Chiral-Scale Perturbation Theory $\chi PT_\sigma$
and the Renormalization Group

Based on work with R.J. Crewther (Adelaide)
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1. Can we make sense of the lowest $0^{++}$ QCD resonance $f_0(500)$?

   The mass and width have been precisely determined

   $$m_{f_0} = 441^{+16}_{-8} \text{ MeV} \quad \Gamma_{f_0} = 544^{+18}_{-25} \text{ MeV}$$

   [Caprini, Colangelo & Leutwyler (06); García-Martin et al. (11)]

   but what is the quark content: $q\bar{q}, q\bar{q}q\bar{q} \ldots$ (something else)? [Peláez (15)]

Related puzzles:

- $m_{f_0} \lesssim m_K \Rightarrow$ poor $\chi^{PT}_3$ convergence in $0^{++}$ channel [Meißner (91)]

- role of $f_0(500)$ in $\Delta I=1/2$ rule for non-leptonic kaon decays?
A variety of non-perturbative approaches indicate that $\alpha_s$ “freezes” in the IR to a constant value $\alpha_{\text{IR}}$ for small values of $N_f \leq 3$.

Main problems:
- scheme / definition dependence of $\alpha_s$ ⇒ the existence of $\alpha_{\text{IR}}$ for small $N_f$ is not entirely settled
  [For a catalog of existing results see: Deur, Brodsky & de Teramond (16)]
- when is $\alpha_{\text{IR}}$ physical?

Q: If $\alpha_{\text{IR}}$ exists, what are the implications for low energy physics, especially $\chi$PT?

[Diagram from RJ Crewther]
Decouple t, b, c $\Rightarrow$ relevance of scale (or dilatation) invariance determined by trace anomaly of $N_f = 3$ theory:

$$\partial^\mu \{ x^\alpha \theta_{\alpha\mu} \} = \theta^\mu = \frac{\beta(\alpha_s)}{4\alpha_s} G^2 + (1 + \gamma_m) \sum_{q=u,d,s} m_q \bar{q}q$$

IR fixed point: $\beta(\alpha_{IR}) = 0$

$\chi_{PT}\sigma = \text{asymptotic expansion in}$

$$\alpha_s \lesssim \alpha_{IR} \quad \text{and} \quad m_{u,d,s} \sim 0$$

about scale-dependent $|\text{vac}\rangle$

$\Rightarrow$ 9 NG bosons $\pi$, $K$, $\eta$, $\sigma$

In limit $\theta^\mu \rightarrow 0$ dilaton $\sigma$ couples to masses in non-NG sector [Gell-Mann (62)]

$$\langle \text{vac}|\theta_{\mu\nu}|\sigma(q)\rangle = \frac{1}{3} (q_\mu q_\nu - q^2 g_{\mu\nu}) F_\sigma \quad \Rightarrow \quad \text{e.g.} \ F_\sigma g_{\sigma NN} = M_N$$
1 | Chiral-scale perturbation theory

Key results

1. Associate dilaton $\sigma$ with $f_0(500)$: the $O(m_K)$ mass understood due to $\sim m_s\bar{s}s$ term in $\theta^{\mu}_{\mu}$ while the width is explained by $O(p^2)$ relation:

$$F_\sigma g_{\sigma\pi\pi} \simeq -m_\sigma^2$$

[Ellis (70); Crewther (70)]

NB. In chiral-scale limit $\theta^{\mu}_{\mu} \rightarrow 0$ get massless $\pi$, $K$, $\eta$ and stable $\sigma/f_0$

(phrase space = 0) $\Rightarrow$ OK to use local field $\sigma$ in $\mathcal{L}_{\text{eff}}$

2. Promotion of $f_0(500)$ to NG sector $\Rightarrow$ scale separation restored

Expect improved convergence in $0^{++}$ channels ($O(p^4)$ analysis in progress)
1 | Chiral-scale perturbation theory

3 Lowest order $O(p^2) \chi PT_\sigma$ explains the $\Delta I=1/2$ rule for kaon decays

4 Relation between $\sigma\gamma\gamma$ coupling and electromagnetic trace anomaly

$$\tilde{\theta}_\mu^\mu = \theta_\mu^\mu + (R\alpha/6\pi)F^2$$

[ Crewther (72); Chanowitz & Ellis (72) ]

Fit data on $\pi\pi \rightarrow \gamma\gamma$, $K_S \rightarrow \gamma\gamma$, and $\sigma \rightarrow \gamma\gamma \Rightarrow$ bounds on asymptotic Drell-Yan ratio at $\alpha_s = \alpha_{IR}$:

$$2.5 \lesssim R_{IR} \lesssim 5$$

This talk: some formal aspects concerning hidden scale invariance & a proposal to test $\chi PT_\sigma$ on the lattice
The idea that particles can have mass in the scale invariant limit $\theta_\mu^\mu \to 0$ seems counter-intuitive, especially for gauge theories like QCD.

