Detecting magnetic defects, ensembles, and gluon topological confinement

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Dual superconductivity (’t Hooft, Nambu, Mandelstam)

- Search for **chromomagnetic quantum** d.o.f. in pure YM that could capture the path integral measure:
  
  Abelian projected monopoles, center vortices, correlated chains formed by them: Di Giacomo, Engelhardt & Reinhardt, Faber, Greensite & Olejnik...

- Search for effective dual models in a Higgs phase where the **chromoelectric** confining string is represented by a **classical vortex solution**: Baker, Konishi, Shifman, Suzuki, Tong...
Ensembles of Abelian monopoles and related models

- Abelian ensembles of monopole loops lead to Abelian effective field models

\[ \sum_{n} Z_{n-\text{loops}} = \sum_{n} Z_{n-\text{worldlines}} = 2^{\text{nd}} - Q = \text{Field} - \text{path} - \text{int} \]

(Halpern & Siegel '77, Bardakci & Samuel '78)

- In $SU(3)$ YM: Linearizing with $\Lambda_{\mu\nu}$, Abelian projection + Abelian dominance hypothesis (D. Antonov, 2000)

\[ \Lambda_{\mu\nu} = \left( \partial_{\mu} \vec{\Lambda}_{\nu} - \partial_{\nu} \vec{\Lambda}_{\mu} \right) \cdot \vec{T} + \vec{B}_{\mu\nu} \cdot \vec{T} , \quad \vec{\Lambda} \cdot \vec{T} = \Lambda_{\mu}^{1} T_{1} + \Lambda_{\mu}^{2} T_{2} \]

\[ (\partial_{\nu} \vec{B}_{\mu\nu} = 0) \]
\[ \langle W \rangle = \int [D \Lambda] e^{-\int d^4x \frac{1}{4g^2}(\partial_\mu \tilde{\Lambda}_\nu - \partial_\nu \tilde{\Lambda}_\mu - \tilde{J}_{\mu\nu})^2} Z_{\alpha_1} Z_{\alpha_2} \ldots \]

**Abelian model:**

\[ Z_\alpha = \int [D \psi_\alpha][D \bar{\psi}_\alpha] e^{-\int d^4x \bar{\psi}_\alpha [-D_\alpha^2 + m^2] \psi_\alpha} , \quad D_\mu^\alpha = \partial_\mu - i \tilde{\alpha} \cdot \tilde{\Lambda}_\mu . \]

- \( \tilde{\alpha} \): the three positive roots (tuples) ,  \( \tilde{J}_{\mu\nu} = 2\pi 2N \tilde{w}_e s_{\mu\nu} \)
- \( \tilde{w}_e \) is a weight of the quark representation

(A weight \( \tilde{w} \) is defined by the eigenvalues of the Cartan generators \( T_q \) corresponding to one common eigenvector.

\[ [T_q, T_p] = 0 , \quad T_q \text{ eigenvector} = \tilde{w} |_q \text{ eigenvector} \]

- \( g = 4\pi N/g_e \)
- SSB for \( m^2 < 0 \), after including density-density interactions
- Center vortices couple with \( B_{\mu\nu} \)
  Random surface models (Engelhardt & Reinhardt, 2000)
Via ensembles in the $SU_e(N)$ YM theory

- **Center vortices** (two-dimensional worldsheets) and chains formed by center vortices correlated with monopoles (loops).

$$W_f[A^e_\mu] = \left(e^{i \frac{2\pi}{N}}\right)^\text{link} W_f[P] \text{ vs. } W_a[A^e_\mu] = W_a[P]$$

**Via effective dual YMH models**

- $SU(N) \rightarrow Z(N) \text{ SSB}$
  
- $\mathcal{M} = SU(N)/Z(N) = Ad(SU(N))$
  
- $\Pi_1(Ad(SU(N))) = Z(N) \Rightarrow$ the confining string is a **Center string**: 3D static vortices that minimize an effective energy functional
  
- they confine fundamental quarks to form normal hadrons
M. Baker, J. S. Ball & F. Zachariasen ’97, introduced a dual model with gauge group $SU(3)$ and three adjoint Higgs fields, and computed the interquark potential.

