsymbol{\phi}] = 0$
The equivalence between the Yang-Mills-Higgs theory and the Yang-Mills theory is expected to hold only when the scalar field is radially fixed. If we include the radial degree of freedom for the scalar field, the equivalence is lost. Indeed, the radial degree of freedom for the scalar field corresponds to the Higgs particle with a non-zero mass.

§ Conclusion and discussion

$\odot$ We have reconsidered the description of the Brout-Englert-Higgs mechanism or Higgs mechanism by which a gauge boson acquires the mass.

• We have given a manifestly gauge-invariant description of the Higgs mechanism in the operator level, which does not rely on spontaneous symmetry breaking which is signaled by a non-vanishing vacuum expectation value of the scalar field. This gives a gauge-independent explanation of the Higgs phenomenon.

• For the Higgs mechanism to work, the spontaneous symmetry breaking is sufficient but not necessary.

$\odot$ This enables us to discuss the confinement-Higgs complementarity from a new perspective. Because the original gauge symmetry is retained at any stage.

• The $SU(2)$ Yang-Mills theory in the gapped (confined) phase is "equivalent" to the Yang-Mills-Higgs theory with a radially fixed adjoint scalar field in the Higgs phase which was considered to be associated to the SSB: $G = SU(2) \to H = U(1)$.

• The "Abelian" dominance in quark confinement of the Yang-Mills theory is understood as a consequence of the Higgs mechanism for the "equivalent" Yang-Mills-Higgs model.

Perspective:

• The case of larger gauge groups $SU(N)$ $(N \geq 3)$ will be treated in a subsequent paper. In particular, some interesting cases

$SU(3) \to U(1) \times U(1)$, $SU(3) \to U(2)$,

and

$SU(2) \times U(1) \to U(1)$ will be discussed in detail.

Thank you very much for your attention.

$\odot$ The Yang-Mills-Higgs model includes the parameters specifying the potential besides the gauge coupling. They are arbitrary and hence the mass gap of the theory is not uniquely determined.

In the Yang-Mills theory, indeed, the mass M_W can be generated in a dynamical way, e.g., by a gauge-invariant vacuum condensation $\langle \mathscr{W}^\mu \cdot \mathscr{W}_\mu \rangle$ so that $M_W^2 \simeq \langle \mathscr{W}^\mu \cdot \mathscr{W}_\mu \rangle$ due to the quartic self-interactions $-\frac{1}{4}(ig[\mathscr{W}_\mu(x), \mathscr{W}_\nu(x)])^2$ among $\mathscr{W}_\mu(x)$ field, in sharp contrast to the ordinary Yang-Mills-Higgs model. The analytical calculation for such a condensate was done in [**?**]. Moreover, the mass M_W has been measured by numerical simulations on the lattice in [**?**] (see also section 9.4 of [**?**]) as

$$M_W \simeq 2.69 \sqrt{\sigma_{\rm phys}} \simeq 1.19 \text{GeV}, \tag{1}$$

where $\sigma_{\rm phys}$ is the string tension of the linear potential in the quark-antiquark potential.

$\odot$ The mass M_W is used to show the existence of confinement-deconfinement phase transition at a finite critical temperature T_c, separating confinement phase with vanishing Polyakov loop average at low temperature and deconfinement phase with non-vanishing Polyakov loop average at high temperature [**?**]. The critical temperature T_c is obtained from the calculated ratio T_c/M_W for a given value of M_W, which provides a reasonable estimate.

$\odot$ Notice that we cannot introduce the ordinary mass term for the field $\mathscr{V}_\mu$, since it breaks the original gauge invariance. But, another mechanism of generating mass for the Abelian gauge field $G_\mu := c_\mu + h_\mu$ could be available, e.g., magnetic mass for photon due to the Debye screening caused by magnetic monopoles, which yields confinement and mass gap in three-dimensional Yang-Mills-Higgs theory as shown in [**?**]. Moreover, the Abelian gauge field must be confined, which is a problem of gluon confinement. In view of these, the full propagator of the Abelian gauge field must have a quite complicated form, as has been discussed in e.g., [**?**].

