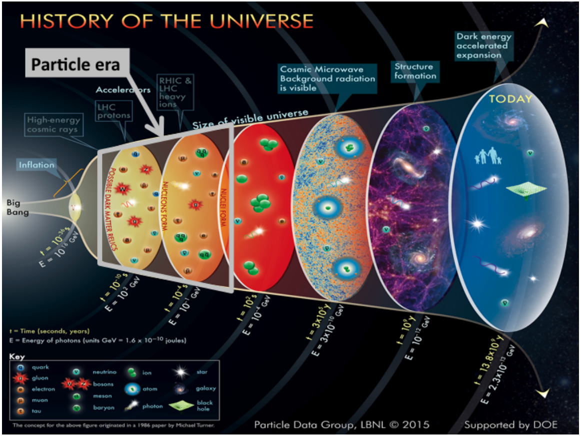
Frontiers of finite temperature lattice field theory

Confinement XII — Sep 2, 2016, Thessaloniki

Szabolcs Borsanyi

Wuppertal-Budapest collaboration



Friedmann equations:

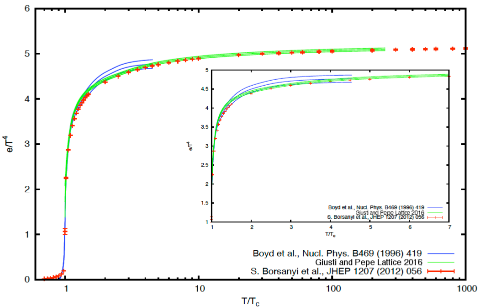
$$H^2 = \frac{8\pi}{3M_{pl}^2}\rho \qquad \frac{d\rho}{dt} = -3HsT$$

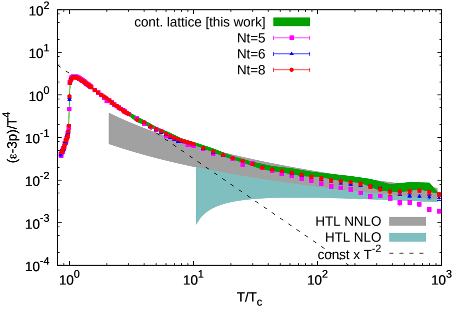
$$\frac{dT}{dt} = \frac{d\rho/dt}{d\rho/dT} = -\frac{3HsT}{c} = -\sqrt{\frac{24\pi}{M_{pl}^2}}\frac{sT\sqrt{\rho}}{c}$$

The equation of state is needed

(c: heat capacity, s entropy density, ρ energy density as a function of the temperature T)

The SU(3) Yang-Mills theory's equation of state





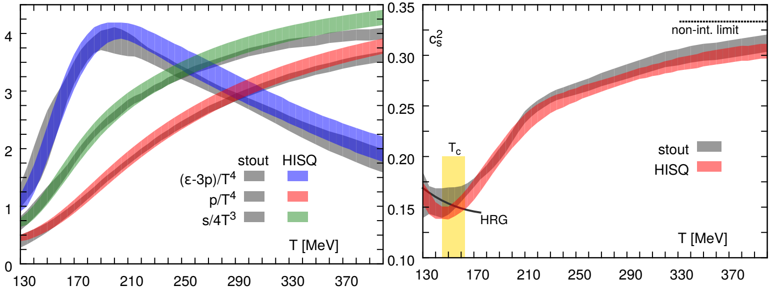
Lattice data: [Wuppertal-Budapest JHEP 1207 (2012) 056] and [Giusti-Pepe Lattice'16]

HTL NLO: [Andersen et al PRD 66,085016 (2002)]

HTL NNLO: [Andersen et al PRL 104,122003 (2010)]

Equation of state with up,down and strange quarks

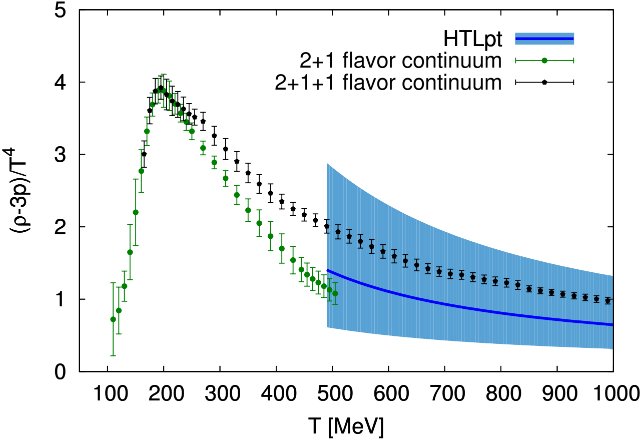
stout result: Wuppertal-Budapest group [1309.5258]
HISQ result: Bielefeld-Brookhaven group [1407.6397]



Around the transition temperature $(\epsilon - 3p)/T^4$ has a steepest point, the speed-of-sound has a minimum

Equation of state at high temperatures

Effect of the charm quark: mostly relevant for **cosmology**

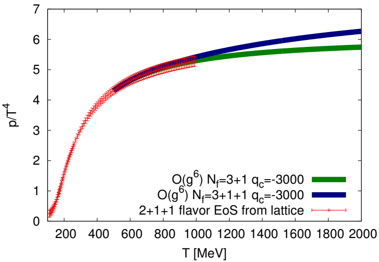


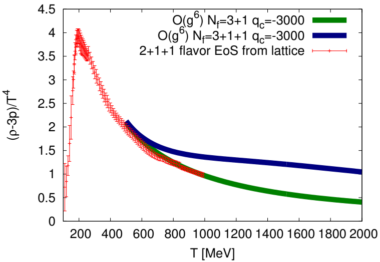
The ratio of the two results can be described by a tree-level threshold function.

[Wuppertal-Budapest: 1606.07494]

Perturbative parametrization at high temperatures

- Applying the tree-level charm threshold to the perturbative pressure
$$\frac{p}{T^4} = \# + \# g^2 + \# g^3 + \# g^4 + \# g^4 \log(g) + \# g^5 + \# g^6 \log(g) + ? g^6$$
[Kajantie 2002]
- The g^6 term is fitted to lattice ($-3200 < q_c < -2700$).
- The fit describes the pressure **and** trace anomaly from 500 MeV.
- Next we can introduce the bottom quark treshold keeping q_c fixed.