Usually one considers fixed points in the Wigner-Weyl mode, as in e.g. the conformal window: two-loop $\beta = -b_1 \alpha_s^2 - b_2 \alpha_s^3 \Rightarrow$ IR fixed point

$$\alpha_{CBZ} = -\frac{b_2}{b_1} \quad \text{if} \quad b_2 < 0, \quad \text{i.e. for} \quad 9 \leq N_f \leq 16$$

[Caswell (74); Banks & Zaks (82)]

But ... if IR fixed point is non-perturbative dimensional transmutation may cause $|\text{vac}\rangle$ to break scale invariance $\Rightarrow$ symmetry is hidden

General theory of the NG mode for scale and conformal invariance established long ago [Salam & Strathdee (69); Zumino (70); Carruthers (71)]

Let us apply these ideas to $N_f = 3$ QCD ...
Recall **chiral** Ward identities for e.g. $J_{\mu 5}^a = \bar{q} \gamma_{\mu} \gamma_5 T^a q$ and its divergence:

$$\partial^\mu J_{\mu 5}^a = D_5^a = 2mi\bar{q}\gamma_5 T^a q$$

Let $\mathcal{O}$ be any operator which is not chiral invariant and consider amplitude

$$\mathcal{A}_{\mu 5}^a(q) = \int d^4x \ e^{iq \cdot x} T\langle \text{vac} | J_{\mu 5}^a(x) \mathcal{O}(0) | \text{vac} \rangle$$

Consider $q = 0$ limit of the chiral Ward identities

$$q^\mu \mathcal{A}_{\mu 5}^a = \text{commutators} + \int_x e^{iq \cdot x} D_5^a \text{ insertion}$$

$$\Rightarrow \quad 0 = \langle \text{vac} | \delta_5^a \mathcal{O} | \text{vac} \rangle + i \int d^4x \ T\langle \text{vac} | D_5^a(x) \mathcal{O}(0) | \text{vac} \rangle$$

The term $T\langle D_5 \mathcal{O} \rangle$ does not vanish in limit $D_5^a \to 0$ if $\exists \ q = 0$ pion pole

$$\Rightarrow \quad \langle \delta \mathcal{O} \rangle \to iF_\pi \langle \text{vac} | \mathcal{O} | \pi \rangle \neq 0, \quad m \to 0, \quad F_\pi \simeq 93 \text{ MeV}$$

So e.g. $\langle \bar{q}q \rangle \neq 0$ **hides** the symmetry; $M_B \neq 0$ and mesons not P-doubled
3 | Chiral and scale condensates

For scale transformations, the relevant divergence is

\[
\partial^\mu \{ x^\alpha \theta_{\alpha \mu} \} = \theta^\mu = \frac{\beta (\alpha_s)}{4 \alpha_s} G^2 + (1 + \gamma_m (\alpha_s)) \sum_{q=u,d,s} m_q \bar{q} q
\]

so the \( q=0 \) Ward identities are the Callan-Symanzik equations

\[
\left\{ \mu \frac{\partial}{\partial \mu} + \beta (\alpha_s) \frac{\partial}{\partial \alpha_s} + \gamma O (\alpha_s) + (1 + \gamma_m (\alpha_s)) \sum_{u,d,s} m_q \frac{\partial}{\partial m_q} \right\} \langle O \rangle_{vac} = 0
\]

The terms \( \beta \partial / \partial \alpha_s \) and \( (1 + \gamma_m) \sum_q m_q \partial / \partial m_q \) correspond to \( q = 0 \) insertion of \( \theta^\mu \):

\[
\left\{ \mu \frac{\partial}{\partial \mu} + \gamma O (\alpha_s) \right\} \langle vac | O(0) | vac \rangle = i \int d^4 x \ T \langle vac | \theta^\mu (x) O(0) | vac \rangle \quad (*)
\]

What happens to RH side of \( (*) \) if \( \alpha_{IR} \) exists and limit \( \theta^\mu \rightarrow 0 \) is taken?
The standard procedure is to set all amplitudes involving $\int \theta^\mu_\mu$ to zero