A class of Yang-Mills-Higgs (YMH)

$$\frac{1}{2} \langle D_\mu \psi_I, D^\mu \psi_I \rangle + \frac{1}{4g^2} \langle F_{\mu\nu} - J_{\mu\nu}, F^{\mu\nu} - J^{\mu\nu} \rangle - V_{\text{Higgs}}(\psi_I)$$

$$D_\mu = \partial_\mu - i [\Lambda_\mu, ] , \quad F_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu - i [\Lambda_\mu, \Lambda_\nu]$$

$\psi_I \in \mathfrak{su}(N)$ is a set of adjoint Higgs fields. $I$ is a flavour index

For $SU(N) \to Z(N)$, the manifold of absolute minima $(\phi_1, \phi_2, \ldots) \in \mathcal{M} / U \phi_I U^{-1} = \phi_I$ iff $U \in Z(N)$

Minimum number of flavours is $N$
Center strings can be labelled by the magnetic weights $\vec{\beta}$ of the different group representations, Konishi-Spanu (2001). The asymptotic behavior is *locally* a pure gauge (but not *globally*),

$$S = e^{i \varphi \vec{\beta} \cdot \vec{T}}, \quad \vec{\beta} = 2N \vec{w}, \quad \vec{\beta} \cdot \vec{T} = \vec{\beta} |_q T_q$$

For the fundamental representation we have $N$ weights $\vec{w}_i$ (fundamental colours), $\vec{\beta}_1 + \cdots + \vec{\beta}_N = \vec{0}$

They are associated with the simplest center strings

$$e^{i 2\pi \vec{\beta}_i \cdot \vec{T}} = e^{i 2\pi / N} I$$

$N$-ality for **Infinite** adjoint center string: An asymptotic behavior $S \sim e^{i \varphi 2N \vec{\alpha} \cdot \vec{T}}$ is a closed loop in $SU(N)$. As $\Pi_1(SU(N)) = 0$, it can be continuously deformed into $S \sim I$ at the origin.
Finite center string induced by fundamental sources $\vec{w}, -\vec{w}$

or by three fundamental sources $\vec{w}_1, \vec{w}_2, \vec{w}_3$, $S = e^{i\chi_1 \vec{\beta}_1 \cdot \vec{T}} e^{i\chi_2 \vec{\beta}_2 \cdot \vec{T}}$

$$\vec{\beta}_{c_1} + \vec{\beta}_{c_2} + \vec{\beta}_{c_3} = 0 \quad (\vec{\beta}_1 + \vec{\beta}_2 = -\vec{\beta}_3)$$
\[ \vec{J}_{\mu\nu} = 2\pi 2N \vec{w} s_{\mu\nu}. \] Static sources:

\[ s_{0i} = 0, \quad s_{ij} = -\epsilon_{ijk} \int ds \frac{dx_k}{ds} \delta^{(3)}(x - x(s)) \]
\[ S = e^{i\varphi \vec{\beta} \cdot \vec{T}} \]  \[ \vec{\beta} = 2N \vec{w} \]  \[ \Lambda_i = a(\rho) i S \partial_i S^{-1} \quad \text{(locally)} \]
Adjoint sources $\vec{\alpha}, -\vec{\alpha}$ at a finite distance. $\vec{J}_{\mu\nu} = 2\pi 2N \vec{\alpha} S_{\mu\nu}$

$S_{\text{Abe}} = e^{i\varphi 2N \vec{\alpha} \cdot \vec{T}}$
The roots are the weights of the adjoint representation, which acts via commutators

\[ [T_q, E_\alpha] = \bar{\alpha}|_q E_\alpha \quad , \quad \bar{\alpha} = \vec{w} - \vec{w}' \]

\[ S_{\text{non-Abe}} = e^{i\varphi \vec{\beta} \cdot \vec{T}} W(\gamma) e^{-i\varphi \vec{\beta}' \cdot \vec{T}} \]

\[ W(\gamma) = e^{i\gamma \sqrt{N} T_\alpha} \]