⊙ Notice that $H_{\mu\nu}(x)$ is *locally* closed $(dH = 0)$ and hence it can be *locally* exact $(H = dh)$ due to the Poincaré lemma. Then $H_{\mu\nu}(x)$ has the Abelian potential $h_\mu(x)$:

$$H_{\mu\nu}(x) = \partial_\mu h_\nu(x) - \partial_\nu h_\mu(x). \tag{2}$$

Therefore, the $SU(2)$ gauge-invariant Abelian-like field strength $f_{\mu\nu}$ is rewritten as

$$f_{\mu\nu}(x) = \partial_\mu G_\nu(x) - \partial_\nu G_\mu(x), \ G_\mu(x) := c_\mu(x) + h_\mu(x). \tag{3}$$

We call c_μ the **electric potential** and h_μ the **magnetic potential**. Indeed, h_μ agrees with the Dirac magnetic potential, see section 6.10 of the review[Physics Reports].
We can define the *magnetic–monopole current* $k^\mu(x)$ in a gauge-invariant way:

$$k^\mu(x) = \partial_\nu {}^*f^{\mu\nu}(x), \tag{4}$$

where * denotes the Hodge dual. The magnetic current $k^\mu(x)$ is not identically zero, since the Bianchi identity valid for c_μ is violated by h_μ.

§ Decomposition of the Lagrangian

By using the expression for $\mathscr{V}_\mu$, the Lagrangian density is further rewritten into the form which is completely rewritten in terms of the independent field variables $\{G_\mu(x), W_\mu^a(x), \hat{\boldsymbol{\phi}}^A(x)\}$:

$$\begin{aligned}\mathscr{L}_{\mathrm{YMH}} = & -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} - \frac{1}{2} W^{\mu a} Q_{\mu\nu}^{ab} W^{\nu b}, \\ & - \frac{1}{4} g^2 \epsilon^{ab3} \epsilon^{cd3} W_\mu^a W_\nu^b W^{\mu c} W^{\nu d} \\ & + \frac{1}{2} M_W^2 W^{\mu a} W_\mu^a,\end{aligned} \tag{1}$$

where $f_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu$ is the $SU(2)$ invariant "Abelian-like" field strength and $Q_{\mu\nu}^{ab}$ $(a, b = 1, 2)$ is defined by (see section 7 of the review)

$$\begin{aligned}Q_{\mu\nu}^{ab} :=& g_{\mu\nu} K^{ab} + 2ig f_{\mu\nu} \boldsymbol{\phi}^A (T_A)^{ab}, \\ K^{ab} :=& [-\partial^\rho \partial_\rho + g^2 G^\rho G_\rho] \delta^{ab} + g[2G_\rho \partial^\rho + \partial^\rho G_\rho] \epsilon^{ab}.\end{aligned} \tag{2}$$

§ Lattice Higgs-Confinement phase: SU(2)