$\Rightarrow$ no NG mechanism and scale invariance realised in WW mode

$$\left\{ \mu \frac{\partial}{\partial \mu} + \gamma_\mathcal{O}(\alpha_{WW}) \right\} \langle \text{vac} | \mathcal{O}(0) | \text{vac} \rangle_{WW} = 0$$

$\therefore$ theory at a WW fixed point $\alpha_{WW}$ is manifestly scale invariant

$\Rightarrow$ Green’s functions scale according to power-laws $\sim \mu^{-\gamma_\mathcal{O}(\alpha_{WW})}$

$\Rightarrow$ No mass gap and dimensional transmutation does not occur. In particular, fermions cannot condense at $\alpha_{WW}$. Must assume that fermion condensation is consequence of explicit scale symmetry breaking, e.g. in walking gauge theories [Appelquist et al (97)]
3 | Chiral and scale condensates

Nambu-Goldstone (NG) mode

\[
\left\{ \mu \frac{\partial}{\partial \mu} + \gamma_\mathcal{O}(\alpha_s) \right\} \langle \text{vac} | \mathcal{O}(0) | \text{vac} \rangle = i \int d^4 x \, \mathcal{T} \langle \text{vac} | \theta_\mu^\mu(x) \mathcal{O}(0) | \text{vac} \rangle \quad (\star)
\]

In this case the RH side of (\star) **does not vanish** at \( \alpha_{\text{IR}} \) as \( \theta_\mu^\mu \to 0 \)

Occurs if \( \exists \) dilaton \( \sigma \) where \( \langle \text{vac} | \theta_\mu^\mu | \sigma \rangle = -m_\sigma^2 F_\sigma \), so key result follows:

\[
\left\{ \mu \frac{\partial}{\partial \mu} + \gamma_\mathcal{O}(\alpha_s) \right\} \langle \text{vac} | \mathcal{O}(0) | \text{vac} \rangle_{\text{NG}} \to -F_\sigma \langle \sigma(q = 0) | \mathcal{O}(0) | \text{vac} \rangle , \quad \theta_\mu^\mu \to 0
\]

Then \( \langle \mathcal{O} \rangle_{\text{vac}} \) is a **scale condensate** (e.g. \( \langle \bar{q} q \rangle_{\text{vac}} \neq 0 \)) and the vacuum breaks scale invariance.

**NB.**
- all particles except \( \pi, K, \eta, \sigma \) can **remain massive** at \( \alpha_{\text{IR}} \)
- Green’s functions do not exhibit power-law behaviour
- \( |\text{vac}\rangle \) carries \( \mu \) dependence of theory and “knows” about dimensional transmutation; driven by strength of interactions in \( \mathcal{H} \)
If $\alpha_{IR}$ exists in $N_f = 3$ QCD, it is **necessary** to extend $\chi$PT$_3$ to account for non-linearly realised scale invariance [Salam et al (69) & (70); Ellis (70)].

In **physical region** $0 < \alpha_s < \alpha_{IR}$ consider the combined limit

$$m_{u,d,s} \sim 0 \quad \text{and} \quad \alpha_s \lesssim \alpha_{IR}$$

Effective chiral-scale Lagrangian composed of terms with differing scale dimension

$$\mathcal{L}_{\chi\text{PT}_\sigma} = \mathcal{L}_{\text{inv}}^{d=4} + \mathcal{L}_{\text{anom}}^{d>4} + \mathcal{L}_{\text{mass}}^{d<4}$$

Operator dimensions satisfy

$$d_{\text{inv}} = 4 \quad \text{and} \quad 1 \leq 3 - \gamma_{m}(\alpha_{IR}) < 4 \quad [\text{Wilson (69)}]$$

Callan-Symanzik equations for QCD amplitudes implies that at LO

$$\beta(\alpha_s) \simeq \beta'(\alpha_{IR})(\alpha_s - \alpha_{IR}) \quad \Rightarrow \quad d_{\text{anom}} = 4 + \beta' > 4$$
4 | Chiral-scale effective Lagrangian

Want $O(p^2)$ expression for $\mathcal{L}_{\chi\text{PT}_\sigma} = \mathcal{L}_{\text{inv}}^{d=4} + \mathcal{L}_{\text{anom}}^{d>4} + \mathcal{L}_{\text{mass}}^{d<4}$

Use covariant field $e^{\sigma/F_\sigma}$ with scale dimension 1 for

$$\sigma \to \sigma - \frac{1}{4} F_\sigma \log \left| \det(\partial x'/\partial x) \right|$$