\( \gamma, \varphi \) are bipolar angles
$e^{i\varphi \vec{\beta} \cdot \vec{T}} W(\gamma) e^{-i\varphi \vec{\beta}' \cdot \vec{T}}$

$W(\gamma) = e^{i\gamma \sqrt{N} T_\alpha}$

- $\gamma \sim 0$: $S_{\text{non-Abe}} \sim e^{i\varphi 2N \vec{\alpha} \cdot \vec{T}}$
\[ e^{i\phi \vec{\beta} \cdot \vec{T}} W(\gamma) e^{-i\phi \vec{\beta}' \cdot \vec{T}} \]

\[ W(\gamma) = e^{i\gamma \sqrt{N} T_\alpha} \]

\[ \gamma \sim \pi: \]

\[ W^{-1}(\pi) \vec{w} \cdot \vec{T} W(\pi) = \vec{w}' \cdot \vec{T} \]

Weyl transformation

\[ S_{\text{non-Abe}} \sim W(\pi) \]

(complete screening by dynamical adjoint monopoles)
Hybrid QCD spectrum

- Lattice calculations predict a rich spectrum of exotic mesons
- Some of them correspond to $qg\bar{q}'$ hybrid mesons: a nonsinglet colour pair and a valence gluon that form a colourless state.
- Currently searched by a collaboration based at the Jefferson Lab (GlueX)

**Figure:** From B. Ketzer 2012
Induced by fundamental sources $\vec{w}, -\vec{w}'$, with $\vec{w} \neq \vec{w}'$

LEO, 2012
Infinite object: \((\theta, \phi)\) are spherical angles

\[
S = e^{i\varphi \vec{\beta} \cdot \vec{T}} W(\theta), \quad W(\theta) = e^{i\theta \sqrt{NT}\alpha}, \quad \vec{\alpha} = \vec{w} - \vec{w}'
\]

Around the north pole,

\[
S \sim e^{i\varphi \vec{\beta} \cdot \vec{T}}
\]

Around the south pole, \(W(\pi)\) is a Weyl reflection,

\[
S \sim W(\pi) e^{i\varphi \vec{\beta}' \cdot \vec{T}}
\]

(gauge transformations act on the left)
A well-defined mapping for the local Cartan directions on $S^2$

$$n_q = ST_q S^{-1}, \quad q = 1, \ldots, N - 1$$

The gauge invariant monopole charge is obtained from the (dual) field strength projection along the *local* Cartan directions $n_q$

$$\tilde{Q}_m = 2\pi 2N (\vec{w} - \vec{w}') = 2\pi 2N \vec{\alpha}$$

This monopole can be identified with a valence gluon with adjoint colour $\vec{\alpha} = \vec{w} - \vec{w}'$

**Valence gluons are confined:** $\mathcal{M}$ is a compact group.

$$\Pi_2(\mathcal{M}) = \Pi_2(Ad(G)) = 0, \quad G = SU(N) \Rightarrow \text{there are no isolated dynamical adjoint monopoles}$$

Stable: One-to-one mapping between $\Pi_2 \left( \frac{Ad(G)}{Ad(H)} \right)$ and loops in $\Pi_1(Ad(H))$, $H = U(1)^{N-1}$, that are trivial when seen as loops in $\Pi_1(Ad(G)) = Z(N)$
Double sources (double Wilson loops) in \( SU(2) \)

Two negative (up) and two positive (down) fundamental sources:

\[
S_{\text{non---Abe}} = e^{i\varphi \beta T_1} W(\gamma) e^{i\varphi \beta T_1}
\]

- \( \gamma \sim 0 \): \( S_{\text{non---Abe}} \sim e^{i\varphi 2\beta T_1} \)

- \( \gamma \sim \pi \): \( S_{\text{non---Abe}} \sim W(\pi) \)

\[
W^{-1}(\pi) T_1 W(\pi) = -T_1
\]

The inner fundamental quarks are partially screened by dynamical adjoint monopoles
Gauge fixing in pure YM