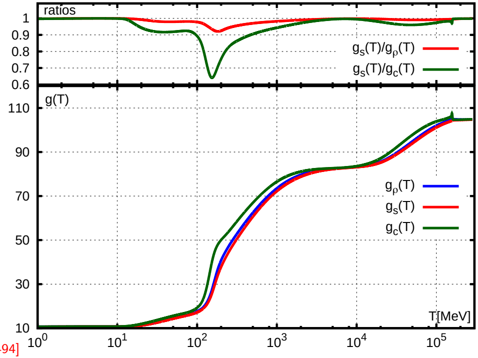




The full equation of state

Adding QCD to the free light particles and the electroweak theory:

number of effective degrees of freedom:

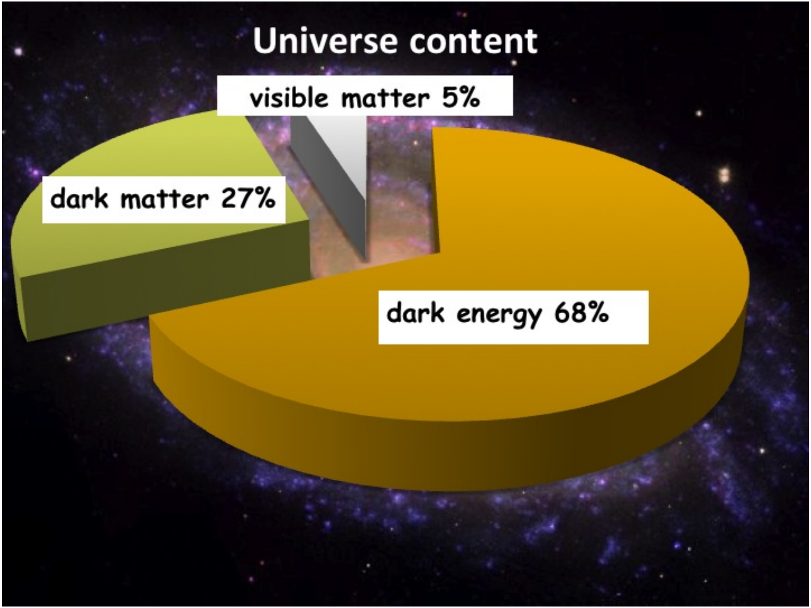


[Wuppertal-Budapest 1606.07494]

energy dens. $\rho = g_\rho \frac{\pi^2}{30} T^4$ entropy dens. $s = g_s \frac{2\pi^2}{45} T^3$ heat cap. $c = g_c \frac{2\pi^2}{15} T^3$

cooling rate in early universe $\frac{dT}{dt} = -\frac{T^3}{M_{pl}} \frac{2\pi^{3/2}}{3\sqrt{5}} \frac{\sqrt{g_\rho} g_s}{g_c}$

What can QCD say about the dark matter?



Axions are hypothetical particles that might make up the dark matter.

The strong CP problem

The Standard Model breaks the CP symmetry.

QCD could break this symmetry with a CP-odd term: $q(x) = \frac{g^2}{64\pi^2} F^a_{\mu\nu} \tilde{F}^{a,\mu\nu}$

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + i\bar{\psi}(D_\mu \gamma^\mu - m)\psi + \theta \frac{g^2}{64\pi^2} F^a_{\mu\nu} \tilde{F}^{a,\mu\nu}$$

$$Z = \int \mathcal{D}\mathcal{A}\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S^E_{QCD}} e^{i\theta Q} \sim \sum_Q P(Q) e^{i\theta Q}$$

Q is the winding number of gauge the configuration.
Q is integer and *topological*

$$Q = \int d^4 q(x)$$

Constraint form the neutron electric dipole moment: $|\theta| < 10^{-10}$

Fine tuning?

If at least one quark mass is zero a $U(1)_A$ rotation can erase θ.

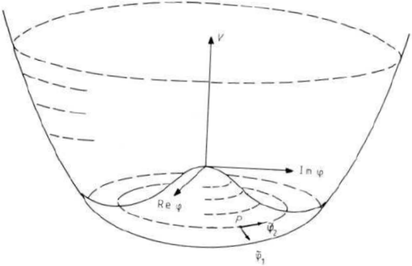
[Peccei-Quinn 1977]: new particle field

Axion potential

Effective lagrangian with a new, spontaneously broken $U_{PQ}(1)$ symmetry

$$\mathcal{L} = \partial_\mu \phi \partial_\mu \phi + i(\frac{\phi}{f_A} + \theta)q + \partial_\mu \theta \cdot (\dots) + \mathcal{L}_{QCD}$$

The emerging pseudo-Goldstone boson (ϕ) is called **axion**.



The potential is tilted according to θ

$\rightarrow$ the axion will roll to the minimum and cancel θ

The axion zero mode will oscillate around the minimum:

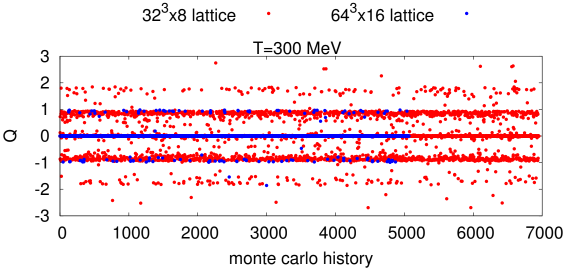
$$m_\phi^2(T) = 1/f_A^2 \int d^4x \langle q(x)q(0)\rangle = \chi(T)/f_A^2$$

