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Dispersive approach to QCD and hadronic contributions to electroweak observables

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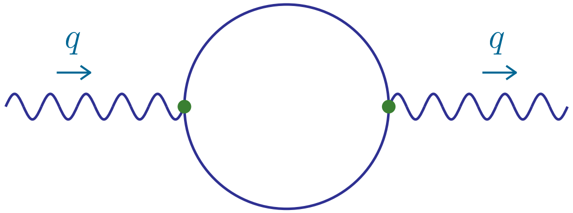
INTRODUCTION

Hadronic vacuum polarization function $\Pi(q^2)$ plays a central role in various issues of QCD and Standard Model. In particular, the theoretical description of some strong interaction processes and of hadronic contributions to electroweak observables is inherently based on $\Pi(q^2)$:

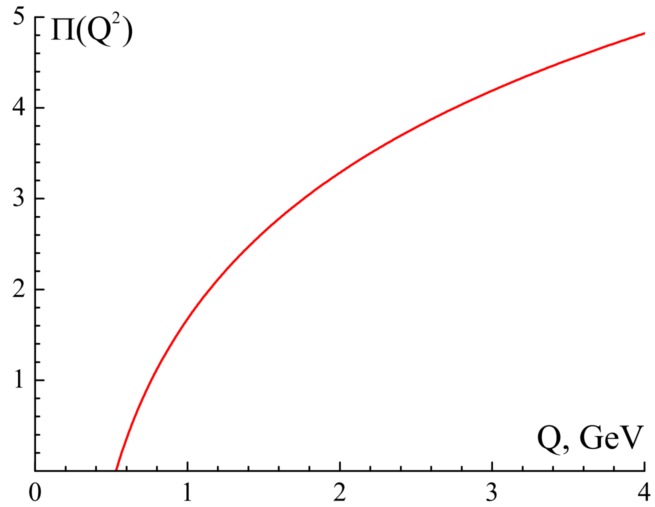
- electron–positron annihilation into hadrons
- inclusive τ lepton hadronic decay
- muon anomalous magnetic moment
- running of the electromagnetic coupling

QCD PERTURBATIVE PREDICTIONS

Leading order:

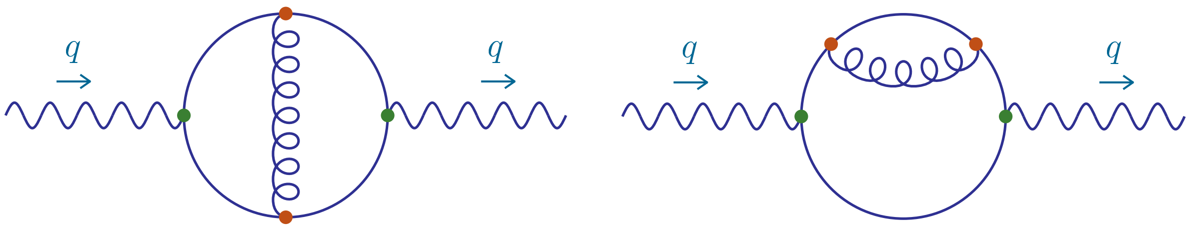


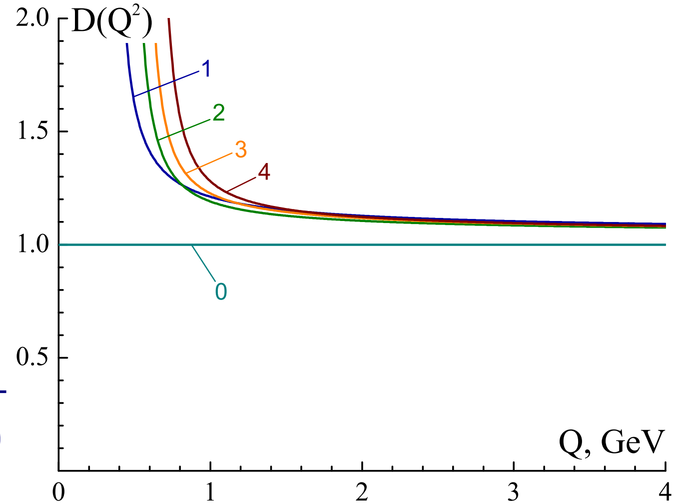
$$\Pi^{(0)}(q^2) = -\ln\left(\frac{-q^2}{-q_0^2}\right)$$



$$D^{(0)}(Q^2) = -\frac{d\Pi^{(0)}(-Q^2)}{d\ln Q^2} = 1$$

Strong corrections:



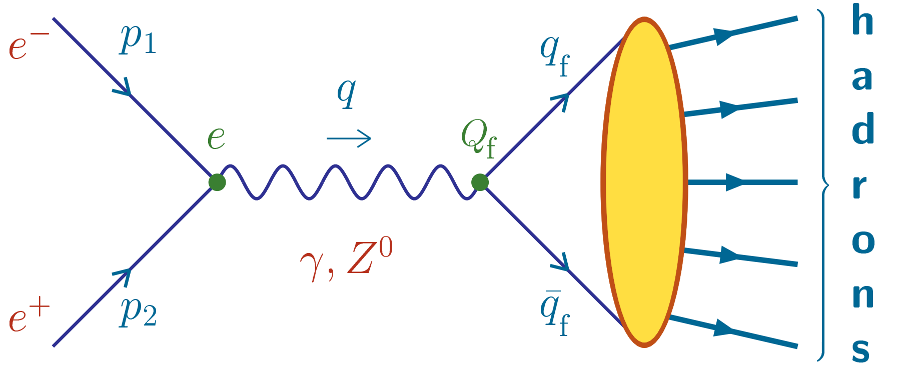


$$D^{(1)}(Q^2) = 1 + \frac{1}{\pi}\alpha_s^{(1)}(Q^2) = 1 + \frac{4}{\beta_0}\frac{1}{\ln(Q^2/\Lambda^2)}$$

$$D^{(\ell)}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j\left[\alpha_s^{(\ell)}(Q^2)\beta_0/(4\pi)\right]^j = 1 + \sum_{j=1}^{\ell} d_j\left[a_s^{(\ell)}(Q^2)\right]^j$$

