Pion-Nucleon Interactions: Status and Outlook

Ulf-G. Meißner, Univ. Bonn & FZ Jülich

Supported by BMBF 05P15PCFN1 and by DFG, SFB/TR-110 and by CAS, PIFI by HGF VIQCD VH-VI-417 and by VW Stiftung
CONTENTS

• Introduction
• Roy-Steiner equations for pion-nucleon scattering
• Results
• Comparison with recent lattice results
• Summary & outlook
Introduction
**TRACE ANOMALY**

- Classical masses QCD is invariant under scale transformations

\[ x \rightarrow \lambda x, \quad q(x) \rightarrow \lambda^{3/2} q(\lambda x), \quad A_\mu(x) \rightarrow \lambda A_\mu(\lambda x) \quad \text{(dilatations)} \]

- Quantization/renormalization generates a scale \( \Lambda_{\text{QCD}} \) that breaks scale invariance: **dimensional transmutation**

\[ \Rightarrow \text{trace anomaly} \]

\[ \theta_\mu^\mu = \frac{\beta_{\text{QCD}}}{2g} G^a_{\mu \nu} G^{a \mu \nu} + m_u \bar{u} u + m_d \bar{d} d + m_s \bar{s} s + \ldots \]

* trace anomaly = signal for the *generation of hadron masses*

* the mass of any hadron made of light quarks mass is essentially **field energy** (“binding”)

“**Mass without mass**” (Wheeler, 1962)
ANATOMY of the NUCLEON MASS

\[ m_N = \langle N(p)|\theta_\mu^\mu|N(p)\rangle = \langle N(p)\rangle \frac{\beta_{\text{QCD}}}{2g} G_{\mu\nu}^a G^{\mu\nu}_a + m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s |N(p)\rangle \]

- Dissect the various contributions:
  - \( \langle N(p)|m_u\bar{u}u + m_d\bar{d}d|N(p)\rangle = 40 \ldots 70 \text{ MeV} \triangleq \sigma_{\pi N} \)
  - \( \langle N(p)|m_s\bar{s}s|N(p)\rangle = 0 \ldots 150 \text{ MeV} \)

  from the analysis of the pion-nucleon sigma term & lattice QCD (before 2015)

\[ \Rightarrow \text{bulk of the nucleon mass is generated by the gluon fields / field energy} \]
\[ \Rightarrow \text{this is a central result of QCD} \]
\[ \Rightarrow \text{requires better Roy-Steiner analysis of } \pi N \text{ and lattice data} \]

\[ \leftrightarrow \text{this talk} \]
ROLE of the PION-NUCLEON $\sigma$-TERM

- Scalar couplings of the nucleon:

  $$\langle N | m_q \bar{q} q | N \rangle = f_q^N \ m_N \quad (N = p, n)$$
  $$\quad (q = u, d, s)$$

  $\rightarrow$ Dark Matter detection

  $\rightarrow$ $\mu \rightarrow e$ conversion in nuclei

- Condensates in nuclear matter

  $$\frac{\langle \bar{q} q \rangle(\rho)}{\langle 0 | \bar{q} q | 0 \rangle} = 1 - \frac{\rho \sigma_{\pi N}}{F_\pi^2 M_\pi^2} + \ldots$$

- CP-violating $\pi N$ couplings

  $\rightarrow$ hadronic EDMs (nucleon, nuclei)
Low-energy constants (LECs) relate many processes

e.g. the dimension-two LECs $c_i$ in $\pi N$, $NN$, $NNN$, ...

Here: determine the $c_i$ from the purest process $\pi N \rightarrow \pi N$

→ make parameter-free predictions for long-ranged nuclear forces
Roy-Steiner equations

Ditsche, Hoferichter, Kubis, UGM, JHEP 1206 (2012) 043
Hoferichter, Ditsche, Kubis, UGM, JHEP 1206 (2012) 063
HYPERBOLIC DISPERSION RELATIONS

- make use of hyperbolic dispersion relations (HDRs):

\[(s - a)(u - a) = b\, , \quad a, b \in \mathbb{R} \quad [b = b(s, t, a)]\]

Steiner (1968), Hite, Steiner (1973)

- why HDRs?

  \[\leftarrow\] combine all physical regions
  very important for a reliable continuation to the subthreshold region

Stahov (1999)

  \[\leftarrow\] especially powerful for the determination of the \(\sigma\)-term

Koch (1982)

  \[\leftarrow\] \(s \leftrightarrow u\) crossing is explicit

  \[\leftarrow\] absorptive parts are only needed in regions where the corresponding PW expansions converge

  \[\leftarrow\] judicious choice of \(a\) allows to increase the range of convergence
**RS EQUATIONS: s-CHANNEL**

