QCD-like theories at finite density

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Outline

- Intro & Motivation
- Two-Color QCD with 2 Flavors of Staggered Quarks
- Effective Lattice Theory for Heavy Quarks
- Two-Color QCD in Two Dimensions
- $G_2$-QCD
- Conclusion
Introduction
QCD-like Theories

• compare lattice simulations with functional methods and effective models where there’s no sign problem

• apply to ultracold fermi gases exploit analogies and more experimental data

• strongly correlated fermions in 2+1 dimensions electronic properties of graphene

QCD at finite isospin density

Kamikado, Strodthoff, LvS, PLB 718 (2013) 1044

polarised fermi gas at unitarity

Gubbels, Stooof, arXiv:1205.0568

polarised fermi gas at unitarity

Confinement 12
1 September 2016 | Lorenz von Smekal | p. 4
Fermion-Sign Problem

sign problem:

\[ (\text{Det } D(\mu_f))^* = \text{Det } D(-\mu_f) \]

• except if:

(a) two degenerate flavors with isospin chemical potential

fermion determinant

\[ \rightsquigarrow \text{Det}(D(\mu_I)D(-\mu_I)) \]

Dyson index:

\[ \beta = 2 \]

QCD at finite isospin density

(b) anti-unitary symmetry

\[ TD(\mu)T^{-1} = D(\mu)^* \quad T^2 = \pm 1 \]

fermion color representation:

(i) pseudo-real \[ T^2 = 1 \]

\[ \text{two-color QCD} \]

\[ \beta = 1 \]

(ii) real \[ T^2 = -1 \]

\[ \text{adjoint QCD, or G}_2\text{-QCD} \]

\[ \beta = 4 \]
Phase Diagram of QC$_2$D

- **Quark-Meson-Diquark-Modell:**

- **MC-Simulationen, Gitter:**

**Figure 13.** T. Brauner, K. Fukushima and Y. Hidaka, Phys. Rev. D85 (2012) 074007

**Figure 12.** Tamer Boz et al.: Phase transitions in 2-colour matter plotted versus four-momentum $a_j$ with the help of Edinburgh Parallel Computing Centre funded extreme Computing Initiative. The simulation code was adapted to the DEISA Consortium (www.deisa.eu), funded through the Facilities Capital Fund of BIS and Swansea University. We thank Thomas Gieseke and Jan Pawlowski for stimulating discussions and advice.

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References

• extended flavor symmetry (Pauli-Gürsey), at $\mu = 0$

$$SU(N_f) \times SU(N_f) \times U(1) \rightarrow SU(2N_f)$$

$N_f = 2$: connects pions and $\sigma$-meson with scalar (anti)diquarks.

• Dirac mass (quark condensate)

$$SU(4) \rightarrow Sp(2)$$

or $$SO(6) \rightarrow SO(5)$$

Coset: $S^5$ 5 Goldstone bosons: pions and scalar (anti)diquarks

• color-singlet diquarks (bosonic baryons)
Vacuum Realignment

zero temperature condensates

\[ \langle \bar{q}q \rangle = 2N_f G \cos \alpha \]

\[ \langle qq \rangle = 2N_f G \sin \alpha \]

\[ n_B = 8N_f F^2 \mu \sin^2 \alpha \]


• QMD model phase diagram

QCD with Isospin Chemical Potential

- $T = 0$ isospin density - FRG vs. lattice QCD:

\[
\begin{array}{c}
\text{isospin chemical potential} \\
2\mu/m_\pi \text{ vs. } \rho_1 \text{ [fm}^{-3}\text{]} \\
\end{array}
\]

Kamikado, Strodthoff, LvS, PLB 718 (2013) 1044
Baryon & Isospin Chemical Potential

- Quark Meson Model

\[ T = 0 \]

\[ \mu_I = m_\pi \]

\( \mu \) for (up/anti-down) imbalance
Up-Antidown Imbalance


\[ \mu_I = m_\pi \]

\[ \mu_q = \Delta \]

Sarma

TCP

pion condensation

Chandrasekhar-Clogston

\[ \Delta + \mu_q \]

\[ \Delta - \mu_q \]

\[ h_\omega_- \]

\[ h_\omega_+ \]

\[ \varepsilon_k \]

\[ \mu_I \]

- compare:

Gubbels, Stoof, 2012

Imbalanced Fermi Gases
Two Color QCD - QC$_2$D

Lattice MC Simulations

Hands, Montvay, Scorzato & Skullerud,

Hands, Kenny, Kim & Skullerud,
Eur. Phys. J. A 47 (2011) 60, ...

