Dispersive approach to QCD and hadronic contributions to electroweak observables

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Hadronic vacuum polarization function $\Pi(q^2)$ plays a central role in various issues of QCD and Standard Model. In particular, the theoretical description of some strong interaction processes and of hadronic contributions to electroweak observables is inherently based on $\Pi(q^2)$:

- electron–positron annihilation into hadrons
- inclusive $\tau$ lepton hadronic decay
- muon anomalous magnetic moment
- running of the electromagnetic coupling
QCD PERTURBATIVE PREDICTIONS

**Leading order:**

\[ \Pi^{(0)}(q^2) = -\ln\left(\frac{-q^2}{-q_0^2}\right) \]

\[ D^{(0)}(Q^2) = -\frac{d\Pi^{(0)}(-Q^2)}{d\ln Q^2} = 1 \]

**Strong corrections:**

\[ D^{(1)}(Q^2) = 1 + \frac{1}{\pi} \alpha_s^{(1)}(Q^2) = 1 + \frac{4}{\beta_0} \frac{1}{\ln(Q^2/\Lambda^2)} \]

\[ D^{(\ell)}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[ \alpha_s^{(\ell)}(Q^2) \beta_0/(4\pi) \right]^j = 1 + \sum_{j=1}^{\ell} d_j \left[ a_s^{(\ell)}(Q^2) \right]^j \]
Cross-section of $e^+e^- \rightarrow \text{hadrons}$: 

$$
\sigma = 4\pi^2 \frac{2\alpha^2}{s^3} L_{\mu\nu} \Delta_{\mu\nu},
$$

where $s = q^2 = (p_1 + p_2)^2 > 0$,

$$
L_{\mu\nu} = \frac{1}{2} \left[ q_\mu q_\nu - g_{\mu\nu} q^2 - (p_1 - p_2)_\mu (p_1 - p_2)_\nu \right],
$$

$$
\Delta_{\mu\nu} = (2\pi)^4 \sum_\Gamma \delta(p_1 + p_2 - p_\Gamma) \langle 0| J_\mu(-q) |\Gamma \rangle \langle \Gamma | J_\nu(q) | 0 \rangle,
$$

and $J_\mu = \sum_f Q_f : \bar{q} \gamma_\mu q :$ is the electromagnetic quark current.

**Kinematic restriction**: the hadronic tensor $\Delta_{\mu\nu}(q^2)$ assumes non–zero values only for $q^2 \geq 4m_\pi^2 = m^2$, since otherwise no hadron state $\Gamma$ could be excited

[Feynman (1972); Adler (1974).]
The hadronic tensor can be represented as $\Delta_{\mu\nu} = 2 \text{Im} \, \Pi_{\mu\nu}$,

$$\Pi_{\mu\nu}(q^2) = i \int e^{iqx} \langle 0 \left| T\{ J_\mu(x) J_\nu(0) \} \right| 0 \rangle \, d^4x = i(q_\mu q_\nu - g_{\mu\nu}q^2) \frac{\Pi(q^2)}{12\pi^2}.$$ 

**Kinematic restriction**: $\Pi(q^2)$ has the only cut $q^2 \geq m^2$

**Dispersion relation for $\Pi(q^2)$**:

$$\Delta \Pi(q^2, q_0^2) = \frac{1}{2\pi i} (q^2 - q_0^2) \int_C \frac{\Pi(\xi)}{(\xi - q^2)(\xi - q_0^2)} \, d\xi$$

$$= (q^2 - q_0^2) \int_{m^2}^{\infty} \frac{R(s)}{(s - q^2)(s - q_0^2)} \, ds,$$

where $\Delta \Pi(q^2, q_0^2) = \Pi(q^2) - \Pi(q_0^2)$ and $R(s)$ denotes the measurable ratio of two cross-sections

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \left[ \Pi(s + i\varepsilon) - \Pi(s - i\varepsilon) \right] = \frac{\sigma(e^+ e^- \to \text{hadrons}; s)}{\sigma(e^+ e^- \to \mu^+ \mu^-; s)}$$

**Kinematic restriction**: $R(s) = 0$ for $s < m^2$
In general, it is also convenient to employ the so-called Adler function \( Q^2 = -q^2 > 0 \)

\[
D(Q^2) = -\frac{d \Pi(-Q^2)}{d \ln Q^2}, \quad D(Q^2) = Q^2 \int_{m^2}^{\infty} \frac{R(s)}{(s + Q^2)^2} ds
\]

Adler (1974); De Rujula, Georgi (1976); Bjorken (1989).

This dispersion relation provides a link between experimentally measurable and theoretically computable quantities.