- **s-channel part of the full RS system:**

\[
f^I_{l+}(W) = N^I_{l+}(W) + n \text{ subtractions around } \nu = t = 0
\]

\[
+ \frac{1}{\pi} \int_{W_+}^{\infty} dW' \sum_{l'=0}^{\infty} \left\{ K^I_{l'l'}(W, W') \text{Im} f^I_{l'+1+}(W') + K^I_{l'l'}(W, -W') \text{Im} f^I_{(l'+1)-}(W') \right\}
\]

\[
+ \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \sum_{J} \left\{ G_{lJ}(W, t') \text{Im} f^J_+(t') + H_{lJ}(W, t') \text{Im} f^J_-(t') \right\}
\]

\[\leftrightarrow N^I_{l+}(W) \text{ nucleon Born term contribution}\]

\[\leftrightarrow \text{coupling to } s\text{-channel absorptive parts } \sim K^I_{l'l'} = \frac{\delta_{l'l'}}{W' - W} + \ldots\]

\[\leftrightarrow \text{coupling to } t\text{-channel absorptive parts } \sim G_{lJ}, H_{lJ}\]

\[\leftrightarrow \text{range of convergence: } a = -23.19 \, M^2_\pi\]

\[\Rightarrow s \in \left[ (m_N + M_\pi)^2, 97.30 \, M^2_\pi \right] \Leftrightarrow W \in [1.08, 1.38] \text{ GeV}\]
RS EQUATIONS: $t$-CHANNEL

- $t$-channel part of the full RS system (only show $f^J_+(t)$):

$$
\begin{align*}
f^J_+(t) &= \tilde{N}^J_+(t) + \frac{1}{\pi} \int_{W_+}^\infty dW' \sum_{l=0}^\infty \left\{ \tilde{G}_{Jl}(t, W') \Im f^I_{l+}(W') + \tilde{G}_{Jl}(t, -W') \Im f^I_{(l+1)-}(W') \right\} \\
&\quad + \frac{1}{\pi} \int_{t_\pi}^\infty dt' \sum_{J'} \left\{ \tilde{K}^1_{JJ'}(t, t') \Im f^J_+(t') + \tilde{K}^2_{JJ'}(t, t') \Im f^J_-(t') \right\} + n_{\text{sub.}} \quad \forall J \geq 0
\end{align*}
$$

$\hookrightarrow \tilde{N}^I_{l+}(W)$ nucleon Born term contribution (no kinematical singularity)

$\hookrightarrow$ coupling to $t$-channel absorptive parts $\sim \tilde{K}^{1,2,3}_{JJ'} = \frac{\delta_{JJ'}}{t' - t} + \ldots$

$\hookrightarrow$ only higher $t$-channel PWs couple to lower ones!

$\hookrightarrow$ coupling to $s$-channel absorptive parts $\sim \tilde{G}_{Jl}, \tilde{H}_{Jl}$

$\hookrightarrow$ range of convergence: $a = -2.71 \, M^2_{\pi}$

$$
\Rightarrow t \in \left[ 4M^2_{\pi}, 205.45 \, M^2_{\pi} \right] \iff \sqrt{t} \in [0.28, 2.00] \text{ GeV}
$$
UNITARITY RELATIONS

• $s$-channel elastic unitarity ($I_s = 1/2, 3/2$):

\[
\text{Im } f_{I_s}^{I_s}(W) = \sqrt{\frac{\lambda(s,m_N^2,M^2_\pi)}{4s}} \left| f_{I_s}^{I_s}(W) \right|^2 \theta(W - (m_N + M_\pi))
\]

• $t$-channel extended unitarity:

  (two-body intermediate states)

\[
\text{Im } f_{I_s}^{I_s}(t) = T_{I_s}^{I_s}(t)^* f_{I_s}^{I_s} \theta(t - 4M^2_\pi) + \bar{K}K - \text{loop} + \ldots
\]

$\Rightarrow$ Muskhelishvili-Omnès (MO) problem!