GENERAL DISPERSION RELATIONS

Cross–section of $e^+e^- \to$ hadrons:



$$\sigma = 4\pi^2 \frac{2\alpha^2}{s^3} L^{\mu\nu} \Delta_{\mu\nu},$$

where $s = q^2 = (p_1 + p_2)^2 > 0,$

$$L_{\mu\nu} = \frac{1}{2}\Big[q_\mu q_\nu - g_{\mu\nu}q^2 - (p_1 - p_2)_\mu (p_1 - p_2)_\nu\Big],$$

$$\Delta_{\mu\nu} = (2\pi)^4 \sum_\Gamma \delta(p_1 + p_2 - p_\Gamma)\big\langle 0 \big| J_\mu(-q) \big| \Gamma \big\rangle \big\langle \Gamma \big| J_\nu(q) \big| 0 \big\rangle,$$

and $J_\mu = \sum_f Q_f : \bar{q}\, \gamma_\mu\, q :$ **is the electromagnetic quark current.**

Kinematic restriction: the hadronic tensor $\Delta_{\mu\nu}(q^2)$ assumes non–zero values only for $q^2 \geq 4m_\pi^2 = m^2$, since otherwise no hadron state Γ could be excited

■ **Feynman (1972); Adler (1974).**

The hadronic tensor can be represented as $\Delta_{\mu\nu} = 2\,\mathrm{Im}\,\Pi_{\mu\nu}$,

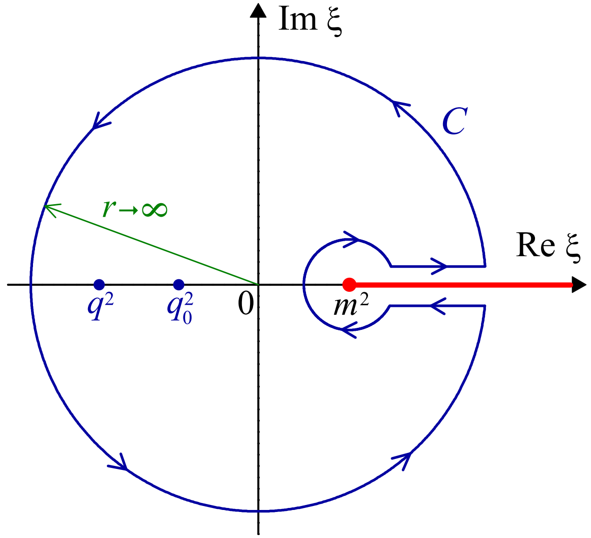
$$\Pi_{\mu\nu}(q^2) = i\int e^{iqx}\langle 0|\,T\{J_\mu(x)\,J_\nu(0)\}\,|0\rangle\,d^4x = i(q_\mu q_\nu - g_{\mu\nu}q^2)\frac{\Pi(q^2)}{12\pi^2}.$$

Kinematic restriction: $\Pi(q^2)$ **has the only cut** $q^2 \geq m^2$

Dispersion relation for $\Pi(q^2)$**:**

$$\Delta\Pi(q^2,\,q_0^2) = \frac{1}{2\pi i}(q^2 - q_0^2)\oint_C \frac{\Pi(\xi)}{(\xi - q^2)(\xi - q_0^2)}\,d\xi$$

$$= (q^2 - q_0^2)\int_{m^2}^{\infty}\frac{R(s)}{(s - q^2)(s - q_0^2)}\,ds,$$



where $\Delta\Pi(q^2,\,q_0^2) = \Pi(q^2) - \Pi(q_0^2)$ **and** $R(s)$ **denotes the measurable ratio of two cross–sections**

$$R(s) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\Big[\Pi(s + i\varepsilon) - \Pi(s - i\varepsilon)\Big] = \frac{\sigma(e^+e^- \to \text{hadrons};\, s)}{\sigma(e^+e^- \to \mu^+\mu^-;\, s)}$$

Kinematic restriction: $R(s) = 0$ **for** $s < m^2$

In general, it is also convenient to employ the so–called Adler function ($Q^2 = -q^2 > 0$)

$$D(Q^2) = -\frac{d\,\Pi(-Q^2)}{d\ln Q^2}, \qquad D(Q^2) = Q^2 \int_{m^2}^{\infty} \frac{R(s)}{(s+Q^2)^2}\, ds$$

■ Adler (1974); De Rujula, Georgi (1976); Bjorken (1989).

This dispersion relation provides a link between experimentally measurable and theoretically computable quantities.