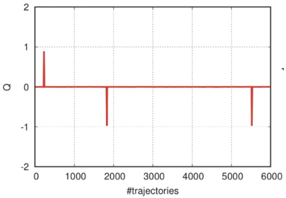
At zero T we know (χpt):

$$m_\phi^2(T=0) f_A^2 = \chi(T=0) = [75.6(1.8)(0.9)MeV]^4$$

At high temperatures (dilute instanton gas): $\chi(T) \sim T^{-b}$

Challenges with the topological susceptibility on the lattice

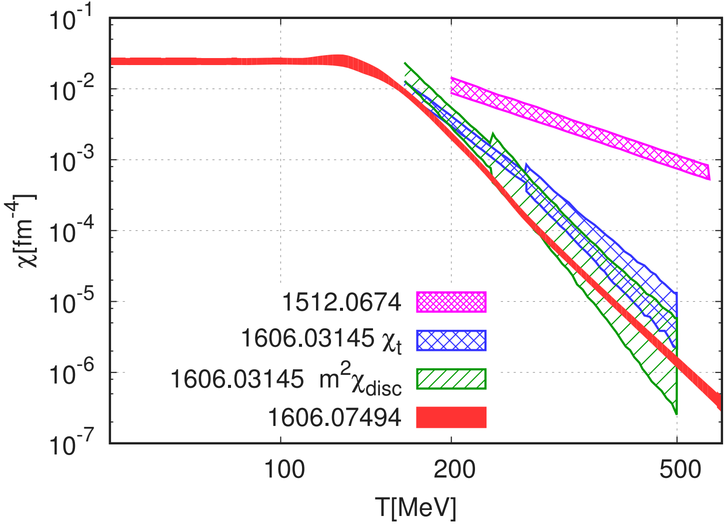




$\langle Q^2 \rangle$ has to be mesaured in the continuum limit.

- Large cut-off effects with dynamical *staggered* fermions: on coarse lattices the $Q \neq 0$ configurations are over-populated.
- At high T $\chi(T)$ is very small, we very seldom get $Q \neq 0$ configurations

Ideas for solutions: Parallel talks by M. D'Elia, S. Sharma, S. D. Katz.



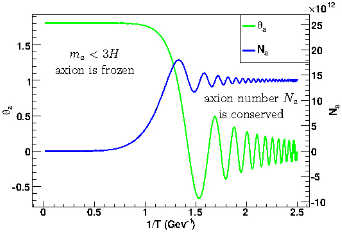
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
$\chi[\mathrm{fm}^{-4}]$
100
200
500
T[MeV]
1512.0674
1606.03145 χ_t
1606.03145 $m^2\chi_{disc}$
1606.07494

Axion dynamics

Using $\chi(T) = f_A^2 m_A^2(T),\ \varepsilon(T),\ s(T)$
Axion equations of motion ($\vartheta = \phi/f_A$):

$$\frac{d^2\vartheta}{dt^2} + 3H(T)\frac{d\vartheta}{dt} + m_A^2(T)\frac{d}{d\vartheta}(1-\cos\vartheta) = 0$$

The ϑ field cannot roll into the pontential minimum until $3H(T) > m_a(T)$, because of the high "friction" H.



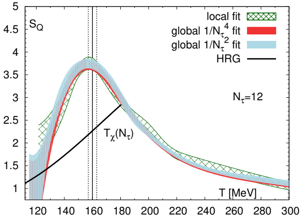
[Wantz&Shellard 0910.1066]

Key assumption: $\theta(x) = \theta(t)$ spatially constant, no strings, domain walls, etc
The evolution is adiabatic: $\mathrm{entropy}(T)/n_{\mathrm{axion}}(T)$ is constant.
Post-inflation scenario: all angles are present, one has to average. $\rightarrow\ \theta_{\mathrm{eff}} \approx 2.16$.

If this mechanism is responsible for all dark matter: $m_A = 28(2)\mu eV$
If this mechanism is responsible for 50% dark matter: $m_A = 50(4)\mu eV$
If this mechanism is responsible for 1% dark matter: $m_A \sim 1500\mu eV$
Strings, domain walls, etc may also play a role within the axion picture.

[Wuppertal-Budapest 1606.07494]

Renormalized Polyakov loop



$$P(\mathbf{x}) = \frac{1}{3}\mathrm{Tr}\prod_{x_0=0}^{N_\tau-1} U_0(\mathbf{x}, x_0),$$

$$\langle P\rangle = e^{-(F_Q+\Delta_{\mathrm{scheme,UV}})/T}$$

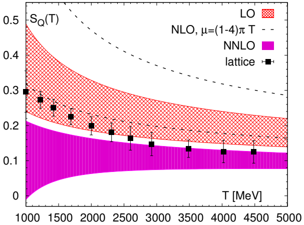
$$S_Q(T) = -\frac{\partial F_Q(T)}{\partial T}$$

[TUMQCD 1603.06637]

See also previous works:

[Wuppertal-Budapest 1005.3508,1501.02173], [Bazavov et al. 1111.1710.1301.394]

Compared to the NNLO weak coupling result:



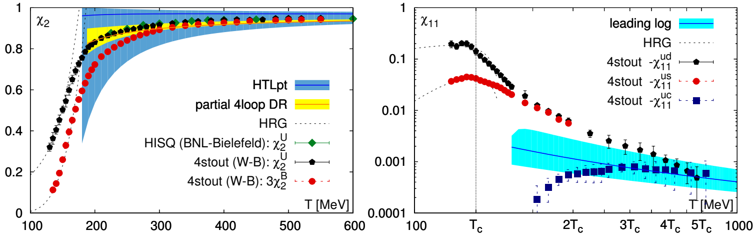
NLO: [Burnier et al 0911.3480]

NNLO: [Berwein et al 1512.08443, Berwein Thu]

Fluctuations of conserved charges: agreement with HTL perturbation theory

The response of a *grand canonical* system to the thermodynamic force μ_q is proportional to the fluctuation of the conserved charge:

$$\frac{\partial \langle N_i \rangle}{\partial \mu_j} = T \frac{\partial^2 \log Z(T, V, \{\mu_q\})}{\partial \mu_j \partial \mu_i} = \frac{1}{T} (\langle N_i N_j \rangle - \langle N_i \rangle \langle N_j \rangle)$$