$\Rightarrow$ precise $\pi\pi$ input ($T_{I_s}^{I_s}$) from Roy equations, Berne and Madrid/Cracow groups
SOLUTION STRATEGY

Higher partial waves
\[ \text{Im } f_{l \pm}, \ l \geq 2, \ s \leq s_m \]

Inelasticities
\[ \eta_{l \pm}, \ l \leq 1, \ s \leq s_m \]

\( s \)-channel partial waves
solve Roy–Steiner equations for \( s \leq s_m \)

\[ f_0^+, f_1^+, f_1^- \]

\( f_0^+ \)
\[ f_1^+ \]
\[ f_1^- \]

High-energy region
\[ \text{Im } f_{l \pm}, \ s \geq s_m \]

Subtraction constants
\( \pi N \) coupling constant

\( t \)-channel partial waves
solve Roy–Steiner equations for \( t \leq t_m \)

\[ f_0^+, f_1^+, f_2^+, f_3^+, f_4^+, \ldots \]

\( \pi \pi \) scattering phases
\[ \delta_j^I \]

High-energy region
\[ \text{Im } f_{l \pm}, \ t \geq t_m \]
SOLUTION STRATEGY continued

- RS equations have a limited range of validity:

\[ \sqrt{s} \leq \sqrt{s_m} = 1.38 \text{ GeV} \quad \text{and} \quad \sqrt{t} \leq \sqrt{t_m} = 2.00 \text{ GeV} \]

- Input/Constraints
  - S- and P-waves above the matching point \((s > s_m, t > t_m)\)
  - Inelasticities
  - Higher waves (D-, F-, ...)
  - Scattering lengths from hadronic atoms

- Output
  - S- and P-wave phase shifts below the matching point \((s \leq s_m, t \leq t_m)\)
  - Subthreshold parameters
    - \(\rightarrow\) Pion-nucleon \(\sigma\)-term
    - \(\rightarrow\) LECs of pion-nucleon CHPT
    - \(\rightarrow\) N form factor spectral functions

- \(\pi N\) input from SAID/GWU, \(\pi\pi\) input from Bern and Madrid/Cracow
- important check: recover KH80 phases with appropriate input
PION–NUCLEON SCATTERING LENGTHS

- Superb experiments performed at PSI
  
  - Hadronic atom theory (Bern, Bonn, Jülich)

- \[ \tilde{a}^+ = a^+ + \frac{1}{1 + M_\pi/m_p} \begin{pmatrix} M_\pi^2 - M_\pi^0 \pi^0 F_2^2 \sin^2 \alpha \end{pmatrix} \]

- \( \tilde{a}^- = a^- + \frac{1}{1 + M_\pi/m_p} \begin{pmatrix} M_\pi^2 - M_\pi^0 \pi F_2^2 \sin^2 \alpha \end{pmatrix} \)

- \( \tilde{a}^+ \) level shift \( \Rightarrow \pi^- p \rightarrow \pi^- p \)

- \( \pi^0 N \rightarrow \pi^- N \)

- \( \pi^0 n \)

- \( a^- = (86.1 \pm 0.9) \cdot 10^{-3}/M_\pi \)

- \( a^+ = (7.6 \pm 3.1) \cdot 10^{-3}/M_\pi \)

\( \Rightarrow \) very precise value for \( a^- \) & first time definite sign for \( a^+ \)
ERROR ANALYSIS

- Variation of the input:
  - use KH80 input instead of GWU/SAID (higher PWs, inelasticities) → small effect
  - very small effect from s-channel PWs with $\ell > 5$
  - small effect from the S-wave extrapolation for $t > 1.3$ GeV
  - negligible effect of the the $\rho'$ and the $\rho''$
  - very significant effect of the D-waves (esp. $f_2(1270)$)
  - F-waves shown to be negligible

- Other sources of uncertainty:
  - statistical errors (shallow fit minima)
  - matching conditions (close to $W_m$) [no error on SAID, use smoothed KH80]
  - scattering lengths errors (important for $\sigma_{\pi N}$)

⇒ First time this has been achieved in a dispersive analysis of $\pi N$ scattering!
Results

Hoferichter, Ruiz de Elvira, Kubis, UGM
PHASE SHIFTS

- S- and P-waves up to the matching point [Notation: $L_{2I_s2J}$]
PHASE SHIFTS II