Kogut, Toublan, Sinclair,


• Talk by A. Kotov, Fri 15:00, D10

$N_f = 2$ Flavors of Staggered Quarks
Two-Colour QCD at Finite Density with Two Flavours of Staggered Quarks

**Introduction**

Introduction

\[ S_f = \bar{\psi} D(\mu) \psi + \frac{\lambda}{2} \left( \psi^T (C \gamma_5) \tau_2 \psi + \bar{\psi} (C \gamma_5) \tau_2 \bar{\psi}^T \right) \]

- **diquark source**

Kogut, Toublan & Sinclair, PRD 68 (2003) 054507

**Figure 10.3.**

The quark number density.

These results look very similar to the expectations from leading order chiral perturbation theory (see section 9.2). We see a non-vanishing diquark condensate at \( \mu = 0 \) due to the explicit symmetry breaking of the included diquark source term. The diquark condensate and the quark number density begin to increase at a critical value of the chemical potential \( \mu_c \), where the chiral condensate begins to decrease. To obtain the according value of \( \mu_c \), we fitted our numerical data to the expectations from leading order chiral perturbation theory, equations (9.2.6)-(9.2.8). The resulting fits for both diquark sources are shown in figure (10.4) and the results for the fit parameters are shown in table (10.1).

**Table 10.1.**

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \langle \bar{\psi} \psi \rangle / N_f )</th>
<th>( \langle \bar{\psi} \psi \rangle / N_f )</th>
<th>( \langle n \rangle / N_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0025</td>
<td>0.41135(88)</td>
<td>0.3430(68)</td>
<td>0.18889(45)</td>
</tr>
<tr>
<td>0.0050</td>
<td>0.41235(73)</td>
<td>0.3335(46)</td>
<td>0.18931(47)</td>
</tr>
</tbody>
</table>

**Figure 10.4.**

Fits to leading order chiral perturbation theory for \( \lambda = 0.0025 \) (left panel) and \( \lambda = 0.0050 \) (right panel).
Goldstone Spectrum - QC$_2$D

- **mixing at finite density:**

$f_0/qq$: \( \frac{1}{2} (\chi^T \tau_2 \chi + \bar{\chi} \tau_2 \bar{\chi}^T) \cos \alpha + \bar{\chi} \chi \sin \alpha \)

\(\pi/\epsilon qq\): \( \bar{\chi} \epsilon \chi \cos \alpha + \frac{1}{2} (\chi^T \tau_2 \epsilon \chi + \bar{\chi} \tau_2 \epsilon \bar{\chi}^T) \sin \alpha \)

\[\begin{array}{c}
\pi/\epsilon qq \\
\epsilon qq \\
f_0/qq
\end{array}\]

**Dyson index:**

\[\beta = 4\]

\[N_f = 2, \beta = 1.5, m = 0.025, \lambda = 0.0025, 12^3 \times 24\] lattice
Bulk Phase of SU(2)

\[
\langle z \rangle = 1 - \frac{1}{N_c} \sum_C \prod_{P \in \partial C} \text{sgn} \text{ tr } P
\]