At high temperature HTL result [Haque et al 1309.3968] agrees with lattice [Wuppertal-Budapest 1112.4416,1210.6901,1507.04627, BNL-Bielefeld 1309.2317,1507.06637,1509.08887]

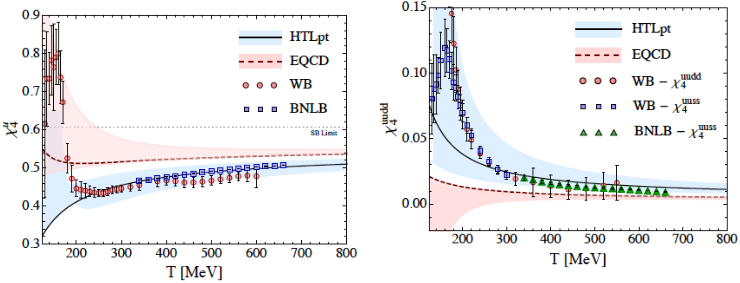
[Nan Su Thu]

χ_{11}: The quark flavor correlation is still relevant above deconfinement.

Fluctuations of conserved charges: agreement with HTL perturbation theory

The response of a *grand canonical* system to the thermodynamic force μ_q is proportional to the fluctuation of the conserved charge:

$$\frac{\partial \langle N_i \rangle}{(\partial \mu_j)^3} = T \frac{\partial^4 \log Z(T, V, \{\mu_q\})}{(\partial \mu_i)^4} \sim \langle N^4 \rangle - 3\langle N^2 \rangle^2 \sim \text{Kurtosis}$$



For the higher moments HTLpt gives very accurate estimates, [Haque et al 1309.3968] and these agree with the lattice results.

[Wuppertal-Budapest 1507.04627] [BNL-Bielefeld 1507.06637] [Nan Su Thu]

Equation of state at finite density

Taylor approach: $\frac{p(\mu_B)}{T^4} = \frac{p(0)}{T^4} + \chi_2^B \frac{\mu_B^2}{2!T^2} + \chi_4^B \frac{\mu_B^4}{4!T^4} + \chi_6^B \frac{\mu_B^6}{6!T^6} + \dots$

$$\chi_n^B = \frac{\partial^n \log Z}{[\partial(\mu_B/T)]^n}$$

MILC and BNL-Bielefeld: $N_t = 6$ 3rd order [1003.5682, hep-lat/0512040]

In *heavy ion phenomenology* at RHIC $\mu_B > 0$ but the strangeness vanishes.

$$M_S = 0, \qquad M_Q = M_B \frac{Z}{A}, \qquad r = \frac{Z}{A} = \frac{79}{197} \approx 0.4 \quad \text{gold}$$

This turns χ_B into comlpicated mixed derivatives [BNL-Bielefeld 1208.1220].

---

1. Taylor method

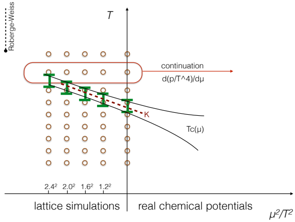
μ_B derivatives at $\mu_B = 0$ simulations;

more statistics are required

2. Analytical continuation

$\mu_B^2 \leq 0$ simulations;

careful analysis of systematics is required

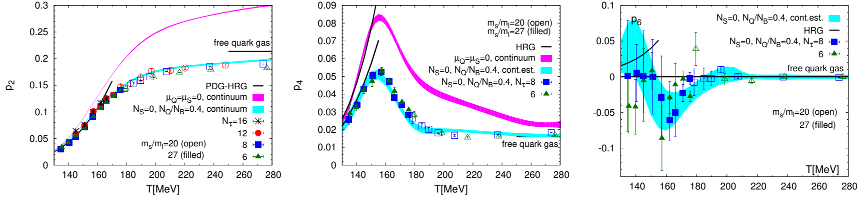


Roberge-Weiss temperature:

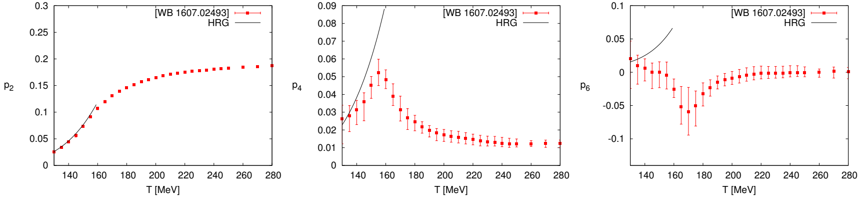
$T_{RW}^{2+1f} = 208(5)$ MeV [Bonati et al. 1602.01426]

Pressure coefficients

Taylor method: *mostly $N_t = 8$, $\mathcal{O}(10^5)$ configurations point,* [C. Schmidt Fr 18.00]



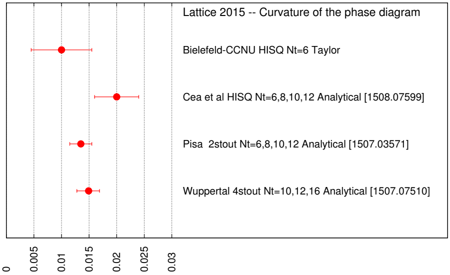
Analytical method: *continuum, $\mathcal{O}(10^4)$ configurations/point, errors include systematics* [J. Günther Fr 19.10; 1607.02493]

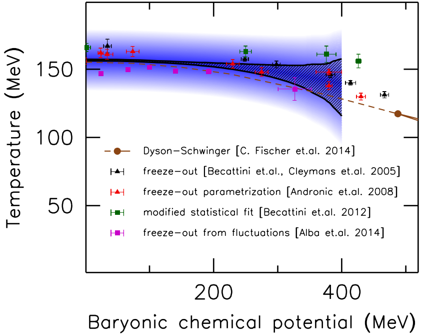


The $T_c(\mu_B)$ can be expandedd around $\mu_B = 0$ (Taylor method) or found through analytical continuation with $\mathrm{Im}\ \mu_B > 0$.

$$\frac{T_c(\mu_B)}{T_c} = 1 - \kappa\left(\frac{\mu_B}{T_c(\mu)}\right)^2 + \dots$$