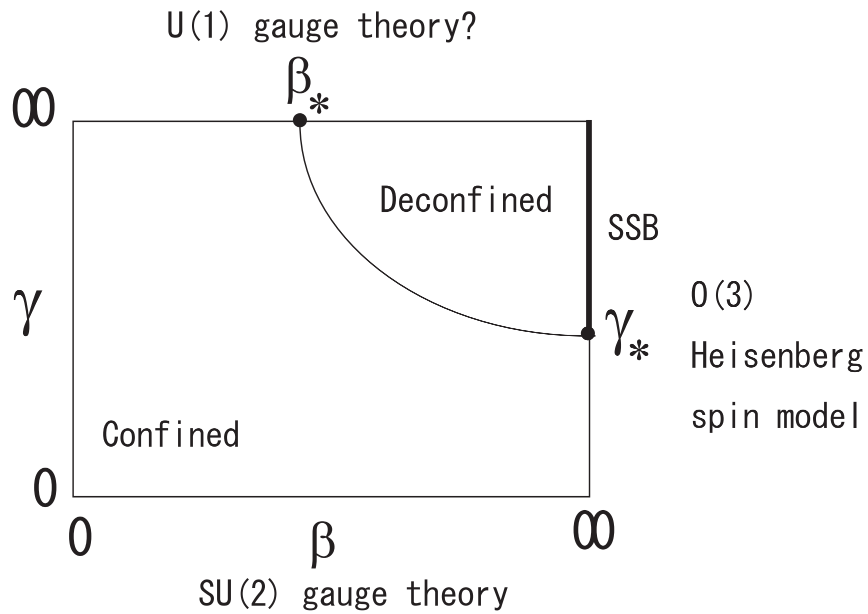


Figure 1: The phase diagram of $SU(2)$ gauge-Higgs model with the adjoint Higgs field.

$$Z(\beta,\gamma) = \int \prod_{n\in L} d\phi(n) \prod_{\ell\in L} dU_\ell \exp\{-\beta S_W[U] - \gamma S_H[\phi,U]\}.$$

$$S_W[U] = \sum_{P\in L}[1 - \mathrm{tr}(U_P)/\mathrm{tr}(\mathbf{1})], \quad U_P := \prod_{\ell\in\partial P} U_\ell,$$

$$S_H[\phi,U] = \sum_{n\in L}\sum_{\mu}[1 - \phi^A(n)\mathcal{U}^\mu_{AB}(n)\phi^B(n+\mu)], \mathcal{U}^\mu_{AB}(n) := \mathrm{tr}[U_\mu(n)T_A U^\dagger_\mu(n)T_B]/\mathrm{tr}(\mathbf{1}$$

§ Higgs mechanism in $U(1)$ case

We use the representation: **polar decomposition** for a radially fixed scalar field:

$$\phi(x) = \frac{v}{\sqrt{2}} e^{i\pi(x)/v} \in \mathbb{C}, \quad \pi(x) \in \mathbb{R},$$

$$\Longrightarrow D_\mu \phi = (\partial_\mu - ieA_\mu)\phi(x) = -\frac{1}{\sqrt{2}} iev \left(A_\mu - \frac{1}{ev}\partial_\mu \pi \right) e^{i\pi/v}.$$

$$\Longrightarrow (D_\mu \phi)^*(D^\mu \phi) = \frac{1}{2}(ev)^2 \left(A_\mu - \frac{1}{ev}\partial_\mu \pi \right)^2. \tag{1}$$

The $\mathscr{L}_{\rm AH}$ is completely rewritten in terms of the massive field

$$W_\mu := A_\mu(x) - m^{-1}\partial_\mu \pi(x) \to \mathscr{L}_{\rm AH} = -\frac{1}{4}(\partial_\mu W_\nu - \partial_\nu W_\mu)^2 + \frac{1}{2}m^2 W_\mu W^\mu, \tag{2}$$

The field W_μ has a gauge invariant representation written in terms of ϕ and A_μ:

$$W_\mu(x) = ie^{-1}\hat{\phi}^*(x) D_\mu \hat{\phi}(x) = -ie^{-1}\hat{\phi}(x) D_\mu \hat{\phi}^*(x), \quad \hat{\phi}(x) := \phi(x)/|\phi(x)|. \tag{3}$$

This representation is parameterization independent. Therefore, we can use also

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \varphi(x) + i\chi(x)]. \tag{4}$$

It is shown that W^μ agrees with the Noether current J^μ associated to the $U(1)$ global gauge symmetry (up to an overall factor).

$$J^\mu = ie\phi(D^\mu\phi)^* = -M^2 W^\mu. \tag{5}$$

Since the Noether current J^μ is conserved $\partial_\mu J^\mu = 0$, the W^μ satisfies the (divergenceless) relation:

$$\partial_\mu W^\mu = 0, \tag{6}$$

which is the subsidiary condition for the massive field W^μ.