- Above the matching point (cf SAID/GWU)
t-CHANNEL PARTIAL WAVES

- Imaginary parts of the t-channel partial waves (cf KH80)

\[ \text{Im} f^0_+ \]
\[ \text{Im} f^1_+ \]
\[ \text{Im} f^1_- \]
\[ \text{Im} f^2_+ \]
\[ \text{Im} f^2_- \]

\[ \sqrt{t} \text{ [GeV]} \]

\[ \text{xxx} = \text{KH80} \]

reproduced if \( a^+ \), \( a^- \)

and \( g_{\pi NN} \) are readjusted
t-CHANNEL PARTIAL WAVES continued

- Real parts of the t-channel partial waves (cf KH80)

\[ \sqrt{t} [\text{GeV}] \]

\[ \text{Re} f_0^0 \]
\[ \text{Re} f_1^1 \]
\[ \text{Re} f_1^1 \]
\[ \text{Re} f_2^2 \]
\[ \text{Re} f_2^2 \]

\[ x = \text{KH80} \]
reproduced if \( a^+ \), \( a^- \)
and \( g_{\pi N} \) are readjusted
THRESHOLD PARAMETERS

- Threshold parameters from:

$$\text{Re} f_{\ell \pm}^I(s) = q^{2\ell} \left( a_{\ell \pm}^I + b_{\ell \pm}^I q^2 + \ldots \right)$$

<table>
<thead>
<tr>
<th></th>
<th>RS</th>
<th>KH80</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{0+}^{1/2}$</td>
<td>169.8 ± 2.0</td>
<td>173 ± 3</td>
</tr>
<tr>
<td>$a_{0+}^{3/2}$</td>
<td>−86.3 ± 1.8</td>
<td>−101 ± 4</td>
</tr>
<tr>
<td>$a_{1+}^{1/2}$</td>
<td>−29.4 ± 1.0</td>
<td>−30 ± 2</td>
</tr>
<tr>
<td>$a_{1+}^{3/2}$</td>
<td>211.5 ± 2.8</td>
<td>214 ± 2</td>
</tr>
<tr>
<td>$a_{1-}^{1/2}$</td>
<td>−70.7 ± 4.1</td>
<td>−81 ± 2</td>
</tr>
<tr>
<td>$a_{1-}^{3/2}$</td>
<td>−41.0 ± 1.1</td>
<td>−45 ± 2</td>
</tr>
<tr>
<td>$b_{0+}^{1/2}$</td>
<td>−35.2 ± 2.2</td>
<td>−18 ± 12</td>
</tr>
<tr>
<td>$b_{0+}^{3/2}$</td>
<td>−49.8 ± 1.1</td>
<td>−58 ± 9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>RS</th>
<th>KH80</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{0+}^+$</td>
<td>−0.9 ± 1.4</td>
<td>−9.7 ± 1.7</td>
</tr>
<tr>
<td>$a_{0-}$</td>
<td>85.4 ± 0.9</td>
<td>91.3 ± 1.7</td>
</tr>
<tr>
<td>$a_{1+}^+$</td>
<td>131.2 ± 1.7</td>
<td>132.7 ± 1.3</td>
</tr>
<tr>
<td>$a_{1-}$</td>
<td>−80.3 ± 1.1</td>
<td>−81.3 ± 1.0</td>
</tr>
<tr>
<td>$a_{1-}^+$</td>
<td>−50.9 ± 1.9</td>
<td>−56.7 ± 1.3</td>
</tr>
<tr>
<td>$a_{1-}^-$</td>
<td>−9.9 ± 1.2</td>
<td>−11.7 ± 1.0</td>
</tr>
<tr>
<td>$b_{0+}^+$</td>
<td>−45.0 ± 1.0</td>
<td>−44.3 ± 6.7</td>
</tr>
<tr>
<td>$b_{0-}$</td>
<td>4.9 ± 0.8</td>
<td>13.3 ± 6.0</td>
</tr>
</tbody>
</table>

- In units of $10^{-3}/M_\pi$ or $10^{-3}/M_{\pi}^3$, respectively

- As $a^+$ is very sensitive to isospin breaking and PWAs measure $\pi^{\pm}p$,
  use $(a_{\pi^-p} + a_{\pi^+p})/2 = (-0.9 \pm 1.4) \cdot 10^{-3}/M_\pi$

- Most striking difference to KH80: S-wave scattering lengths! (here: input)
RESULTS for the SIGMA-TERM

- Basic formula:

\[
\sigma_{\pi N} = F_\pi^2 (d_{00}^+ + 2M_\pi^2 d_{01}^+) + \Delta_D - \Delta_\sigma - \Delta_R
\]

- Subthreshold parameters output of the RS equations:

\[
d_{00}^+ = -1.36(3) M_\pi^{-1} \quad [\text{KH: } -1.46(10) M_\pi^{-1}]
\]

\[
d_{01}^+ = 1.16(3) M_\pi^{-3} \quad [\text{KH: } 1.14(2) M_\pi^{-3}]
\]