Laplacian-center gauge

Faber, Greensite & Olejnik; de Forcrand & Pepe (2001)

\[ A_\mu \rightarrow f_I \rightarrow \text{Ad}(S) \]

\[ U_e A_\mu U_e^{-1} + i U_e \partial_\mu U_e^{-1} \rightarrow U_e f_I U_e^{-1} \rightarrow \text{Ad}(U_e S) \]

Impose a condition on \( S \)

\( S \) is extracted from a polar decomposition \( f_I = Sq_I S^{-1} \) in term of “modulus” \( q_I \) and “phase” \( S \) variables

For three adjoint fields \( f_1, f_2, f_3 \) in \( SU(2) \) this corresponds to the usual polar decomposition of a \( 3 \times 3 \) matrix
Lattice:

- Lowest eigenfunctions of the adjoint covariant Laplacian

\[ \mathcal{D}_\mu \mathcal{D}_\mu f_I = \lambda_I f_I , \quad \mathcal{D}_\mu = \partial_\mu - i [A_\mu, ] \]

Continuum: LEO & Santos-Rosa, 2015

\[ \mathcal{D}_\mu \mathcal{D}_\mu f_I = \ldots \quad \text{set of coupled eqs.} \quad \frac{\delta S_{\text{aux}}}{\delta f_I} = 0 \]

\[ 1 = \int [Df_I] \det \left( \frac{\delta^2 S_{\text{aux}}}{\delta f_I \delta f_j} \right) \delta \left( \frac{\delta S_{\text{aux}}}{\delta f_I} \right) \]

\[ [Df_I] = [Dq_I] [DS] , \quad \sum_I [q_I, u_I] = 0 \]

- \( S \) could have defects
- \( S \sim S' \) if there is a regular \( U_e / S' = U_e S \)
- Gauge fixed config. become partitioned into sectors \( \mathcal{V}(S_0) \)
- They are labelled by representatives \( S_0 \) that could be singular

\[ A_{\mu}^e \rightarrow \text{Ad}(S_0) = R_0 \]
If $P_\mu \in \mathcal{V}(I)$, then, something that is \textit{locally},

$$A_\mu^e \big|_{loc} = S_0 \, P_\mu S_0^{-1} + i \, S_0 \partial_\mu S_0^{-1}$$

($P_\mu$ / $A_\mu^e$ is regular) is in $\mathcal{V}(S_0)$. Defining $\langle , \rangle$ as $\text{Tr}$ in the adjoint:

$$S_{YM} = \int d^4x \frac{1}{4g_e} \, \text{Tr} \left( F_{\mu\nu}^e F_{\mu\nu}^e \right)$$

$$F_{\mu\nu}(A^e) = R_0 \left( F_{\mu\nu}(P) - H_{\mu\nu}(R_0) \right) R_0^{-1}, \quad H_{\mu\nu}(R_0) = i \, R_0^{-1} [\partial_\mu, \partial_\nu] R_0$$

\textbf{Gauge transf. (left action with regular $U_e$):} \quad S_0 \rightarrow U_e \, S_0

$F_{\mu\nu}(A^e)$ rotates and $H_{\mu\nu}(R_0)$ is left invariant

\textbf{Inequiv. config. (right action with regular $\tilde{U}^{-1}$):} \quad S_0' = S_0 \, \tilde{U}^{-1}

- if $\nexists$ regular $U_e / S_0 \, \tilde{U}^{-1} = U_e \, S_0$, \quad $\mathcal{V}(S_0) \rightarrow \mathcal{V}(S_0')$

- $H_{\mu\nu}(R_0') = \tilde{R} \, H_{\mu\nu}(R_0) \, \tilde{R}^{-1}$
Center vortices are also labelled by the weights of the fundamental representation. Example $S_0 = e^{i \chi \vec{\beta} \cdot \vec{T}}$

$$-\mathcal{H}_{\mu \nu}(R_0) = 2\pi \vec{\beta} \cdot \vec{M} \int d^2 \sigma_{\mu \nu} \delta^{(4)}(x - \bar{y}(\sigma_1, \sigma_2)) \ , \ M_q = \text{Ad}(T_q)$$

$$\mathcal{H}_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} H_{\mu \nu}$$