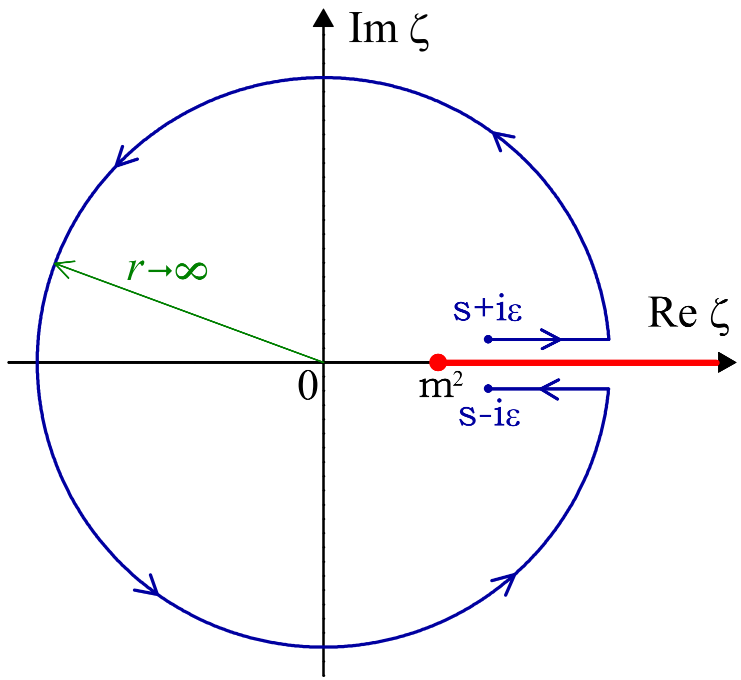
The inverse relations between the functions on hand read

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\, \frac{d\zeta}{\zeta},$$

■ Radyushkin (1982); Krasnikov, Pivovarov (1982)

$$\Delta\Pi(-Q^2, -Q_0^2) = -\int_{Q_0^2}^{Q^2} D(\sigma)\, \frac{d\sigma}{\sigma}$$

■ Pivovarov (1992).



The complete set of relations between $\Pi(q^2)$, $R(s)$, and $D(Q^2)$:

$$\Delta\Pi(q^2,\, q_0^2) = (q^2 - q_0^2) \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma - q^2)(\sigma - q_0^2)}\, d\sigma = - \int_{-q_0^2}^{-q^2} D(\sigma)\, \frac{d\sigma}{\sigma},$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[\Pi(s + i\varepsilon) - \Pi(s - i\varepsilon)\Big] = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\frac{d\zeta}{\zeta},$$

$$D(Q^2) = -\frac{d\,\Pi(-Q^2)}{d\,\ln Q^2} = Q^2 \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma + Q^2)^2}\, d\sigma.$$

Derivation of these relations requires only the location of cut of $\Pi(q^2)$ and its UV asymptotic. Neither additional approximations nor phenomenological assumptions are involved.

Nonperturbative constraints:

- $\Pi(q^2)$: **has the only cut** $q^2 \geq m^2$;
- $R(s)$: **embodies** π^2**–terms, vanishes for** $s < m^2$;
- $D(Q^2)$: **has the only cut** $Q^2 \leq -m^2$, **vanishes at** $Q^2 \to 0$.

DISPERSIVE APPROACH TO QCD

Functions on hand in terms of the common spectral density:

$$\Delta\Pi(q^2,\, q_0^2) = \Delta\Pi^{(0)}(q^2,\, q_0^2) + \int_{m^2}^{\infty} \rho(\sigma) \ln\left(\frac{\sigma - q^2}{\sigma - q_0^2}\,\frac{m^2 - q_0^2}{m^2 - q^2}\right)\frac{d\,\sigma}{\sigma},$$

$$R(s) = R^{(0)}(s) + \theta(s - m^2)\int_{s}^{\infty}\rho(\sigma)\,\frac{d\,\sigma}{\sigma},$$

$$D(Q^2) = D^{(0)}(Q^2) + \frac{Q^2}{Q^2 + m^2}\int_{m^2}^{\infty}\rho(\sigma)\,\frac{\sigma - m^2}{\sigma + Q^2}\,\frac{d\,\sigma}{\sigma},$$

$$\rho(\sigma) = \frac{1}{\pi}\frac{d}{d\,\ln\sigma}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} p(\sigma - i\varepsilon) = -\frac{d\,r(\sigma)}{d\,\ln\sigma} = \frac{1}{\pi}\,\mathrm{Im}\lim_{\varepsilon\to 0_+} d(-\sigma - i\varepsilon),$$

where $\Delta\Pi^{(0)}(q^2,\, q_0^2)$, $R^{(0)}(s)$, $D^{(0)}(Q^2)$ denote the leading–order terms and $p(q^2)$, $r(s)$, $d(Q^2)$ stand for the strong corrections

- Nesterenko, Papavassiliou (2005–2007); Nesterenko (2007–2014).

Derivation of obtained representations involves neither additional approximations nor model–dependent assumptions, with all the nonperturbative constraints being embodied.

The leading–order terms of the functions on hand read

$$\Delta\Pi^{(0)}(q^2,\, q_0^2) = 2\,\frac{\varphi - \tan\varphi}{\tan^3\varphi} - 2\,\frac{\varphi_0 - \tan\varphi_0}{\tan^3\varphi_0}, \qquad \sin^2\varphi = \frac{q^2}{m^2},$$

$$R^{(0)}(s) = \theta(s - m^2)\left(1 - \frac{m^2}{s}\right)^{3/2}, \qquad \sin^2\varphi_0 = \frac{q_0^2}{m^2},$$

$$D^{(0)}(Q^2) = 1 + \frac{3}{\xi}\Bigl[1 - \sqrt{1+\xi^{-1}}\,\sinh^{-1}\bigl(\xi^{1/2}\bigr)\Bigr], \qquad \xi = \frac{Q^2}{m^2}$$

■ Feynman (1972); Akhiezer, Berestetsky (1965).