- \(\Delta_D - \Delta_\sigma = (1.8 \pm 0.2) \text{ MeV}\)  
  \(\text{Hoferichter, Ditsche, Kubis, UGM (2012)}\)

- \(\Delta_R \lesssim 2 \text{ MeV}\)  
  \(\text{Bernard, Kaiser, UGM (1996)}\)

- Isospin breaking in the CD theorem shifts \(\sigma_{\pi N}\) by \(+3.0 \text{ MeV}\)

\(\Rightarrow\) Final result:

\[
\sigma_{\pi N} = (59.1 \pm 1.9_{\text{RS}} \pm 3.0_{\text{LET}}) \text{ MeV} = (59.1 \pm 3.5) \text{ MeV}
\]

[NB: recover \(\sigma_{\pi N} = 45 \text{ MeV}\) if KH80 scattering lengths are used]
RESULTS for the SCALAR NUCLEON COUPLINGS

• WIMP scattering off nuclei sensitive to scalar nucleon couplings:

\[
\langle N | m_q \bar{q} q | N \rangle = f_q^N m_N \quad (N = p, n; \ q = u, d, s)
\]

• Include isospin breaking corrections and use \( m_u/m_d = 0.46 \)

\[
\Rightarrow f_u^p = (20.8 \pm 1.5) \cdot 10^{-3}, \quad f_d^p = (41.1 \pm 2.8) \cdot 10^{-3}
\]

\[
f_u^n = (18.9 \pm 1.4) \cdot 10^{-3}, \quad f_d^n = (45.1 \pm 2.7) \cdot 10^{-3}
\]

\[
\sum_{q=u,...,t} f_q^N = \frac{2}{9} + \frac{7}{9} (f_u^N + f_d^N + f_s^N) = 0.305 \pm 0.009
\]

– sizeable reduction in uncertainties of \( f_{u,d}^N \) due to the precise \( \sigma \)-term

– combination of couplings relevant for Higgs-mediated interactions

– \( f_s^N \) from Lattice QCD

Junnarkar, Walker-Loud (2013)
RESULTS for the LECs

• Chiral expansion expected to work best at the subthreshold point (polynomial, maximal distance to singularities)

• Express subthreshold parameters in terms of LECs → invert system

• LECs $c_i$ of the dimension two chiral effective $\pi N$ Lagrangian:

<table>
<thead>
<tr>
<th>LEC</th>
<th>RS</th>
<th>KGE 2012</th>
<th>UGM 2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$ [GeV$^{-1}$]</td>
<td>$-1.11 \pm 0.03$</td>
<td>$-1.13 \ldots - 0.75$</td>
<td>$-0.9^{+0.2}_{-0.5}$</td>
</tr>
<tr>
<td>$c_2$ [GeV$^{-1}$]</td>
<td>$3.13 \pm 0.03$</td>
<td>$3.49 \ldots 3.69$</td>
<td>$3.3 \pm 0.2$</td>
</tr>
<tr>
<td>$c_3$ [GeV$^{-1}$]</td>
<td>$-5.61 \pm 0.06$</td>
<td>$-5.51 \ldots - 4.77$</td>
<td>$-4.7^{+1.2}_{-1.0}$</td>
</tr>
<tr>
<td>$c_4$ [GeV$^{-1}$]</td>
<td>$4.26 \pm 0.04$</td>
<td>$3.34 \ldots 3.71$</td>
<td>$-3.5^{+0.5}_{-0.2}$</td>
</tr>
</tbody>
</table>


• also results for pertinent dimension three and four LECs
Comparison with recent results from lattice QCD

[arXiv:1602.07688]
RESULTS for the SIGMA-TERM

- Recent results from various LQCD collaborations:

<table>
<thead>
<tr>
<th>collaboration</th>
<th>$\sigma_{\pi N}$ [MeV]</th>
<th>reference</th>
<th>tension to RS</th>
</tr>
</thead>
<tbody>
<tr>
<td>BMW</td>
<td>38(3)(3)</td>
<td>Dürr et al. (2015)</td>
<td>3.8$\sigma$</td>
</tr>
<tr>
<td>$\chi$QCD</td>
<td>45.9(7.4)(2.8)</td>
<td>Yang et al. (2015)</td>
<td>1.5$\sigma$</td>
</tr>
<tr>
<td>ETMC</td>
<td>37.22(2.57)(^{+0.99}_{-0.63})</td>
<td>Abdel-Rehim et al. (2016)</td>
<td>4.9$\sigma$</td>
</tr>
<tr>
<td>CRC 55</td>
<td>35(6)</td>
<td>Bali et al. (2016)</td>
<td>4.0$\sigma$</td>
</tr>
</tbody>
</table>