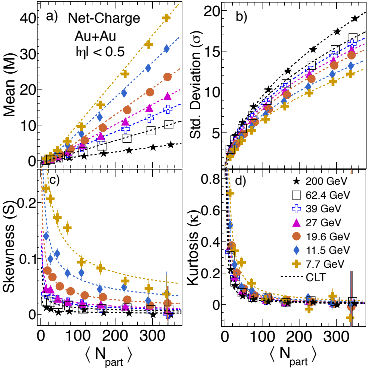
Lattice 2015, Kobe:

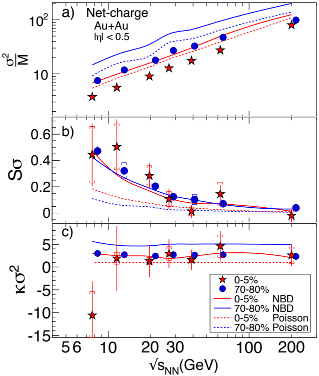




Fluctuations from experiment

At RHIC **STAR** has measured the mean, variance, skewness and kurtosis of the event-by-event **net charge** distribution at various energies and centralities.

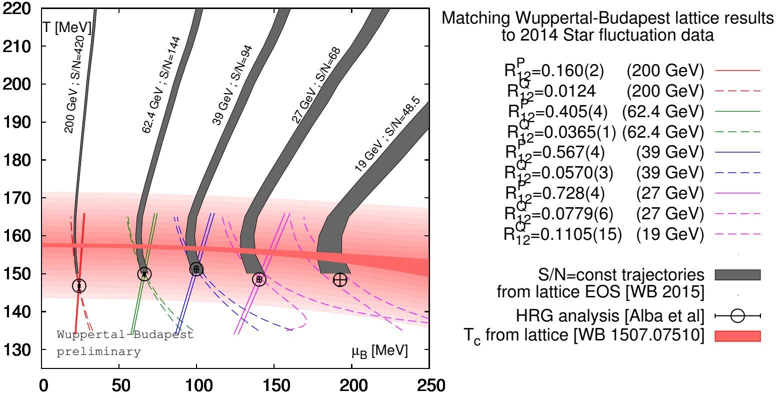




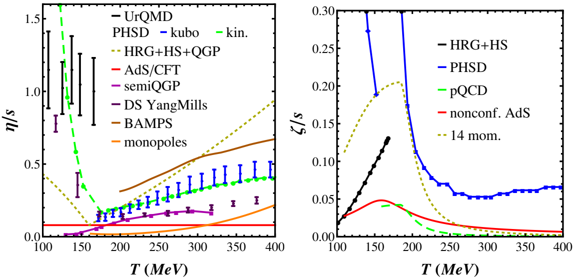
[STAR: 1402.1558]

Lattice QCD: towards the phase diagram

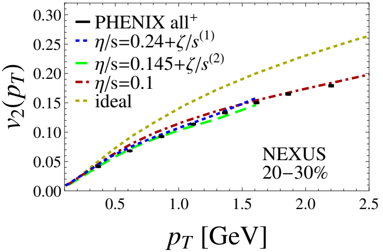
Small μ_B techniques use the smoothness the of the free energy around $\mu_B \approx 0$.
The transition temperature, as well as the fluctuations can be extrapolated to the range covered by the RHIC experiments.



Numbers are preliminary here, listen to J. G̃unther (Fri. 19.10) for an update.



Flow data is best described with non-vanishing (shear and bulk) viscosity.



[Noronha-Hostler 1512.06315]

Viscosity from Euclidean correlators

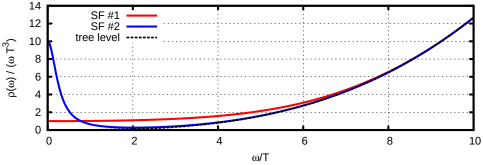
Kubo formulas relevant for the shear viscosity

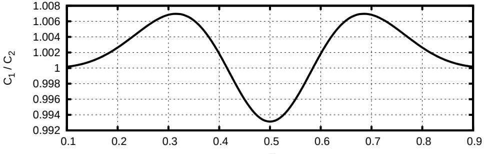
$$\eta(T) = \pi \lim_{\omega\to 0}\lim_{\mathbf{k}\to 0} \frac{\rho^{12,12}(\omega,\mathbf{k},T)}{\omega}$$

$$\frac{4}{3}\eta(T) + \zeta(T) = \pi \lim_{\omega\to 0}\lim_{\mathbf{k}\to 0} \frac{\rho^{11,11}(\omega,\mathbf{k},T)}{\omega}$$

The reconstruction of ρ poses an ill-defined problem.

$$C_{\mu\nu,\rho\sigma}(\tau,\mathbf{p}) = \int_0^\infty \mathrm{d}\omega \rho_{\mu\nu,\rho\sigma}(\omega,\mathbf{p},\mathrm{T}) \frac{\cosh\left(\omega\left(\tau - 1/(2\mathrm{T})\right)\right)}{\sinh\left(\omega/(2\mathrm{T})\right)}$$



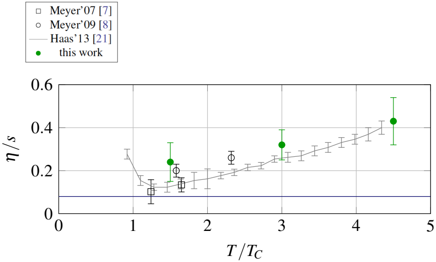


[A. Pásztor Lattice 2016]

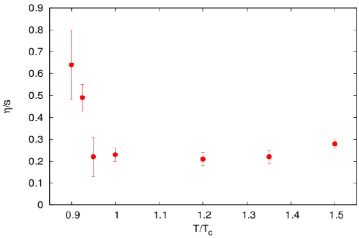
Viscosity results from the lattice

Only quenched results are available so far:

[S Mages: Lattice 2014]



[V Braguta Lattice 2016; N Astrakhantsev Fri 15.20]



For a continuum limit study see: [A. Pasztor Lattice 2016]

Problem with dynamical fermions: *no multilevel algorithm*

For other transport coefficients we face similar problems. But this research is in a more advavced status: e.g. **electric conductivity** with dynamical Wilson fermions [FASTSUM 1412.6411].