Perturbative contribution to the spectral density:

$$\rho_{\text{pert}}(\sigma) = \frac{1}{\pi}\,\frac{d\,\mathrm{Im}\,p_{\text{pert}}(\sigma - i0_+)}{d\,\ln\sigma} = -\frac{d\,r_{\text{pert}}(\sigma)}{d\,\ln\sigma} = \frac{1}{\pi}\,\mathrm{Im}\,d_{\text{pert}}(-\sigma - i0_+),$$

one–loop level: $\rho^{(1)}_{\text{pert}}(\sigma) = (4/\beta_0)[\ln^2(\sigma/\Lambda^2) + \pi^2]^{-1}$,

higher–loops: ■ Nesterenko, Simolo (2010, 2011); Bakulev (2013); Cvetic (2015).

Note on the massless limit

In the limit $m = 0$ the obtained integral representations read

$$\Delta\Pi(q^2,\, q_0^2) = -\ln\left(\frac{-q^2}{-q_0^2}\right) + \int_0^\infty \rho(\sigma) \ln\left[\frac{1-(\sigma/q^2)}{1-(\sigma/q_0^2)}\right] \frac{d\,\sigma}{\sigma},$$

$$R(s) = 1 + \int_s^\infty \rho(\sigma)\, \frac{d\,\sigma}{\sigma},$$

$$D(Q^2) = 1 + \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2}\, d\,\sigma.$$

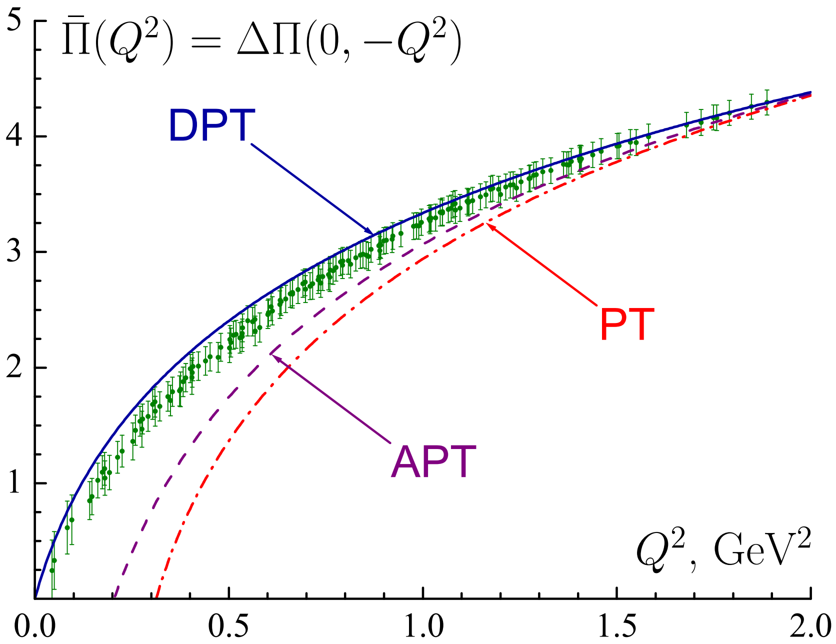
For $\rho(\sigma) = \rho_{\text{pert}}(\sigma)$ two highlighted massless equations become identical to those of the APT ■ Shirkov, Solovtsov, Milton (1997–2007).

But it is essential to keep the threshold m nonvanishing:

- **massless limit loses some of nonperturbative constraints**
- **effects due to $m \neq 0$ become substantial at low energies**

HADRONIC VACUUM POLARIZATION FUNCTION

Comparison of obtained results with lattice simulation data:



Both PT and APT fail to describe $\Pi(q^2)$ at low energies:

PT: $\Pi(q^2)$ **possesses infrared unphysical singularities**

APT: $\Pi(q^2)$ **diverges in IR limit**

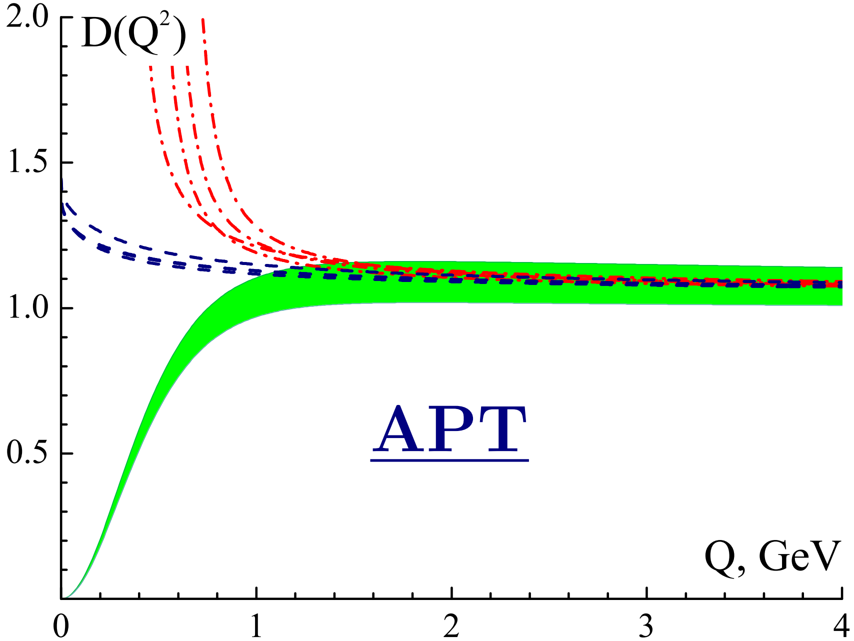
■ Della Morte, Jager, Juttner, Wittig (2011–2015); Nesterenko (2014, 2015).