Open center vortex worldsheets with different weights $\vec{\beta}, \vec{\beta}'$ can be matched by monopoles. Only using a Weyl transf., at each junction, there is a contribution

$$- \partial_{\nu} \mathcal{H}_{\mu \nu}(R_0) = 2\pi 2N \vec{\alpha} \cdot \vec{M} \oint_C dy_{\mu} \delta^{(4)}(x - y)$$

Inequivalent configuration (right action): at each junction,

$$-D_{\nu} (\tilde{Z}) \mathcal{H}_{\mu \nu}(R_0') = 2\pi 2N \tilde{R} \vec{\alpha} \cdot \vec{M} \tilde{R}^{-1} \oint_C dy_{\mu} \delta^{(4)}(x - y)$$

$$D_{\mu} (\tilde{Z}) = \partial_{\mu} - i [\tilde{Z}_{\mu}, \cdot] \ , \ \tilde{Z}_{\mu} = i \tilde{R} \partial_{\mu} \tilde{R}^{-1}$$
Ensemble of monopoles with non-Abelian d.o.f.

Motivated by:

\[-\int d^4x \frac{1}{4g^2} \operatorname{Tr} (F^e_{\mu\nu} F^e_{\mu\nu})\]

\[= \int [D\Lambda_{\mu\nu}] e^{-\int d^4x \frac{1}{4g^2} \operatorname{Tr} \Lambda^2_{\mu\nu} + \frac{i}{4\pi N} \int d^4x \operatorname{Tr} \Lambda_{\mu\nu} (F_{\mu\nu}(P) - \mathcal{H}_{\mu\nu}(R'))}\]

- A non-Abelian Hodge decomposition

\[\Lambda_{\mu\nu} = D_\mu(\tilde{Z})(\Lambda_\nu - \tilde{Z}_\nu) - D_\nu(\tilde{Z})(\Lambda_\mu - \tilde{Z}_\mu) + B_{\mu\nu}, \quad D_\nu(\tilde{Z}) B_{\mu\nu} = 0\]

- Only monopoles (attached to unobservable Dirac worldsheets)
- Dress the monopoles with phenomenological information
- Simplest properties of looplike objects: tension, stiffness and interactions \((S_m)\)
Let us consider

\[ Z = \sum_{\text{ensemble}} e^{-S_m} e^{-S_G - S_H} \]

\[ S_G = \int d^4x \frac{1}{4g^2} \text{Tr} \left( D_\mu(\tilde{Z})(\Lambda_\nu - \tilde{Z}_\nu) - D_\nu(\tilde{Z})(\Lambda_\mu - \tilde{Z}_\mu) \right)^2 \]

\[ S_H = -i \int ds \frac{dx_\mu}{ds} \text{Tr} (\Lambda_\mu - \tilde{Z}_\mu) \tilde{R} (\tilde{\alpha} \cdot \tilde{M}) \tilde{R}^{-1} \]

\( \tilde{\alpha} : N(N - 1)/2 \) positive values.

The variable \( \tilde{U} \) is similar to that appearing in the Skyrme model through the combination \( \tilde{Z}_\mu \). Under,

\[ \tilde{U} \rightarrow U \tilde{U} \quad (\tilde{R} \rightarrow R \tilde{R}) \]

\[ \Lambda_\mu \rightarrow R \Lambda_\mu R^{-1} + i R \partial_\mu R^{-1} , \]

\( S_G \) and \( S_H \) are left invariant.
Non-Abelian coupling

- Analyze one $\vec{\alpha}$-sector. For the highest weight $\vec{\alpha}$,

$$\frac{dx_\mu}{ds} \operatorname{Tr} (\Lambda_\mu - \tilde{Z}_\mu) \tilde{R} (\vec{\alpha} \cdot \vec{M}) \tilde{R}^{-1} \Lambda_\mu^A = \frac{dx_\mu}{ds} I_A \Lambda_\mu^A - iz_c \dot{z}_c$$

$$I_A = M_A|_{cd} \tilde{z}_c z_d$$,  \quad z = \tilde{R} u_\alpha$$, \quad u_\alpha is the weight vector