- We seem to have a problem - do we? [we = RS folks]

- Robust prediction of the RS analysis:

$$
\sigma_{\pi N} = (59.1 \pm 3.1) \text{ MeV} + \sum_{I_s} c_{I_s} (a_{I_s} - \bar{a}_{I_s}) \quad (I_s = \frac{1}{2}, \frac{3}{2})
$$

$$
c_{1/2} = 0.242 \text{ MeV} \times 10^3 M_{\pi}, \quad c_{3/2} = 0.874 \text{ MeV} \times 10^3 M_{\pi}
$$

$$
\bar{a}^{1/2} = (169.8\pm 2.0) \times 10^{-3} M_{\pi}^{-1}, \quad \bar{a}^{3/2} = (-86.3\pm 1.8) \times 10^{-3} M_{\pi}^{-1}
$$

$\rightarrow$ expansion around the reference values from $\pi H$ and $\pi D$
RESULTS for the SIGMA-TERM

- Apply this linear expansion to the lattice data:

\[ \tilde{a} + \left[ \frac{10^{-3} M}{\pi} \right] \text{ level shift of } \pi H \]

⇒ Lattice results clearly at odds with empirical information on the scattering lengths!

⇒ scattering lengths to \([5 \ldots 10]\% \rightarrow \delta\sigma_{\pi N} = [5.0 \ldots 8.5] \text{ MeV}\)
SUMMARY & OUTLOOK

- Derived closed system of RS equations (PWHDRs) for $\pi N \rightarrow \pi N$
- Solved t-channel MO problem for the one- and two-channel approximation
- Numerical solution of the full system of RS equations
  $\rightarrow$ KH80 self-consistent, but at odds with hadronic atom phenomenology
- Complete error analysis (first time!)
- Precise determination of the pion-nucleon $\sigma$-term: $\sigma_{\pi N} = (59.1 \pm 3.5)$ MeV
- Precise determination of threshold parameters, scalar couplings & CHPT LECs
- New determination of the nucleon isovector spectral functions (in the works)
- Open ends:
  $\rightarrow$ lattice determinations of $\sigma_{\pi N}$ at odds with modern scattering lengths
  $\rightarrow$ strangeness content $\sim \langle N | m_s \bar{s}s | N \rangle$
STRANGENESS and the $\sigma$-TERM

- Strangeness content:

$$\sigma_{\pi N} = \frac{\sigma_0}{1 - y}, \quad y = \frac{2 \langle p | \bar{s}s | p \rangle}{\langle p | \bar{u}u + \bar{d}d | p \rangle}$$

- Three-flavor CHPT analysis:

$$\sigma_0 = (33 \pm 5) \text{ MeV} \quad \text{Gasser (1981)}$$

$$\sigma_0 = (36 \pm 7) \text{ MeV} \quad \text{Borasoy, UGM (1997)}$$

$$\sigma_0 = (58 \pm 8) \text{ MeV} \quad \text{Alarcon et al. (2014)}$$

⇒ Situation inconclusive:

First two calculations use cut-off / heavy baryons and no decuplet, third one claims large decuplet and relativistic corrections, but: $\sigma_0 = 27 \text{ MeV}$ at Leading Order from baryon mass splittings
Pion-nucleon scattering: Fundamentals etc.
**PION-NUCLEON SCATTERING**

- **s-channel:** \( \pi(q) + N(p) \to \pi(q') + N(p') \)

- **t-channel:** \( \pi(q) + \pi(-q') \to \bar{N}(-p) + N(p') \)

- **Mandelstam variables:**
  \[
  s = (p + q)^2, 
  t = (p - p')^2, 
  u = (p - q')^2 
  s + t + u = 2m_N^2 + 2M_\pi^2, 
  s = W^2
  \]

- **Isospin structure:**
  \[
  T^{ba}(s, t) = \delta^{ba} T^+(s, t) + i\epsilon_{abc}\tau^c T^-(s, t)
  \]

- **Lorentz structure:**
  \[
  8\pi\sqrt{s}T^I(s, t) = \bar{u}(p') \{ A^I(s, t) + \frac{1}{2}(\not{q} + \not{q'})B^I(s, t) \} u(p), 
  I = +, - 
  I = 1/2, 3/2
  \]