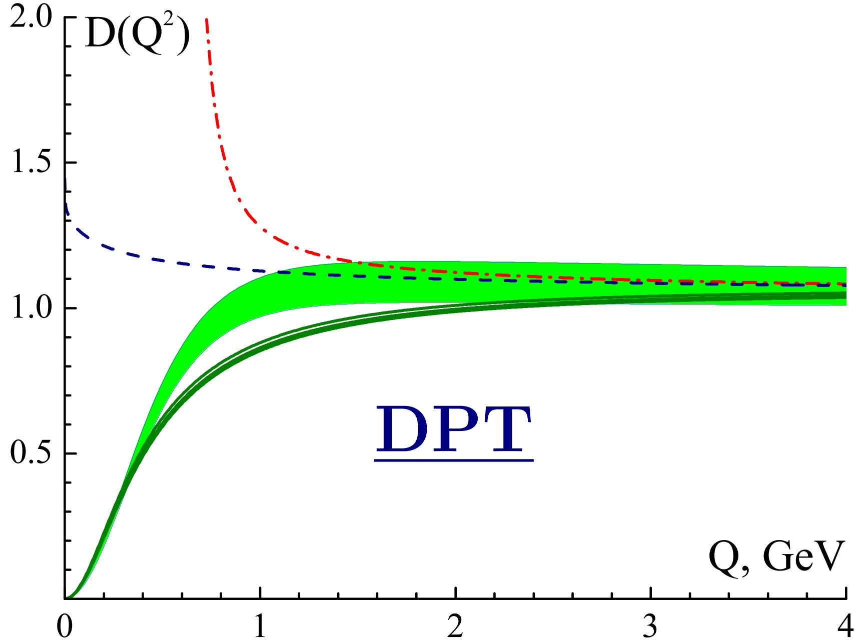
	unphysical singularities	agreement with lattice
PT	contains	disagrees
APT	diverges in IR	disagrees
DPT	free	agrees

ADLER FUNCTION

massless limit ($m = 0$)

realistic case ($m \neq 0$)



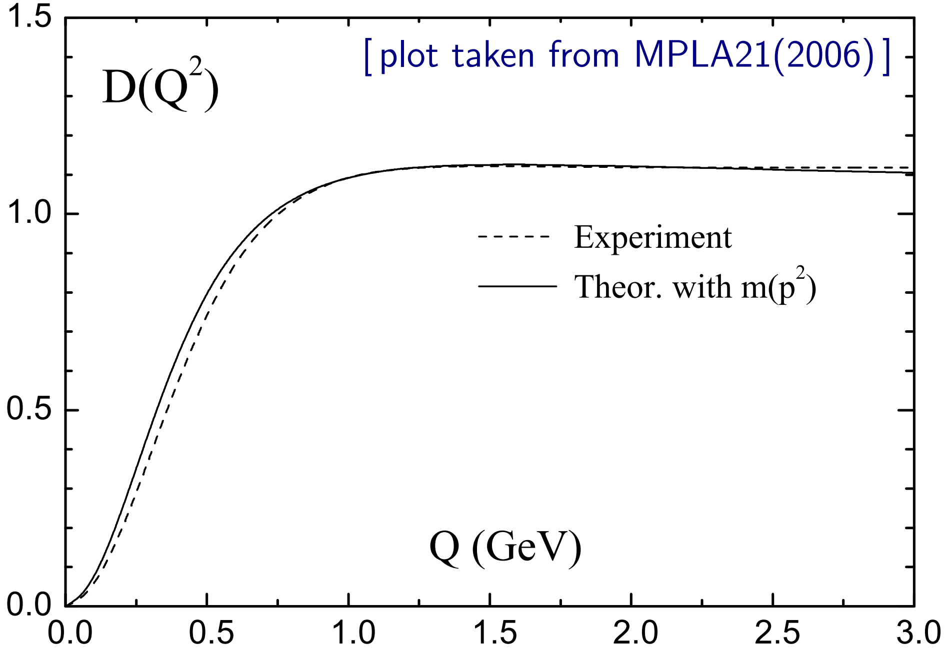


- Nesterenko, Papavassiliou (2006); Nesterenko (2015, 2016).

	unphysical singularities	agreement with data
PT	contains	disagrees
APT	free	disagrees
DPT	free	agrees

Some attempts to improve IR behavior of $D(Q^2)$ within APT:

APT + relativistic quark mass threshold resummation:

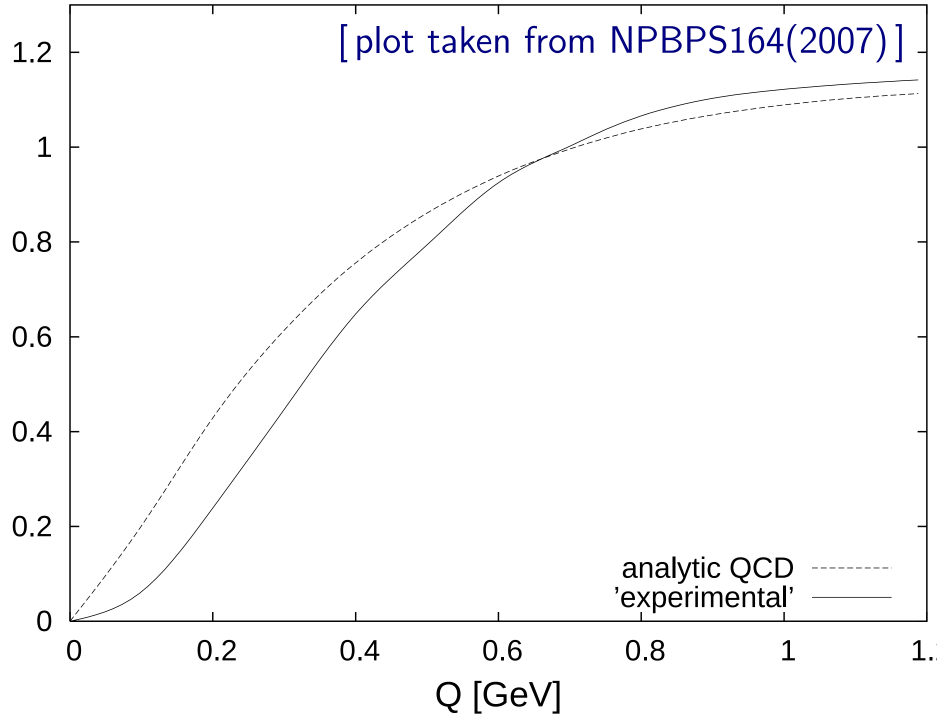


large light quark masses

$2m_{u,d} \simeq 520\,\mathrm{MeV} \simeq 4m_\pi$

■ Milton, Solovtsov, Solovtsova (2001–2006)

APT + vector meson dominance assumption:



VMD NW approximation and cut–off at $M_0 \simeq 740\,\mathrm{MeV}$

■ Cvetic et al. (2005–2015)

MUON ANOMALOUS MAGNETIC MOMENT

The theoretical description of $a_\mu = (g_\mu - 2)/2$ is a long–standing challenging issue of the elementary particle physics.

Experiment: $a_\mu^{\text{exp}} = (11659208.9 \pm 6.3) \times 10^{-10}$ $(0.54\,\text{ppm})$

- Muon (g-2) Collaboration (2006); Roberts (2010).

Theory: $a_\mu^{\text{theor}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{HLO}} + a_\mu^{\text{HHO}} + a_\mu^{\text{Hlbl}}$

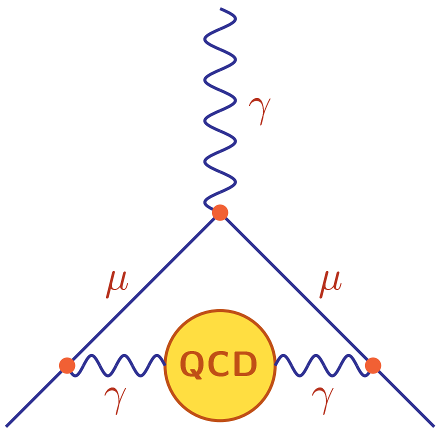
$a_\mu^{\text{QED}} = (11658471.8951 \pm 0.0080) \times 10^{-10}$ Aoyama, Hayakawa, Kinoshita, Nio (2012)

$a_\mu^{\text{EW}} = (15.36 \pm 0.10) \times 10^{-10}$ Gnendiger, Stockinger, Stockinger–Kim (2013)

$a_\mu^{\text{HHO}} = (-9.84 \pm 0.07) \times 10^{-10}$ Hagiwara, Liao, Martin, Nomura, Teubner (2011)

$a_\mu^{\text{Hlbl}} = (11.6 \pm 4.0) \times 10^{-10}$ Nyffeler (2014).

The uncertainty of theoretical estimation of a_μ is mainly dominated by the leading–order hadronic contribution a_μ^{HLO} →



The latter involves the integration of $\Pi(q^2)$ over low energies:

$$a_\mu^{\rm HLO} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty f\left(\frac{\zeta}{4m_\mu^2}\right)\bar{\Pi}(\zeta)\,\frac{d\zeta}{4m_\mu^2}, \qquad f(x) = \frac{1}{x^3}\frac{y^5(x)}{1-y(x)},$$

where $y(x) = x\left(\sqrt{1+x^{-1}} - 1\right)$ ■ Lautrup, Peterman, de Rafael (1972).