What to expect at Confinement XIII?

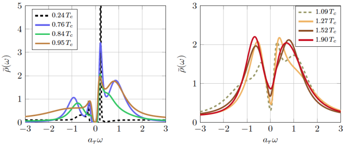
- Cleaning up details:
  quenched tests for Euclidean correlators and reconstructions
- Non-staggered simulations with physical quark masses
  Today: Domain-Wall: m_π =135 MeV (HotQCD) Wilson: m_π =285 MeV (BW), Overlap: m_π =350 MeV (BW)
- Spectral reconstruction analyses towards physical point
  Today: 392 MeV Wilson or physical point asqtad,. . .
- Heavy ion physics: better thermometer
  Simplest variance/mean-like ratios do not work at LHC
- First attempts for the viscosity with dynamical fermions
  Yang-Mills results are available today
- Phase diagram, Columbia plot
  Elusive transition line in the lower left corner
- Finite density. . . further orders?
  sign program remains a challenge

backup

more of spectral reconstructions

Chiral symmetry restoration:
Nucleon ($\omega > 0$) & parity partner ($\omega < 0$)
spectral function reconstruction

[De Boni, Tue, 1607.05082]

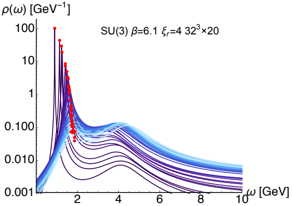


---

pNRQCD effective theory: $m_Q \gg T$, $m_Q \gg |\vec{p}|$

$$i\partial_t \psi = \left(V(R) + \mathcal{O}(m_Q^{-1}) \right) \psi$$

[Brambilla et al. RMP 77(2005)1423] [Brambilla et al. PRD 78(2008)14017]



$$V(R) = \lim_{t\to\infty} \frac{i\partial_t W_\square(R,t)}{W_\square(R,t)}$$

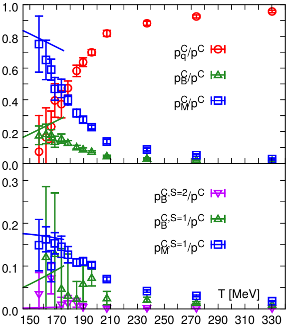
[Laine et al JHEP03(2007)054; Beraudo et al NPA806(2008)312]

[A. Rothkopf Tue]

[Burnier et al.: 1509.07366 1606.06211, 1607.04049]

$$i\partial_t D^>(t,r) = \left(2m_Q - \frac{1}{m_Q}\nabla^2 + V(t,r) \right) D^>(t,r)$$

Thermal D mesons: [J-I Skullerud Fri 16.20]

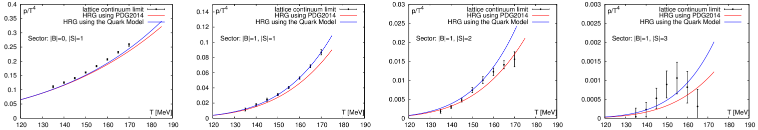


Search for open charm bound states above T_c

$$p^C(T,\mu_C,\mu_B) = p^C_q(T)\cosh\left(\hat\mu_C + \hat\mu_B/3\right) + p^C_B(T)\cosh\left(\hat\mu_C + \hat\mu_B\right) + p^C_M(T)\cosh\left(\hat\mu_C\right) ,$$

[Mukherjee et al 1509.08887]

Search for resonances not covered by the Particle Data Book below T_c

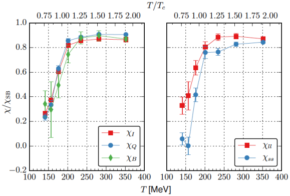


[BNL-Bielefeld 1404.4043, 1404.6511; Wuppertal-Budapest prelim (S.B. Lattice16)]

Non-staggered finite temperature results

Quark number susceptibilities

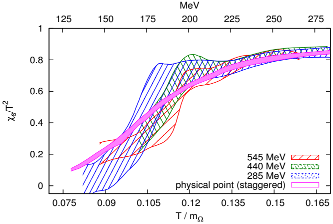
anisotropic wilson $m_\pi = 392$ MeV



[FASTSUM 1309.6253,1412.6411]

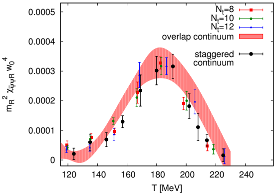
isotropic wilson in continuum

$m_\pi = 545, 440, 285$ MeV.



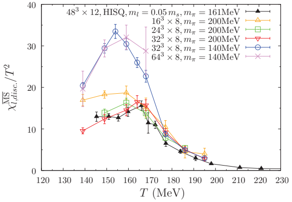
[WB 1205.0440,1504.03676]

Chiral susceptibility



Overlap fermions ($m_\pi = 350$ MeV)

[WB 1204.4089].



Domain wall ($m_\pi = 135$ MeV)

[HotQCD 1402.5175]

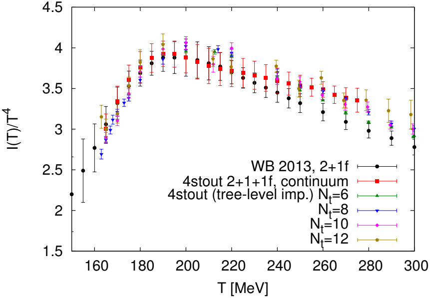
Caveats

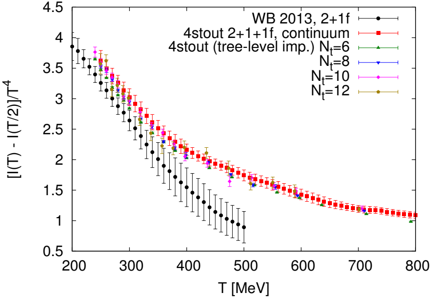
- *Effects due to volume variation because of finite centrality bin width*
  Experimentally corrected by centrality-bin-width correction method
- *Finite reconstruction efficiency*
  Experimentally corrected based on binomial distribution
  [A. Bzdak, V. Koch, PRC (2012)]
- *Spallation protons*
  Experimentally removed with proper cuts in p_T
- *Canonical vs Grand Canonical ensemble*
  Experimental cuts in the kinematics and acceptance
  [V. Koch, S. Jeon, PRL (2000)]
- *Proton multiplicity distributions vs baryon number fluctuations*
  Numerically very similar once protons are properly treated
  [M. Asakawa and M. Kitazawa], [PRC (2012), M. Nahrgang et al., 1402.1238]
- *Final-state interactions in the hadronic phase* [J.Steinheimer et al., PRL (2013)]
  Consistency between different charges = fundamental test