The inverse relations between the functions on hand read

\[
R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta},
\]

Radyushkin (1982); Krasnikov, Pivovarov (1982)

\[
\Delta \Pi(-Q^2, -Q_0^2) = -\int_{Q_0^2}^{Q^2} D(\sigma) \frac{d\sigma}{\sigma}
\]

The complete set of relations between $\Pi(q^2)$, $R(s)$, and $D(Q^2)$:

$$\Delta \Pi(q^2, q_0^2) = (q^2 - q_0^2) \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma - q^2)(\sigma - q_0^2)} d\sigma = - \int_{-q_0^2}^{-q^2} D(\sigma) \frac{d\sigma}{\sigma},$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \left[ \Pi(s + i\varepsilon) - \Pi(s - i\varepsilon) \right] = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta},$$

$$D(Q^2) = -\frac{d \Pi(-Q^2)}{d \ln Q^2} = Q^2 \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma + Q^2)^2} d\sigma.$$
Functions on hand in terms of the common spectral density:

\[
\Delta \Pi(q^2, q_0^2) = \Delta \Pi^{(0)}(q^2, q_0^2) + \int_{m^2}^{\infty} \rho(\sigma) \ln \left( \frac{\sigma - q^2 \, m^2 - q_0^2}{\sigma - q_0^2 \, m^2 - q^2} \right) \frac{d\sigma}{\sigma},
\]

\[
R(s) = R^{(0)}(s) + \theta(s - m^2) \int_{s}^{\infty} \rho(\sigma) \frac{d\sigma}{\sigma},
\]

\[
D(Q^2) = D^{(0)}(Q^2) + \frac{Q^2}{Q^2 + m^2} \int_{m^2}^{\infty} \rho(\sigma) \frac{\sigma - m^2}{\sigma + Q^2} \frac{d\sigma}{\sigma},
\]

\[
\rho(\sigma) = \frac{1}{\pi} \frac{d}{d \ln \sigma} \text{Im} \lim_{\varepsilon \to 0^+} p(\sigma - i\varepsilon) = -\frac{d}{d \ln \sigma} r(\sigma) = \frac{1}{\pi} \text{Im} \lim_{\varepsilon \to 0^+} d(-\sigma - i\varepsilon),
\]

where \( \Delta \Pi^{(0)}(q^2, q_0^2), R^{(0)}(s), D^{(0)}(Q^2) \) denote the leading–order terms and \( p(q^2), r(s), d(Q^2) \) stand for the strong corrections.

Derivation of obtained representations involves neither additional approximations nor model-dependent assumptions, with all the nonperturbative constraints being embodied.

The leading-order terms of the functions on hand read

\[ \Delta \Pi^{(0)}(q^2, q_0^2) = 2 \frac{\varphi - \tan \varphi}{\tan^3 \varphi} - 2 \frac{\varphi_0 - \tan \varphi_0}{\tan^3 \varphi_0}, \quad \sin^2 \varphi = \frac{q^2}{m^2}, \]

\[ R^{(0)}(s) = \theta(s - m^2) \left(1 - \frac{m^2}{s}\right)^{3/2}, \quad \sin^2 \varphi_0 = \frac{q_0^2}{m^2}, \]

\[ D^{(0)}(Q^2) = 1 + \frac{3}{\xi} \left[1 - \sqrt{1 + \xi^{-1} \sinh^{-1}(\xi^{1/2})}\right], \quad \xi = \frac{Q^2}{m^2}. \]

\[ \text{Perturbative contribution to the spectral density:}\]

\[ \rho_{\text{pert}}(\sigma) = \frac{1}{\pi} \frac{d \text{Im} p_{\text{pert}}(\sigma - i0^+)}{d \ln \sigma} = - \frac{d r_{\text{pert}}(\sigma)}{d \ln \sigma} = \frac{1}{\pi} \text{Im} d_{\text{pert}}(-\sigma - i0^+), \]

\[ \text{one-loop level:}\]

\[ \rho_{\text{pert}}^{(1)}(\sigma) = \left(4/\beta_0\right)\left[\ln^2(\sigma/\Lambda^2) + \pi^2\right]^{-1}, \]

\[ \text{higher-loops:}\]

\[ \text{Nesterenko, Simolo (2010, 2011); Bakulev (2013); Cvetić (2015).} \]
Note on the massless limit

In the limit \( m = 0 \) the obtained integral representations read

\[
\Delta \Pi(q^2, q_0^2) = -\ln \left( \frac{-q^2}{-q_0^2} \right) + \int_0^\infty \rho(\sigma) \ln \left[ \frac{1 - (\sigma/q^2)}{1 - (\sigma/q_0^2)} \right] \frac{d\sigma}{\sigma},
\]

\[
R(s) = 1 + \int_s^\infty \rho(\sigma) \frac{d\sigma}{\sigma},
\]

\[
D(Q^2) = 1 + \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2} d\sigma.
\]