- **Crossing:**
  \[
  A^\pm(\nu, t) = \pm A^\pm(-\nu, t), 
  B^\pm(\nu, t) = \mp B^\pm(-\nu, t), 
  \nu = \frac{s - u}{4m_N}
  \]
PION-NUCLEON SCATTERING continued

• Partial wave projection:

\[
X^{I}_{\ell}(s) = \frac{1}{\pi} \int_{-1}^{+1} dz_s P_{\ell}(z_s) X^{I}(s, t) \bigg|_{t = -2q^2(1-z_s)} , \quad X \in \{A, B\}
\]

⇒ partial wave expansion (total isospin \(I\), ang. mom. \(\ell\), \(j = \ell \pm 1/2\)):

\[
f^{I}_{\ell \pm}(W) = \frac{1}{16\pi W} \times \left\{ (E + m)[A^{I}_{\ell}(s) + (W - m)B^{I}_{\ell}(s)] + (E - m)[-A^{I}_{\ell \pm 1}(s) + (W + m)B^{I}_{\ell \pm 1}(s)] \right\}
\]

• MacDowell symmetry:

\[
f^{I}_{\ell+}(W) = -f^{I}_{(\ell+1)-}(-W) \quad \forall \ l \geq 0 \quad \text{MacDowell (1959)}
\]

• Low-energy region: only S- and P-waves are relevant

\[
\begin{align*}
f^{\pm}_{0+}, f^{\pm}_{1+}, f^{\pm}_{1-}
\end{align*}
\]

⇒ low-energy amplitude can eventually be matched to chiral perturbation theory

Büttiker, Fettes, UGM, Steiniger; Ellis, Tang; Becher, Leutwyler, . . .
SUBTHRESHOLD EXPANSION

- For the $\sigma$-term extraction, the $\pi N$ amplitude $D = A + \nu B$ is most useful:

$$\bar{D}^+(\nu, t) = D^+(\nu, t) - \frac{g_{\pi N}^2}{m_N} - \nu g_{\pi N}^2 \left( \frac{1}{m_N^2 - s} - \frac{1}{m_N^2 - u} \right)$$

★ subtraction of pseudovector Born terms $\rightarrow \bar{D}$

- Subthreshold expansion: expand around $\nu = t = 0$:

$$X(\nu, t) = \sum_{m,n} x_{mn} \nu^{2m} t^n, \quad X \in \left\{ \bar{A}^+, \frac{\bar{A}^-}{\nu}, \frac{\bar{B}^+}{\nu}, \bar{B}^-, \bar{D}^+, \frac{\bar{D}^-}{\nu} \right\}$$

★ $x_{mn}$ are the subthreshold parameters $\rightarrow$ can be calculated via sum rules

★ inside the Mandelstam triangle, scattering amplitudes are real polynomials
**σ-TERM BASICS**

- Scalar form factor of the nucleon (isospin limit \( \hat{m} = (m_u + m_d)/2 \)):

  \[
  \sigma_{\pi N}(t) = \langle N(p')|\hat{m}(\bar{u}u + \bar{d}d)|N(p)\rangle, \quad t = (p' - p)^2
  \]

- Cheng-Dashen Low-Energy Theorem (LET):

  \[
  \bar{D}^+(0, 2M^2_\pi) = \sigma(2M^2_\pi) + \Delta_R \\
  |\Delta_R| \lesssim 2 \text{ MeV}
  \]

  Cheng, Dashen (1971)  

- Standard decomposition of the σ-term:  \( \sigma_{\pi N} = \sigma_{\pi N}(0) \)

  \[
  \sigma_{\pi N} = \Sigma_d + \Delta_D - \Delta_\sigma - \Delta_R \\
  \Sigma_d = F^2_\pi(d^{+}_{00} + 2M^2_\pi d^{+}_{01}) \quad \rightarrow \text{full RS analysis} \\
  \Delta_D = \bar{D}^+(0, 2M^2_\pi) - \Sigma_d \\
  \Delta_\sigma = \sigma(2M^2_\pi) - \sigma_{\pi N}
  \]

  \[ \}
  \rightarrow \text{RS t-channel analysis} \]

- Strong ππ rescattering in \( \Delta_D \) and \( \Delta_\sigma \), the difference is small!

  Gasser, Leutwyler, Sainio (1991)
Hadronic Atoms
WHY HADRONIC ATOMS?

• Hadronic atoms: bound by the static Coulomb force (QED)

• Many species: $\pi^+\pi^−$, $\pi^+K^−$, $\pi^-p$, $\pi^-d$, $K^-p$, $K^-d$, . . .