$2+1+1$ flavor equation of state – lattice data

For low temperatures $I(T)/T^4$ is calculated using vacuum subtraction.

For higher temperatures $[I(T) - I(T/2)]/T^4$ is calculated and continuum extrapolated.

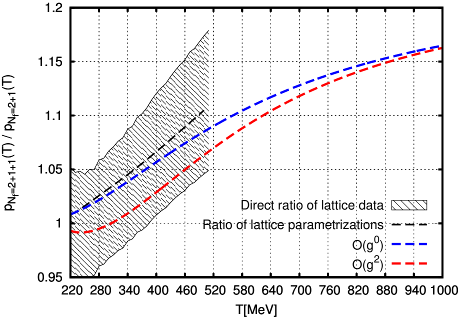




The final trace anomaly is calculated from the continuum extrapolated terms using the formula

$$\frac{I(T)}{T^4} = \sum_{k=0}^{n-1} 2^{-4k} \frac{I(T/2^k) - I(T/2^{k+1})}{(T/2^k)^4} + 2^{-4n} \frac{I(T/2^n)}{(T/2^n)^4}$$

Charm quark threshold



The tree level formula gives a very good approximation:

$$\frac{p^{(3+1)}(T)}{p^{(3)}(T)} = \frac{SB(3) + F_Q(m_c, T)}{SB(3)}$$

where $F_Q(m, T)$ is dimensionless free energy density of a free massive quark. [Laine&Schröder hep-ph/0603048]

Dilute instanton gas approximation

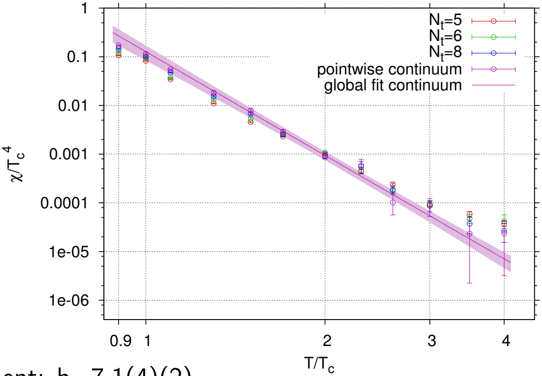
[Gross, Pisarski, Yaffe 1981]

Single instanton action: $2\pi/\alpha_s$

At high T this pertrubative estimate is expeted to be valid and give

$$\chi(T) \sim T^4 \left(\frac{m}{T}\right)^{N_f} e^{-2\pi/\alpha_s} \sim m^{N_f} / T^{4-11+1/3N_f}$$

Comparison to $N_f = 0$ lattice data



Fitted exponent: b=7.1(4)(2) [Wuppertal Budapest 1508.06917]

[see also $b = 5.65(4)$ [Berkowitz et al 1505.07455]]

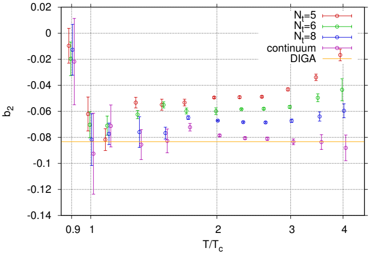
Higher moments of the topological susceptibility

$$F(\theta, T) - F(0, T) = \frac{1}{2}\chi(T)\theta^2\left[1 + b_2(T)\theta^2 + b_4(T)\theta^4 + \dots\right]$$

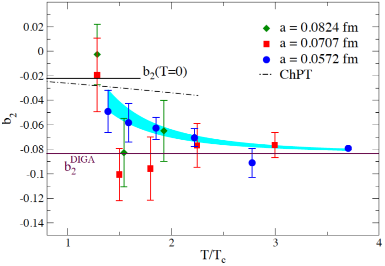
If multi-instanton configurations are rare: $\exp[-V/T \cdot V_{\text{eff}}(\vartheta)] = \langle \exp iQ\vartheta \rangle$

$$F(\theta, T) - F(0, T) \approx \chi(T)(1 - \cos\theta) \quad \Longrightarrow b_2 = -1/12,\ b_4 = 1/360 \dots$$

Chiral perturbation theory: $b_2 = -0.0022(1)$ (for $m_u = m_d$)

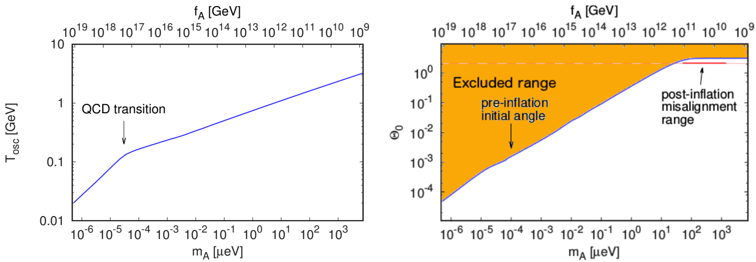


[Wuppertal-Budapest 1508.06917]



[Bonati et al 1512.06746]

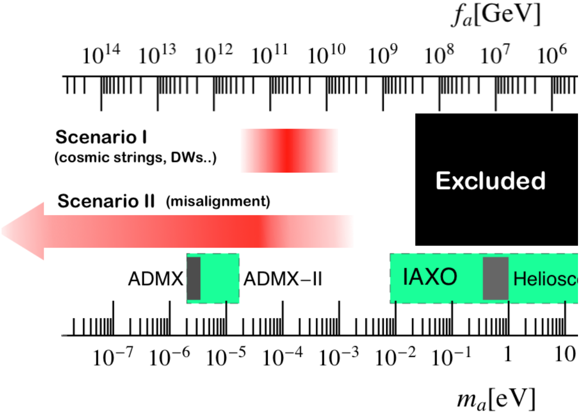
On the pre-inflationary scenario



Depending the reheat temperature ($< f_A$), the likely temperature range of interest is around the QCD transition. This, however, reintroduces the fine tuning problem for θ_0.