For \( \rho(\sigma) = \rho_{\text{pert}}(\sigma) \) two highlighted massless equations become identical to those of the APT \( \text{Shirkov, Solovtsov, Milton (1997–2007).} \)

But it is \textbf{essential} to keep the threshold \( m \) nonvanishing:

- massless limit loses some of nonperturbative constraints
- effects due to \( m \neq 0 \) become substantial at low energies
Comparison of obtained results with lattice simulation data:

Both PT and APT fail to describe $\Pi(q^2)$ at low energies:

**PT:** $\Pi(q^2)$ possesses infrared unphysical singularities

**APT:** $\Pi(q^2)$ diverges in IR limit

Unphysical singularities

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**Della Morte, Jager, Juttner, Wittig (2011–2015); Nesterenko (2014, 2015).**
ADLER FUNCTION

massless limit ($m = 0$)  
realistic case ($m \neq 0$)

- **APT**
  - unphysical singularities: contains
  - agreement with data: disagrees

- **DPT**
  - unphysical singularities: free
  - agreement with data: agrees

Some attempts to improve IR behavior of $D(Q^2)$ within APT:

**APT + relativistic quark mass threshold resummation:**

APT + vector meson dominance assumption:

large light quark masses

$2m_{u,d} \approx 520 \text{ MeV} \approx 4m_\pi$


VMD NW approximation and cut–off at $M_0 \approx 740 \text{ MeV}$

- Cvetic et al. (2005–2015)
The theoretical description of $a_\mu = (g_\mu - 2)/2$ is a long-standing challenging issue of the elementary particle physics.

**Experiment:** \( a_\mu^{\text{exp}} = (11659208.9 \pm 6.3) \times 10^{-10} \) (0.54 ppm)

- Muon (g-2) Collaboration (2006); Roberts (2010).

**Theory:**

\[
a_\mu^{\text{theor}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{HLO}} + a_\mu^{\text{HHO}} + a_\mu^{\text{Hlbl}}
\]

\[
a_\mu^{\text{QED}} = (11658471.8951 \pm 0.0080) \times 10^{-10} \quad \text{Aoyama, Hayakawa, Kinoshita, Nio (2012)}
\]

\[
a_\mu^{\text{EW}} = (15.36 \pm 0.10) \times 10^{-10} \quad \text{Gnendiger, Stockinger, Stockinger–Kim (2013)}
\]

\[
a_\mu^{\text{HHO}} = (-9.84 \pm 0.07) \times 10^{-10} \quad \text{Hagiwara, Liao, Martin, Nomura, Teubner (2011)}
\]

\[
a_\mu^{\text{Hlbl}} = (11.6 \pm 4.0) \times 10^{-10} \quad \text{Nyffeler (2014)}.
\]

The uncertainty of theoretical estimation of $a_\mu$ is mainly dominated by the leading–order hadronic contribution $a_\mu^{\text{HLO}}$. 
The latter involves the integration of $\Pi(q^2)$ over low energies:

$$a_{\mu}^{\text{HLO}} = \frac{1}{3} \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty f\left(\frac{\zeta}{4m_{\mu}^2}\right) \Pi(\zeta) \frac{d\zeta}{4m_{\mu}^2}, \quad f(x) = \frac{1}{x^3} \frac{y^5(x)}{1 - y(x)},$$

where $y(x) = x\left(\sqrt{1 + x^{-1}} - 1\right)$. \textbf{Lautrup, Peterman, de Rafael (1972).}

Dispersive approach enables one to evaluate $a_{\mu}^{\text{HLO}}$ without invoking experimental data on $R(s)$:

$$a_{\mu}^{\text{HLO}} = (696.1 \pm 9.5) \times 10^{-10}.$$

This result agrees fairly well with recent assessments of $a_{\mu}^{\text{HLO}}$.

The complete SM prediction

$$a_{\mu} = (11659185.1 \pm 10.3) \times 10^{-10}$$

differs from $a_{\mu}^{\text{exp}}$ by two standard deviations. \textbf{Nesterenko (2015).}
ELECTROMAGNETIC FINE STRUCTURE CONSTANT

The electromagnetic running coupling $\alpha_{\text{em}}(q^2)$ plays a central role in a variety of issues of precision particle physics:

$$\alpha_{\text{em}}(q^2) = \frac{\alpha}{1 - \Delta\alpha_{\text{lep}}(q^2) - \Delta\alpha_{\text{had}}(q^2)}$$

with $\alpha = e^2/(4\pi) \simeq 1/137.036$ being the fine structure constant.

Leptonic contribution to $\alpha_{\text{em}}(q^2)$ can be calculated within perturbation theory: $\Delta\alpha_{\text{lep}}(M_Z^2) = (314.979\pm0.002) \times 10^{-4}$ \cite{Sturm2013}.