- A worldline with non-Abelian coupling: Balachandran et al. '77:

$$I_A = R (T_A)|_{cd} \tilde{z}_c z_d$$

- An ensemble with non-Abelian coupling:

Santos Rosa, Teixeira & LEO (2014)

$$Z = \int [D\phi] e^{-W[\phi]} \sum_n Z_n$$

$$Z_n = \int [Dm]_n \exp \left[ - \sum_{k=1}^n \int_0^{L_k} ds_k (\cdot)_k \right] , \quad u_\mu = \frac{dx_\mu}{ds} \in S^3$$

$$(\cdot) = \mu + \frac{1}{2} (\tilde{z}_c \dot{z}_c - \dot{\tilde{z}}_c z_c) + \frac{1}{2\kappa} \dot{u}_\mu \dot{u}_\mu - i u_\mu I_A \Lambda_\mu^A(x) + \phi(x) + I_A \phi^A(x)$$
\[ [Dm]_n \equiv \frac{1}{n!} \int_0^\infty \frac{dL_1}{L_1} \frac{dL_2}{L_2} \ldots \frac{dL_n}{L_n} \int dv_1 dv_2 \ldots dv_n \int [Dv(s_1)_{v_1,v_1} \ldots [Dv(s_n)]_{v_n,v_n} \]

\[ v : x, u, z \quad , \quad dv = d^4x d^3u dz d\bar{z} \]

For smooth closed monopole worldlines

\[ \sum_n Z_n = e^{\int_0^\infty \frac{dl}{L} \int dv q(v,v,L)} \quad , \quad q(v,v_0,L) = \int [Dv(s)]_{v,v_0} e^{-\int_0^L ds (\cdot)} \]

\( q(v,v_0,L) \): end-to-end probability, for a line of length \( L \), to start at \( x_0 \), with tangent \( u_0 \) and \( z_0 \), and end at \( x \) with \( u, z \)
• \([Dz(s)]_{z,z_0}\) gives \(\langle z | e^{-\hat{H}L} | z_0 \rangle\)

\[
\hat{H} = -i \, u_{\mu} A_{\mu} \Lambda^A \hat{a}_c^\dagger \hat{a}_d + \phi^A M_{cd} \hat{a}_c^\dagger \hat{a}_d
\]

\(\implies\) in the exponent of \(\sum_n Z_n\), there is a \(\text{Tr} \, e^{-\hat{H}L}\)

• \(\langle \phi | e^{-\hat{H}L} | \psi \rangle\):

\[
\int dzd\bar{z} \, dz_0d\bar{z}_0 \, e^{-\frac{\bar{z} \cdot z}{2}} \, e^{-\frac{\bar{z}_0 \cdot z_0}{2}} \, \phi(z) \, \psi(\bar{z}_0) \, q(v, v_0, L)
\]

• \(\langle b | e^{-\hat{H}L} | a \rangle\):

\[
Q^{ba} = \int dzd\bar{z} \, dz_0d\bar{z}_0 \, e^{-\frac{\bar{z} \cdot z}{2}} \, e^{-\frac{\bar{z}_0 \cdot z_0}{2}} \, z^b \, \bar{z}_0^a \, q(v, v_0, L)
\]

\[
\sum_n Z_n = e^{\int_0^\infty \frac{dl}{L}} \int d^4x \, d^3u \, \text{tr} \, [Q(x,x,u,u,L)] + \ldots
\]

• \(q(v, v_0, L)\) as continuum limit of a Chapman-Kolmogorov recurrence relation for diffusion in \(v\)-space