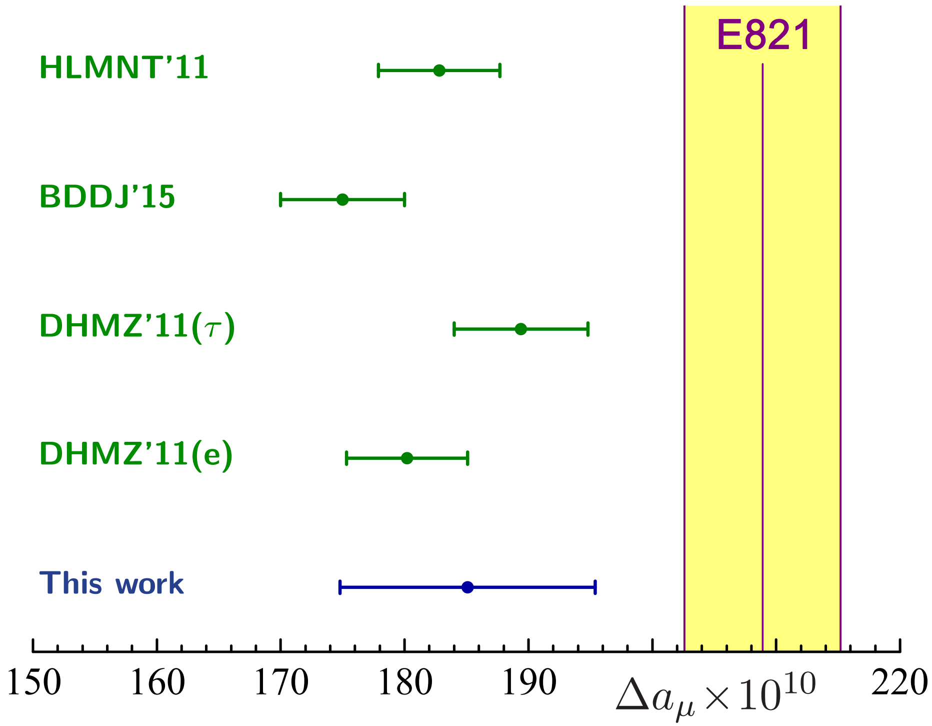
Dispersive approach enables one to evaluate $a_\mu^{\rm HLO}$ without invoking experimental data on $R(s)$:

$$a_\mu^{\rm HLO} = (696.1 \pm 9.5) \times 10^{-10}.$$

This result agrees fairly well with recent assessments of $a_\mu^{\rm HLO}$.

The complete SM prediction

$$a_\mu = (11659185.1 \pm 10.3) \times 10^{-10}$$



$[\Delta a_\mu = a_\mu - a_0, \quad a_0 = 11659 \times 10^{-7}]$

differs from $a_\mu^{\rm exp}$ by two standard deviations ■ Nesterenko (2015).

ELECTROMAGNETIC FINE STRUCTURE CONSTANT

The electromagnetic running coupling $\alpha_{\rm em}(q^2)$ plays a central role in a variety of issues of precision particle physics:

$$\alpha_{\rm em}(q^2) = \frac{\alpha}{1 - \Delta\alpha_{\rm lep}(q^2) - \Delta\alpha_{\rm had}(q^2)}$$

with $\alpha = e^2/(4\pi) \simeq 1/137.036$ being the fine structure constant.

Leptonic contribution to $\alpha_{\rm em}(q^2)$ can be calculated within perturbation theory: $\Delta\alpha_{\rm lep}(M_{\rm Z}^2) = (314.979 \pm 0.002) \times 10^{-4}$ ■ Sturm (2013).

However, the respective hadronic contribution involves the integration over the low–energy range

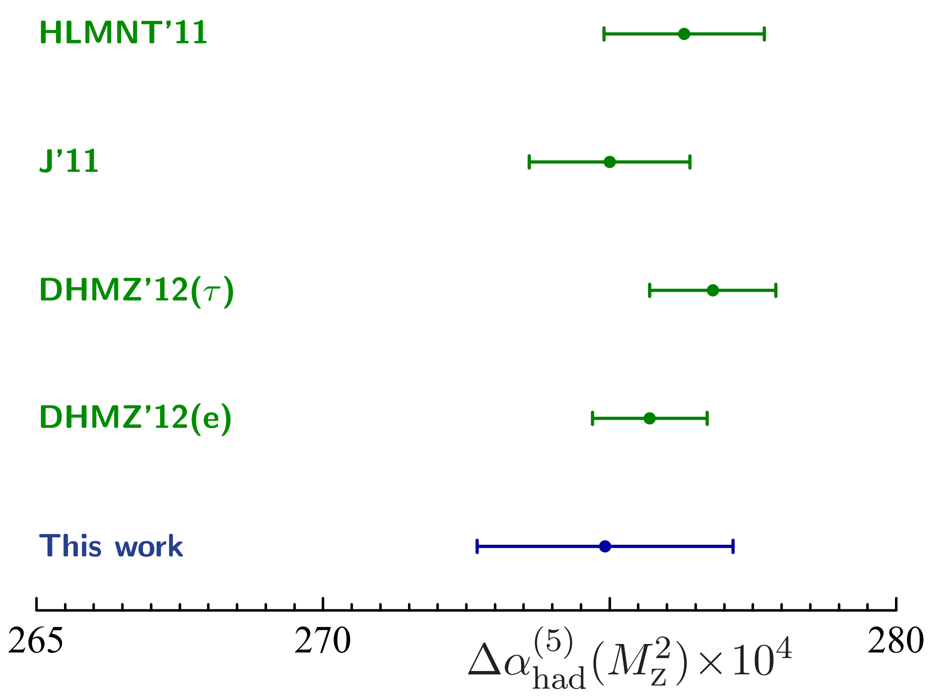
$$\Delta\alpha_{\rm had}(M_{\rm Z}^2) = -\frac{\alpha}{3\pi} M_{\rm Z}^2 \int_{m^2}^{\infty} \frac{R(s)}{s - M_{\rm Z}^2} \frac{d\,s}{s}$$

and constitutes the prevalent source of uncertainty of $\alpha_{\rm em}(M_{\rm Z}^2)$.