- $\beta = 1.5 \rightarrow \langle z \rangle \approx 0.95$
- gauge action Symanzik improvement
After noticing the large value of the $Z(2)$ monopole density for the used parameters of the last section, we wanted to run a simulation outside the artificial bulk phase. For this, we will now use the improved gauge action (2.4.11) as we saw in section {2.7} that it suppresses the $Z(2)$ monopole density. Furthermore, we need to increase the inverse coupling to further reduce the $Z(2)$ monopole density and increase the lattice volume to suppress finite volume effects, as explained in section {2.7}. We decided to use a lattice volume of $16^3 \times 32$ and a quark mass of $m = 0.01$ to keep the computing time endurable. At this lattice size and quark mass, we found the inverse coupling of $\beta = 1.7$, $m_\pi / m_\rho = 0.5816(27)$ to be a good compromise as for the $Z(2)$ monopole density is about $0.27$ and the lattice volume is large enough so that the scalar meson and the pion are not degenerated due to finite size effects (see figure 10.13). Furthermore, the pion and the scalar meson, as well as the $\rho$ meson and the $a_1$ meson, are not degenerate at this inverse coupling, meaning that chiral symmetry is broken. Hence at vanishing chemical potential we simulate inside the confined regime.

Goldstone Spectrum

\[ \lambda = 0.0025 \]

\[ \lambda = 0.0001 \]

Figure 10.28: Comparing the scalar diquark, the scalar anti-diquark and the pion to their leading order chiral perturbation theory predictions for \( \lambda = 0.0050 \) (top), \( \lambda = 0.0025 \) (middle) and \( \lambda = 0.0010 \) (bottom).

J. Wilhelm, MSc thesis, JLU Giessen (2016)
Goldstone Spectrum

Dyson: $\beta = 4$

J. Wilhelm, MSc thesis, JLU Giessen (2016)
masses have to be determined from the zeroes of the mixes with the two diquark modes, i.e., the respective Renormalization Group (RG) calculation, where the calibration will make use of this property to fix the pion mass in the masses be, also that because... This is true at all temperatures in the normal phase. Note they must from diquark screening masses (..., antidiquark screening masses C5/C6, and as required by the Silver Blaze... From below.

In the diquark-condensation phase the sigma-meson pole masses we have the general exact zero-symmetry. Moreover, the gap equation for the chiral condensate reduces in the chiral limit... inequality (40), on the other hand, for the linear-sigma model... multiplication of the chiral condensate vanishes C5/C6, on the other hand, for the linear-sigma model masses we obtain s2/C5, for the pion and... results from the mass formulas at C22/C25. While their screen-... and C27, for the pion and... above the onset of diquark condensation phase show a considerable dependence on C22/C25, on the other hand, for the linear-sigma model masses we obtain...
Density & Diquark Condensate

The diquark condensate (left panel) and the chiral condensate without the UV-divergent contribution. It has been shown in [35] that at a smaller value of $\mu$ is why we did not see this effect. Remember, a larger inverse coupling $\lambda$ at $\mu=0$.

Due to our normalization factor of the quark number density, it saturates at one, see equation (6.3.2). In the continuum there is no limit on the rise of the quark number density with the chemical potential. Thus, the diquark condensate as for the other observables it is consistent with the result for the chemical potential. We only include the linear extrapolation to $\mu=0$, which we expect to be due to a UV-divergent contribution to the chiral susceptibility.

To explore if the expected behaviour of the renormalized condensate in the limit can be seen. The results are shown in figures (10.17) to (10.19), where we also show the condensates and the quark number density for this limited range of $\lambda$.

First, notice that the quark number density saturates at $\mu=1$. Saturation is reached, given by $\lambda=0.0010$, $\lambda=0.0025$, and $\lambda=0.0050$. Hence we now need to investigate a renormalized chiral condensate. The resulting fit of the perturbation theory prediction.

The quark number density. As the model study [38], that the Polyakov loop has a peak at the reflection point of the continuum with Wilson fermions [37] and also in an exploratory study. We show quenching effects, too, but these are not visible. To obtain more insight, we measured the local Polyakov loop distribution. We show the distribution for three values of the chemical potential, but it is consistent at all values of $\lambda=1$. In comparison, it has been seen in simulations with Wilson fermions [37] and also in an exploratory study.
Quark Condensate

- additive renormalization:

\[
\langle \bar{\psi} \psi \rangle_{m_q} = \langle \bar{\psi} \psi \rangle_0 + c_2 m_q + \frac{c_{UV}}{a^2} m_q + O(m_q^3)
\]

\[
\chi_{m_q} = c_2 + \frac{c_{UV}}{a^2} + O(m_q^2).
\]
• additive renormalization:

![Graph](image)
Figure 10.24. The resulting fit of the \( \lambda = 0 \) diquark condensate to the chiral perturbation theory prediction.