[Wuppertal-Budapest 1606.07494]

Black: supernova 1987A neutrio flux: $f_A \geq 4 \cdot 10^8$ GeV



Post-inflationary axions: $m_A \geq 28(2)\mu ev$ (this work)
Likely estimate : $m_A = 50 - 1500\mu eV$

[Redondo&Döbrich 1311.5341] [Raffelt hep-ph/06113501]

Integral method for topological susceptibility

$$Z = \int DU e^{-S} = \sum_Q \int DU|_Q \, e^{-S} = \sum_Q Z_Q$$

At high T: $Z_0 \gg Z_1 \gg Z_2 \ldots$; $\mathcal{P}$ symmetry: $Z_1 = Z_{-1}$, $Z_2 = Z_{-2}, \ldots$
Keeping $R = LT = 4$ constant:

$$R^3 \frac{\chi_t}{T^4} = \langle Q^2 \rangle = \frac{\sum_Q Q^2 Z_Q}{\sum_Q Z_Q} = \frac{0 \cdot Z_0 + 1 \cdot Z_1 + 1 \cdot Z_{-1} + \mathrm{small}}{Z_0 + \mathrm{small}} = \frac{2Z_1}{Z_0}$$

$$\frac{\partial \log \chi_t}{\partial \log T} = 4 + \frac{\partial \log Z_1}{\partial \log T} - \frac{\partial \log Z_0}{\partial \log T}$$

This is a very simple observable even in full QCD:

$$\chi_t \sim T^{-b}, \quad \text{with } b = -4 - \frac{d\beta}{dT} \langle S_g \rangle_{1-0} - \sum_f \frac{dm_f}{dT} m_f \langle \bar{\psi} \psi_f \rangle_{1-0}$$

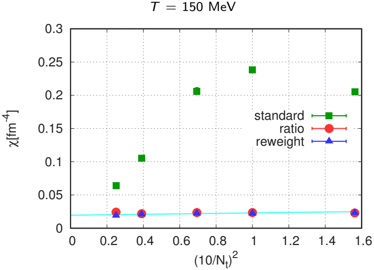
If we know χ_t at some temperature, we can calculate other tempertaures.
for SU3: [Frison et al. 1606.07175], for QCD: [Wuppertal-Budapest 1606.07494]

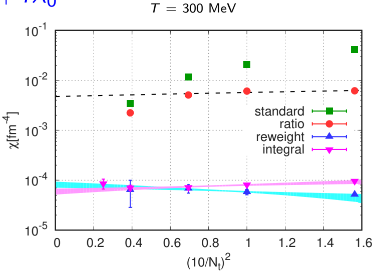
Strong cut-off effects are related to the lack of exact zero-modes.
Zero-modes contirbute with large weight in the path integral – unless there is a lattice artefact on the eigenvalue
Weaker suppression of $Q > 0$: $\chi(T)$ is overestimated

Improvement:

1. we identify the would be zero-modes
2. restore the weight: reweighting factor

$$w[U] \sim \frac{m}{m + i\lambda_0}$$





[Wuppertal-Budapest 1606.07494]

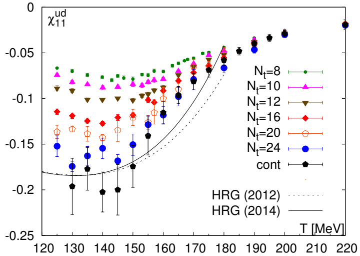
The fermion determinant $\det M = |\det M| e^{i\theta}$ has a fluctuating phase at $\mu > 0$: [Allton hep-lat/02040130]

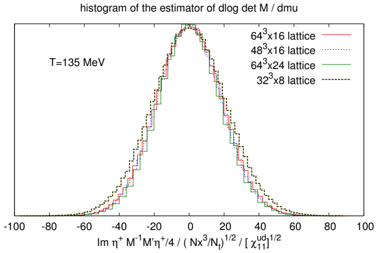
$$\theta = \frac{1}{4} N_f \mathrm{Im}\left[\mu \frac{\partial \ln \det M}{\partial \mu} + \frac{\mu^3}{3!}\frac{\partial \ln \det M}{\partial \mu} + \dots\right]$$

The fluctuation of $A = \partial \ln \det M / \partial \mu$ gives at LO for the phase:

$$\langle \theta^2 \rangle = -\frac{1}{9}\mu_B^2 L^3 T N_f^2 \chi_{11}^{ud}$$

(~ 1 at $\mu_B \approx 100$ MeV, with $T = T_c$ and $LT = 3$.)





[Wuppertal-Budapest 1507.04627]

What do these experimantal data actually mean?

- In heavy ion collisions quark gluon plasma is produced, which is rapidly cooling down across the transition temperature
- Chemical freeze-out: instant of the last inelastic scattering
- Before freeze-out: grand canonical ensemble
- After freeze-out: conserved charges are conserved
- In the detertor: the event by event fluctuations correspond to a grand canonical ensemble at the time of the freeze-out.

On the lattice we calculate

$$\chi_n = d^n log Z / d\mu^n$$

In experiment the ratios are available

$$\frac{M}{\sigma^2} = \frac{V\chi_1}{V\chi_2} \qquad \kappa\sigma^2 = \frac{V\chi_4}{V\chi_2}$$

Available for net proton number and net electric charge.

Lattice can calculate χ_n for small chemical potentials, too.