• The equilibrium theory of inhomogeneous polymers, G. H. Fredrickson (2006)
From the probability to get \( x, u, z \) with \( M \) monomers to the probability to get \( x' = x + u' \Delta L, u', z' \) with \( M + 1 \)

\[
q(v', v_0, L + \Delta L) = \int d^3u dz d\bar{z} \\
\times e^{-\mu \Delta L} e^{-\frac{1}{2\kappa} \Delta L \left( \frac{u' - u}{\Delta L} \right)^2} e^{(\bar{z}' - \bar{z}) \cdot z} \\
\times e^{-\left[ \phi - i u'_\mu \Lambda^A_{\mu} M^A_{cd} \bar{z}'^c z^d + \phi^A M^A_{cd} \bar{z}'^c z^d \right] \Delta L} q(v, v_0, L)
\]

First order in \( \Delta L \) with finite \( \kappa \rightarrow \) Fokker-Plank equation:

\[
\partial_L q = \left[ -\mu - \phi(x) + \frac{\kappa}{\pi} \hat{L}^2_u - u_\mu \partial_\mu + (i u_\mu \Lambda^A_{\mu} - \phi^A) M^A_{cd} \bar{z}'^c \frac{\partial}{\partial \bar{z}'^d} \right] q
\]
Reduced Fokker-Plank equation

\[
\left[ (\partial_L - (\kappa/\pi) \hat{L}_u^2 + (\mu + \phi) 1 + u \cdot D \right] Q(x, x_0, u, u_0, L) = 0
\]

\[
Q(x, x_0, u, u_0, 0) = \delta(x - x_0) \delta(u - u_0) 1
\]

where \( Q^{cd} = Q^{cd} \), 1 is a \( D \times D \) identity matrix (\( D = N^2 - 1 \))

\[
D_\mu = 1 \partial_\mu - i \Lambda^A_\mu M^A
\]

Semiflexible limit:

Small stiffness: disregard the angular momenta \( l \geq 2 \) in an expansion of spherical harmonics on \( S^3 \) (memory loss)
Effective field representation

\[ \int d^4x d^3u Q(x,x,u,u,L) \approx \int d^4x \langle x | e^{-LO} | x \rangle \]

\[ O = -\frac{\pi}{12\kappa} D_\mu D_\mu + (\phi + \mu) 1 + \phi^A M_A \]

\[ Z = \int [D\phi] e^{-W} e^{-Tr \ln O} = \int [D\phi] e^{-W} (Det O)^{-1} \]

\[ = \int [D\phi] e^{-W} \int [D\bar{\zeta}][D\zeta] e^{-\int d^4x \zeta^\dagger O\zeta} \]

- Integrating \( \phi, \phi^A \) with Gaussian weight

\[ Z = \int [D\psi] e^{-\int d^4x L_{\text{eff}}} \]

\[ L_{\text{eff}} = \frac{1}{2} \langle D_\mu \psi_I, D^\mu \psi_I \rangle + \frac{m^2}{2} \langle \psi_I, \psi_I \rangle + \frac{\lambda}{4} \langle \psi_I \wedge \psi_J, \psi_I \wedge \psi_J \rangle + \frac{\eta}{4} \langle \psi_I, \psi_I \rangle \langle \psi_J, \psi_J \rangle \]

- \( \psi_I \): a pair of Hermitian adjoint Higgs fields

- \( m^2 \propto \mu \kappa \)
Natural invariant terms to form $V_{\text{Higgs}}(\psi_I)$

\[
\langle \psi_I, \psi_J \rangle , \quad \langle \psi_I, \psi_J \wedge \psi_K \rangle \\
\langle \psi_I \wedge \psi_J, \psi_K \wedge \psi_L \rangle , \quad \langle \psi_I, \psi_J \rangle \langle \psi_K, \psi_L \rangle
\]

where $\psi_I \wedge \psi_J = -i[\psi_I, \psi_J]$.