The obtained results for the Polyakov loop and the \( Z(2) \) monopole density are shown in figure (10.25).

The \( Z(2) \) monopole density increases with \( \mu \) and saturates at \( \mu = 1 \), as the quark number density. We suppose that in the saturated regime the system corresponds to that obtained by pure gauge theory as there are no fermionic degrees of freedom any more. Thus, the difference in the \( Z(2) \) monopole density between \( \mu = 1.1 \) and \( \mu = 0 \) is the same as between pure gauge theory and including dynamical fermions at \( \mu = 0 \), which we confirmed with the data of figure (2.3). Again, the Polyakov loop is constant within the errorbars over the whole range of \( \mu \). In comparison, it has been seen in simulations with Wilson fermions \cite{37} and also in an effective Polyakov loop model study \cite{38}, that the Polyakov loop has a peak at the reflection point of the quark number density. As the \( Z(2) \) monopole does, we expected the Polyakov loop to show quenching effects, too, but these are not visible. To obtain more insight, we also measured the local Polyakov loop distribution. We show the distribution for three values of the chemical potential, but it is consistent at all values of \( \mu \), see figure 86.

also seen by Braguta et al., arXiv:1605.04090
Heavy Quarks

- **effective lattice theory**: systematic expansion in inverse coupling and inverse quark mass

QCD, simulate despite mild sign problem

→ evidence of liquid-gas transition to nuclear matter

Fromm, Langelage, Lottini, Neumann, Philipsen, PRL 110 (2013) 12

- **characteristic differences, 2 ↔ 3 colors?**

Ph. Scior & LvS, PRD 92 (2015) 094504
Heavy Quarks

- **Effective lattice theory:**
  - Systematic expansion in inverse coupling and inverse quark mass
  - QCD, simulate despite mild sign problem
  - Evidence of liquid-gas transition to nuclear matter

- **Characteristic differences, 2 ↔ 3 colors?**

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**Ph. Scior & LvS, PRD 92 (2015) 094504**

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**Fromm, Langelage, Lottini, Neumann, Philipsen, PRL 110 (2013) 12**
QC$_2$D in Two Dimensions

Setup

- Two flavour $SU(2)$-QCD in $2d$
- $N_t \times 16$ lattice with $N_t = 2 \ldots 128$ at fixed $\beta$ and $\kappa$
- Physical scale set by pion mass $m_\pi = 200$ MeV at $N_t = 32$

  $\Rightarrow a = 0.26(4)$ fm $\sim 0.0013$ MeV$^{-1}$
  $\Rightarrow T = 6 \ldots 385$ MeV
  $\Rightarrow \mu = 0 \ldots 885$ MeV
  $\Rightarrow$ diquark mass $m_{d_0^+} = 200$ MeV
  $\Rightarrow$ vector diquark mass $m_{d_1^+} = 177$ MeV
  $\Rightarrow$ $a$ meson mass $m_a = 254$ MeV
QC$_2$D in Two Dimensions

Quark Number

Chiral condensate

$T = 385$ MeV
QC\textsubscript{2D} in Two Dimensions

**Quark Number**

- \( d_1 \)

**Chiral condensate**

- \( d_1^+ \)

\[ T = 192 \, \text{MeV} \]
QC\textsubscript{2}D in Two Dimensions

Quark Number

Chiral condensate

\[ T = 128 \text{ MeV} \]
$T = 96$ MeV
QC$_2$D in Two Dimensions

Quark Number

Chiral condensate

$T = 77$ MeV
QC$_2$D in Two Dimensions

Quark Number

Chiral condensate

$T = 64$ MeV
QC$_2$D in Two Dimensions

Quark Number

Chiral condensate

$T = 55$ MeV
**QC\(_2\)D in Two Dimensions**

**Quark Number**

![Graph of Quark Number vs. \(\mu\) in MeV]