The conserved Noether charge is a generator of the $U(1)$ global transformation:

$$\delta\phi(x) = [i\theta Q, \phi(x)] = i\theta \int d^d y [J^0(y), \phi(x)] = i\theta e\phi(x). \tag{7}$$

where $J^0 = ie\phi\Pi_\phi$. This is consistent with no SSB:

$$\langle 0|\delta\phi(x)|0\rangle = i\theta e\langle 0|\phi(x)|0\rangle = 0. \tag{8}$$

§ Non-Abelian mass term (complete SSB)

In the non-Abelian case, we add a gauge-invariant mass term $\mathscr{L}_{\mathrm{m}}$ of the form:

$$\mathscr{L}_{\mathrm{m}} = \frac{1}{2}M^2(\mathscr{A}_\mu^A - K_\mu^A)(\mathscr{A}^{\mu A} - K^{\mu A}) = M^2 \mathrm{tr}\left\{(\mathscr{A}_\mu - K_\mu)^2\right\}, \tag{1}$$

where the explicit form of $K_\mu(x) = K_\mu^A(x)T_A$ is determined so that $\mathscr{L}_{\mathrm{m}}$ is gauge invariant. It must reduce to the extended Stückelberg field in the Abelian limit.Using the element of the unitary group U, we define

$$K_\mu(x) = ig^{-1}U^{-1}(x)\partial_\mu U(x). \tag{2}$$

Then such a mass term is indeed written as [Kunimasa and Goto, 1967]

$$\mathscr{L}_{\mathrm{m}} = g^{-2}M^2\mathrm{tr}\left\{(U^{-1}D_\mu[\mathscr{A}]U)^2\right\} = g^{-2}M^2\mathrm{tr}\left\{\left(U^{-1}(\partial_\mu U - ig\mathscr{A}_\mu U)\right)^2\right\}. \tag{3}$$

Here, if U is parameterized as

$$U(x) = e^{i\frac{\chi(x)}{v}},\ \chi(x) = \chi^A(x)T_A, \tag{4}$$

it turns out that χ corresponds to the Stückelberg field.

§ Introducing the Higgs modes

We can introduce the Higgs mode $\rho(x)$ by removing the constraint $\boldsymbol{\phi}(x) \cdot \boldsymbol{\phi}(x) = v^2$. We introduce a unit field $\hat{\boldsymbol{\phi}}(x)$ satisfying $\hat{\boldsymbol{\phi}}(x) \cdot \hat{\boldsymbol{\phi}}(x) = 1$ to separate ρ:

$$\boldsymbol{\phi}(x) = h(x)\hat{\boldsymbol{\phi}}(x) = [v + \rho(x)]\hat{\boldsymbol{\phi}}(x). \tag{1}$$

Then the covariant derivative of $\boldsymbol{\phi}$ reads

$$\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}(x) = (\partial_\mu h(x))\hat{\boldsymbol{\phi}}(x) + h(x)(\mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x)). \tag{2}$$

The kinetic term of $\boldsymbol{\phi}$ yields the mass term of W_μ, the kinetic term of ρ and interactions:

$$\begin{aligned}
&\frac{1}{2}\mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi}(x) \cdot \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}(x) = \frac{1}{2}\partial^\mu h(x)\partial_\mu h(x) + \frac{1}{2}h(x)^2(\mathscr{D}^\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x) \cdot \mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x)) \\
=&\frac{1}{2}\partial^\mu h(x)\partial_\mu h(x) + \frac{1}{2}\frac{h(x)^2}{v^2}(gv)^2\mathscr{W}^\mu(x) \cdot \mathscr{W}_\mu(x), \\
=&\frac{1}{2}\partial^\mu \rho(x)\partial_\mu \rho(x) + \frac{1}{2}M_W^2\mathscr{W}^\mu(x) \cdot \mathscr{W}_\mu(x) + g^2 v\rho(x)\mathscr{W}^\mu(x) \cdot \mathscr{W}_\mu(x) + \ldots,
\end{aligned} \tag{3}$$

and the potential is calculated from

$$\boldsymbol{\phi}(x) \cdot \boldsymbol{\phi}(x) = h(x)^2 = [v + \rho(x)]^2. \tag{4}$$