- Flavour symmetric Higgs potential, $I \rightarrow A = 1, \ldots, N^2 - 1$

LEO, 2012

\[
c + \frac{m^2}{2} \langle \psi_A, \psi_A \rangle + \frac{\gamma}{3} f_{ABC} \langle \psi_A \wedge \psi_B, \psi_C \rangle + \frac{\lambda}{4} \langle \psi_A \wedge \psi_B, \psi_A \wedge \psi_B \rangle
\]

- Degenerate potential

\[
\frac{\lambda}{4} \langle \psi_A \wedge \psi_B - f_{ABC} \nu \psi_C \rangle^2 , \quad \psi_A \wedge \psi_B - \nu f_{ABC} \psi_C = 0
\]

For $m^2 < \frac{2}{9} \frac{\gamma^2}{\lambda}$, the absolute minima are Lie bases:

$SU(N) \rightarrow Z(N)$

- $m^2 = 0$: Eqs. of motion get Abelianized (all $N$), $a$, $h$, $h_1$

- BPS point $\lambda = 1$ ($N = 2$): Starting point to explore parameter space
Figure: Normal potential at BPS point in $SU(2)$
Heavy quark hybrid potentials in lattice YM

Figure:
Motivated an ensemble of monopoles with non-Abelian d.o.f.

Obtained an effective model based on a set of interacting adjoint fields (after assuming phenomenological properties like tension, stiffness, etc.)

Depending on the parameters: $SU(N) \rightarrow Z(N)$

$N$-ality

(via Weyl transformations)

Hybrid glue

Topological confinement of (valence) gluons

Correct minimum energy for doubled fundamental pairs
't Hooft (1978)

- Vortex creation operators. At a given time $t$:

$$\hat{W}(C) \hat{V}(x) = e^{i2\pi \text{link}/N} \hat{V}(x) \hat{W}(C)$$

$$\hat{W}|A\rangle = W|A\rangle \quad , \quad \hat{W} \left( \hat{V}|A\rangle \right) = e^{i2\pi \text{link}/N} W \left( \hat{V}|A\rangle \right)$$

$$\hat{V}|A\rangle = |A + \text{defect}\rangle$$

- $\text{defect} + \text{defect} + \ldots \text{defect} = \text{nothing} \rightarrow \mathbb{Z}(N) \text{ symmetry}$

- Criteria for confinement

$$\langle \hat{V}(x) \rangle = \text{const} \neq 0$$
Understanding the criterion in 3D:

$$\hat{V}(x)|V\rangle = V|V\rangle$$

$$\hat{V}(x) \left( \hat{W}(C)|V\rangle \right) = e^{-i2\pi \text{link}/N} V \left( \hat{W}(C)|V\rangle \right)$$

$$\hat{W}(C)|V\rangle = |V \text{ rotated inside } C\rangle$$

Figure: If there is SSB in the dual theory, $\langle \text{fund}|\hat{W}(C)|\text{fund}\rangle \sim e^{-\sigma \text{Area}}$
$\partial_\mu \bar{V} \partial_\mu V + m^2 \bar{V} V + \alpha (\bar{V} V)^2 + \beta (V^N + \bar{V}^N)$

where $V$ represents the vortex sector. Displays a global magnetic $Z(N)$ symmetry.

- when $m^2 < 0 \Rightarrow$ SSB $\Rightarrow$ Domain walls (confining strings)
Chains of monopoles and center vortices in 3D
Lemos, Teixeira & LEO (2011)

\( N = 2 \)

\[
S_d = \sum_{k=1}^{2n} \int_0^{L_k} ds \left[ \mu + \frac{1}{2\kappa} \dot{u}_{\alpha}^{(k)} \dot{u}_{\alpha}^{(k)} \right] + \frac{1}{2} \sum_{k,k'} \int_0^{L_k} \int_0^{L_{k'}} ds \, ds' \, V \left( x^{(k)}(s), x^{(k')}(s') \right)
\]
Building block

\[ Q(x, x_0) = \int_0^\infty dL \, e^{-\mu L} \int [Dx(s)] \, e^{-\int_0^L ds \left[ \frac{1}{2\kappa} \dot{u}_\alpha \dot{u}_\alpha + \phi(x(s)) \right]} \]

Small stiffness:

\[ \left[ -\frac{1}{3\kappa} \nabla^2 + \phi(x) + \mu \right] Q = \delta(x - x_0) \]

- Summing over the ensemble of closed loops and chains → 't Hooft vortex model with \( m^2 \propto \mu \kappa \)
- Chains are essential to describe typical \( Z(2) \) processes that lead to the \( Z(2) \) terms!