However, the respective hadronic contribution involves the integration over the low–energy range

$$\Delta\alpha_{\text{had}}(M_Z^2) = -\frac{\alpha}{3\pi} M_Z^2 \int_m^\infty \frac{R(s)}{s - M_Z^2} \frac{d s}{s}$$

and constitutes the prevalent source of uncertainty of $\alpha_{\text{em}}(M_Z^2)$. 

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As usual, the top quark contribution to $\alpha_{\text{em}}(q^2)$ is taken into account separately:

$$\Delta \alpha_{\text{had}}^{\text{top}}(M_Z^2) = (-0.70 \pm 0.05) \times 10^{-4}$$


The evaluation of $\Delta \alpha_{\text{had}}^{(5)}(M_Z^2)$ in the framework of dispersive approach leads to

$$\Delta \alpha_{\text{had}}^{(5)}(M_Z^2) = (274.9 \pm 2.2) \times 10^{-4}.$$ 

The obtained assessment appears to be in a good agreement with recent estimations of $\Delta \alpha_{\text{had}}^{(5)}(M_Z^2)$ and eventually yields

$$\alpha_{\text{em}}^{-1}(M_Z^2) = 128.962 \pm 0.030$$

Perturbative approximation of $R$–ratio:

At high energies one commonly re–expands the $R$–ratio:

$$R^{(\ell)}(s) = 1 + \sum_{j=1}^{\ell} r_j \left[ a^{(\ell)}_s(|s|) \right]_j, \quad r_j = d_j - \delta_j, \quad B_j = \frac{\beta_j}{\beta_{j+1}},$$

$$\delta_1 = 0, \quad \delta_2 = 0, \quad \delta_3 = \frac{\pi^2}{3} d_1, \quad \delta_4 = \frac{\pi^2}{3} \left( \frac{5}{2} d_1 B_1 + 3 d_2 \right),$$

$$\delta_5 = \frac{\pi^2}{3} \left[ \frac{3}{2} d_1 (B_1^2 + 2 B_2) + 7 d_2 B_1 + 6 d_3 \right] - \frac{\pi^4}{5} d_1,$$

$$\delta_6 = \frac{\pi^2}{3} \left[ \frac{7}{2} d_1 (B_1 B_2 + B_3) + 4 d_2 (B_1^2 + 2 B_2) + \frac{27}{2} d_3 B_1 + 10 d_4 \right] - \frac{\pi^4}{5} \left( \frac{77}{12} d_1 B_1 + 5 d_2 \right),$$

$$\delta_7 = \frac{\pi^2}{3} \left[ 4 d_1 \left( B_1 B_3 + \frac{1}{2} B_2^2 + B_4 \right) + 9 d_2 (B_1 B_2 + B_3) + \frac{15}{2} d_3 (B_1^2 + 2 B_2) + 22 d_4 B_1 + 15 d_5 \right] - \frac{\pi^4}{5} \left[ \frac{5}{6} d_1 (17 B_1^2 + 12 B_2) + \frac{57}{2} d_2 B_1 + 15 d_3 \right] + \frac{\pi^6}{7} d_1,$$

$$\delta_8 = \frac{\pi^2}{3} \left[ \frac{9}{2} d_1 (B_1 B_4 + B_2 B_3 + B_5) + 10 d_2 \left( B_1 B_3 + \frac{1}{2} B_2^2 + B_4 \right) + \frac{33}{2} d_3 (B_1 B_2 + B_3) + 12 d_4 (B_1^2 + 2 B_2) + \frac{65}{2} d_5 B_1 + 21 d_6 \right] -$$

$$- \frac{\pi^4}{5} \left[ \frac{15}{8} d_1 (7 B_1^3 + 22 B_1 B_2 + 8 B_3) + \frac{5}{12} d_2 (139 B_1^2 + 96 B_2) + \frac{319}{4} d_3 B_1 + 35 d_4 \right] + \frac{\pi^6}{7} \left( \frac{223}{20} d_1 B_1 + 7 d_2 \right).$$

\[ \blacksquare \] Bjorken (1989); Kataev, Starshenko (1995); Nesterenko, Popov (2016).

Dispersion relations “resummate” $\pi^2$–terms to all orders.
The integral representations for $\Pi(q^2)$, $R(s)$, and $D(Q^2)$ are derived in the framework of dispersive approach to QCD.

These representations merge the corresponding perturbative input with intrinsically nonperturbative constraints, which originate in the respective kinematic restrictions.

The obtained results are in a good agreement with relevant lattice data and low–energy experimental predictions.

The developed approach yields reasonable assessments of the hadronic contributions to electroweak observables.