• Observable effects of QCD: strong interactions as small perturbations

  ★ energy shift $\Delta E$

  ★ deacy width $\Gamma$

  ⇒ access to scattering at zero energy!

  = S-wave scattering lengths

  ⇒ best way to determine the scattering lengths!

• can be analyzed in suitable NREFTs

  Pionic hydrogen ($\pi H$)

  Pionic deuterium ($\pi D$)

  Gasser, Rusetsky, . . . 2002

  Baru, Hoferichter, Kubis . . . 2011
GMO SUM RULE

- Goldberger-Miyazawa-Oehme sum rule:

\[
\frac{g_{\pi N}^2}{4\pi} = \left[ \left( \frac{m_p + m_n}{M_\pi} \right)^2 - 1 \right] \left\{ \left( 1 + \frac{M_\pi}{m_p} \right) \frac{M_\pi}{4} \left[ \frac{a_{\pi^- p} - a_{\pi^+ p}}{2} - \frac{M_\pi^2}{2} J^- \right] \right\}
\]

\[
= 13.69 \pm 0.12 \pm 0.15
\]

\[
J^- = \frac{1}{4\pi^2} \int_0^\infty dk \frac{\sigma_{\pi^- p}^{\text{tot}}(k) - \sigma_{\pi^+ p}^{\text{tot}}(k)}{\sqrt{M_\pi^2 + k^2}}
\]

- \( J^- \) is very well determined

- consistent with other determinations:

\[
\begin{align*}
\pi N & \quad 13.75 \pm 0.15 \\
NN & \quad 13.54 \pm 0.05
\end{align*}
\]

Goldberger, Miyazawa, Oehme 1955

Ericson et al. 2002, Abaev et al. 2007

Arndt et al. 1994

de Swart et al. 1997
SUBTHRESHOLD EXPANSION

- expand around $\nu = t = 0$:

$$X(\nu, t) = \sum_{m,n} x_{mn} \nu^{2m} t^n, \quad X \in \left\{ A^+, \frac{A^-}{\nu}, \frac{B^+}{\nu}, B^-, D^+, \frac{D^-}{\nu} \right\}$$

★ subtraction of pseudovector Born terms $\rightarrow \bar{X}$

★ $x_{mn}$ are the **subthreshold parameters** $\rightarrow$ can be calculated via sum rules

- low-energy expansion of the pion-nucleon scattering amplitude ($D = A + \nu B$):

$$A^+(\nu, t) = \frac{g_{\pi N}^2}{m_N} + d_{00}^+ + d_{01}^+ t + \mathcal{O}(\nu^2, \nu^2 t, t^2)$$

$$A^-(\nu, t) = \nu d_{00}^- + \mathcal{O}(\nu^3, \nu t), \quad B^+(\nu, t) = g_{\pi N}^2 \frac{4m_N \nu}{M^2} + \nu b_{00}^+ + \mathcal{O}(\nu^3, \nu t)$$

$$B^-(\nu, t) = -\frac{g_{\pi N}^2}{M^2} \left[ 2 + \frac{t}{M^2} \right] - \frac{g_{\pi N}^2}{2m_N^2} + b_{00}^- + b_{01}^- t + \mathcal{O}(\nu^2, \nu^2 t, t^2)$$
**t-CHANNEL MO PROBLEM**

- One-channel MO problem with finite matching point $t_m$

\[
 f(t) = \Delta(t) + \frac{1}{\pi} \int_{4M_\pi^2}^{t_m} dt' \frac{T(t')*f(t')}{t' - t} + \frac{1}{\pi} \int_{t_m}^{\infty} dt' \frac{\text{Im} f(t')}{t' - t}
\]

$\leftrightarrow \Delta(t)$: pole terms, s-channel imag. parts, other $t$-channel PWs

$\leftrightarrow$ solve for $f(t)$ in $4M_\pi^2 \leq t \leq t_m$ requires

- $\text{Im} f(t)$ for $t \geq t_m$
- $T(t)$ for $4M_\pi^2 \leq t \leq t_m$

- Solution via once-subtracted Omnès function (w/ $\Omega(0) = 1$):

\[
 \Omega(t) = \exp \left\{ \frac{t}{\pi} \int_{4M_\pi^2}^{t_m} dt' \frac{\delta(t')}{t' - t} \right\} = |\Omega(t)| \exp \{i\delta(t)\theta(t - 4M_\pi^2)\theta(t_m - t)\}
\]