As usual, the top quark contribution to $\alpha_{\text{em}}(q^2)$ is taken into account separately:

$$\Delta\alpha^{\text{top}}_{\text{had}}(M_Z^2) = (-0.70 \pm 0.05) \times 10^{-4}$$

■ Kuhn, Steinhauser (1998).



The evaluation of $\Delta\alpha^{(5)}_{\text{had}}(M_Z^2)$ in the framework of dispersive approach leads to

$$\Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = (274.9 \pm 2.2) \times 10^{-4}.$$

The obtained assessment appears to be in a good agreement with recent estimations of $\Delta\alpha^{(5)}_{\text{had}}(M_Z^2)$ and eventually yields

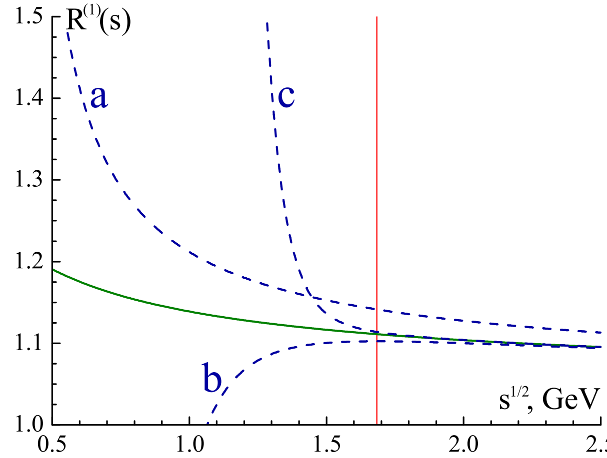
$$\alpha^{-1}_{\text{em}}(M_Z^2) = 128.962 \pm 0.030$$

■ Nesterenko (2015).

Perturbative approximation of R–ratio:

At high energies one commonly re–expands the R–ratio:

$$R^{(\ell)}(s) = 1 + \sum_{j=1}^{\ell} r_j \left[a_s^{(\ell)}(|s|)\right]^j, \qquad r_j = d_j - \delta_j, \qquad B_j = \frac{\beta_j}{\beta_0^{j+1}},$$



$$\delta_1 = 0, \qquad \delta_2 = 0, \qquad \delta_3 = \frac{\pi^2}{3} d_1, \qquad \delta_4 = \frac{\pi^2}{3}\left(\frac{5}{2} d_1 B_1 + 3 d_2\right),$$

$$\delta_5 = \frac{\pi^2}{3}\left[\frac{3}{2} d_1 \left(B_1^2 + 2B_2\right) + 7 d_2 B_1 + 6 d_3\right] - \frac{\pi^4}{5} d_1,$$

$$\delta_6 = \frac{\pi^2}{3}\left[\frac{7}{2} d_1 \left(B_1 B_2 + B_3\right) + 4 d_2 \left(B_1^2 + 2B_2\right) + \frac{27}{2} d_3 B_1 + 10 d_4\right] - \frac{\pi^4}{5}\left(\frac{77}{12} d_1 B_1 + 5 d_2\right),$$

$$\delta_7 = \frac{\pi^2}{3}\left[4 d_1 \left(B_1 B_3 + \frac{1}{2} B_2^2 + B_4\right) + 9 d_2 \left(B_1 B_2 + B_3\right) + \frac{15}{2} d_3 \left(B_1^2 + 2B_2\right) + 22 d_4 B_1 + 15 d_5\right] - \frac{\pi^4}{5}\left[\frac{5}{6} d_1 \left(17 B_1^2 + 12 B_2\right) + \frac{57}{2} d_2 B_1 + 15 d_3\right] + \frac{\pi^6}{7} d_1,$$

$$\delta_8 = \frac{\pi^2}{3}\left[\frac{9}{2} d_1 \left(B_1 B_4 + B_2 B_3 + B_5\right) + 10 d_2 \left(B_1 B_3 + \frac{1}{2} B_2^2 + B_4\right) + \frac{33}{2} d_3 \left(B_1 B_2 + B_3\right) + 12 d_4 \left(B_1^2 + 2B_2\right) + \frac{65}{2} d_5 B_1 + 21 d_6\right] -$$

$$- \frac{\pi^4}{5}\left[\frac{15}{8} d_1 \left(7 B_1^3 + 22 B_1 B_2 + 8 B_3\right) + \frac{5}{12} d_2 \left(139 B_1^2 + 96 B_2\right) + \frac{319}{4} d_3 B_1 + 35 d_4\right] + \frac{\pi^6}{7}\left(\frac{223}{20} d_1 B_1 + 7 d_2\right).$$

- **Bjorken (1989); Kataev, Starshenko (1995); Nesterenko, Popov (2016).**

Dispersion relations “resummate” π^2–terms to all orders.

SUMMARY

- **The integral representations for $\Pi(q^2)$, $R(s)$, and $D(Q^2)$ are derived in the framework of dispersive approach to QCD**
- **These representations merge the corresponding perturbative input with intrinsically nonperturbative constraints, which originate in the respective kinematic restrictions**
- **The obtained results are in a good agreement with relevant lattice data and low–energy experimental predictions**
- **The developed approach yields reasonable assessments of the hadronic contributions to electroweak observables**