**Chiral condensate**

![Graph of Chiral condensate vs. \(\mu\) in MeV]

\[ T = 48 \text{ MeV} \]
QC$_2$D in Two Dimensions

Quark Number

Chiral condensate

$T = 32$ MeV
QC$_2$D in Two Dimensions

Quark Number vs. $\mu$ in MeV

Chiral condensate vs. $\mu$ in MeV

$T = 24$ MeV
QC\textsubscript{2D} in Two Dimensions

Quark Number

\[ \text{Chiral condensate} \]

\[ T = 16 \text{ MeV} \]
$T = 12$ MeV
QC$_2$D in Two Dimensions

Quark Number

Chiral condensate

$T = 6 \text{ MeV}$
Free Lattice Fermions

- ensembles with fixed particle number $k \mod N$:

$$Z_N(k) = \frac{1}{N} \sum_{n=0}^{N-1} e^{\frac{2\pi i}{N} kn} Z\left(\mu - \frac{2\pi i}{N} Tn\right)$$

- $N = 2, k = 0$:

$$Z_{even}(k) = \frac{1}{2} \left(Z(\mu) + Z(\mu - i\pi T)\right)$$

$N_t \times 16$ lattice with $N_t = 4 \ldots 128$
\textit{G}_2\textit{-QCD}

\textit{G}_2\textit{ gauge theory with fundamental fermions, } T = C\gamma_5 \otimes 1

- 7 colors, 14 gluons
- bound states with integer quark number (fermionic and bosonic baryons)

\[ n_q = 1 \sim \text{Hybrid}(H) \sim qggg \]
\[ n_q = 1 \sim \bar{\Delta}, \bar{N} \sim (\bar{q}q)q \]
\[ n_q = 2 \sim \text{diquarks}(d) \sim q^T q \]
\[ n_q = 3 \sim \Delta, N \sim (q^T q)q \]

- gluodynamic very similar to \textit{SU}(3) (first order deconfinement transition)
Goldstone boson becomes the lightest state, with the between parity even and odd states as well as between scalar and vector diquarks. Especially in the light ensemble, the step toward values of the chemical potential plateaus develop where the might expect Bose-Einstein-condensation in a continuous set of states only when the quark chemical potential reaches the mass of the heavier bosonic diquark states divided by their quark number. It appears that the two different bosonic and fermionic phases at finite density, set by the pseudo-Goldstone scale, an intermediate boson scale set by 3/2 representations. In particular, we find three clearly distinct transitions in the heavier ensemble.

G₂-QCD

- simulate at finite density
  250 M core hours →

baryon density

\textbf{G}_2\text{-QCD}

- effective theory for heavy quarks

\[ N_f = 1 \]

\begin{itemize}
  \item Cold and Dense Region
  \item Deconfinement and unphysical lattice saturation
  \item Three regions with different exponential growth
\end{itemize}

\begin{itemize}
  \item \( a^3 n \) model
  \item Full theory
\end{itemize}

\begin{itemize}
  \item Comparison of full 4d G2 QCD simulations to effective theory
\end{itemize}

\begin{itemize}
  \item \( \frac{\mu}{T} \)
  \item \( \frac{2\mu}{T} \)
  \item \( \frac{3\mu}{T} \)
\end{itemize}

\textit{Scior, Wellegehausen & LvS, in preparation (Lattice 2016)}
G$_2$-QCD in Two Dimensions

Quark Number

Chiral condensate

\[ T = 20 \text{ MeV} \]

- nucleon / delta mass decreases above diquark onset (preliminary)

Wellegehausen & LvS, in preparation (Lattice 2016)

\[ N_t = 64 \]
Conclusions

- **Two-Color QCD with Two Flavors of Staggered Quarks**
  improved action, away from bulk phase \( \rightarrow \) continuum Goldstone spectrum

- **Effective Lattice Theory for Heavy Quarks**
  strong-coupling / hopping expansion \( \rightarrow \) continuous transition to finite diquark density

- **Two-Color QCD in Two Dimensions**
  qualitative understanding from statistical confinement

- **G_2-QCD**
  G_2-nuclear matter, effective theory for heavy quarks with nucleons, understand generic features in two dimensions (way cheaper to simulate)

Thank you for your attention!