⇒ use powers of $e^{\sigma/F_\sigma}$ to adjust dimensions of operators like

$$\mathcal{K}[U, U^\dagger] = \frac{1}{4} F_\pi^2 \text{Tr}(\partial_\mu U \partial^\mu U^\dagger)$$

$$\mathcal{K}_\sigma = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma$$

Then strong interactions at lowest order given by

$$\mathcal{L}_{\text{inv}}^{d=4} = \{c_1 \mathcal{K} + c_2 \mathcal{K}_\sigma + c_3 e^{2\sigma/F_\sigma}\} e^{2\sigma/F_\sigma}$$

$$\mathcal{L}_{\text{anom}}^{d>4} = \{(1 - c_1) \mathcal{K} + (1 - c_2) \mathcal{K}_\sigma + c_4 e^{2\sigma/F_\sigma}\} e^{(2+\beta')\sigma/F_\sigma}$$

$$\mathcal{L}_{\text{mass}}^{d<4} = \text{Tr}(MU^\dagger + UM^\dagger) e^{(3-\gamma_m)\sigma/F_\sigma}$$

New set of low energy constants $c_1, \ldots, 4, \quad \beta'(\alpha_{\text{IR}}), \quad \gamma_m(\alpha_{\text{IR}})$
Can we constrain the low energy constants?

\[
\mathcal{L}_{\text{inv}}^{d=4} = \left\{ c_1 \mathcal{K} + c_2 \mathcal{K}_\sigma + c_3 e^{2\sigma/F_\sigma} \right\} e^{2\sigma/F_\sigma}
\]

\[
\mathcal{L}_{\text{anom}}^{d>4} = \left\{ (1 - c_1) \mathcal{K} + (1 - c_2) \mathcal{K}_\sigma + c_4 e^{2\sigma/F_\sigma} \right\} e^{(2+\beta')\sigma/F_\sigma}
\]

\[
\mathcal{L}_{\text{mass}}^{d<4} = \text{Tr}(MU^\dagger + UM^\dagger) e^{(3-\gamma_m)\sigma/F_\sigma}
\]

For consistency, \(c_3\) and \(c_4\) are \(O(M)\). Vacuum stability (no tadpoles) req:

\[
4c_3 + (4 + \beta')c_4 = -(3 - \gamma_m)F_\pi^2 (m_K^2 + \frac{1}{2}m_\pi^2)
\]

Recently, we found a simplification: for

\[
\theta_{\mu\mu}^{\mu} \big|_{\text{eff}} = \sum_d (d - 4) \mathcal{L}_d = : \beta' \mathcal{L}_{\text{anom}}^{d>4} - (1 + \gamma_m) \mathcal{L}_{\text{mass}}^{d<4} :
\]

to vanish in chiral-scale limit, \(c_1\) and \(c_2\) are both \(1 + O(M)\)
5 | Lowest order results

Mass formula

\[ m_\sigma^2 F_\sigma^2 = F_\pi^2 (m_K^2 + \frac{1}{2} m_\pi^2) (3 - \gamma_m)(1 + \gamma_m) - \beta' (4 + \beta') c_4 \]

Factor \( d(d - 4) \) in each term: \( d_{\text{mass}} = 3 - \gamma_m \) and \( d_{\text{anom}} = 4 + \beta' \)

Sign of \( c_4 \) not determined by theory: depends on \( m_K/m_\sigma \):

\[
\Rightarrow \begin{cases} 
  c_4 > 0 \text{ unless } d_{\text{mass}} \text{ close to 0 or 4} \\
  \text{for (say) } d_{\text{mass}} = 2 \text{ and } \beta' = 1: c_4 \approx \frac{3}{4} m_K^2 F_\pi^2 
\end{cases}
\]

Width

\[ \mathcal{L}_{\pi\pi} = \left\{ 2|\partial \pi|^2 - (3 - \gamma_m) m_\pi^2 |\pi|^2 \right\} \sigma/(2F_\sigma) \]

\( \Rightarrow \) small effect on \( \pi\pi \) scattering in \( SU(2) \times SU(2) \) limit with \( \partial = O(m_\pi) \)

Vertex for on shell amplitude \( g_{\sigma\pi\pi} = -\left( m_\sigma^2 + (1 - \gamma_m)m_\pi^2 \right)/F_\sigma \)

Fit to data on \( f_0(500) \) width \( \Rightarrow \) \( |F_\sigma| \approx 50 \text{ MeV} \)
Let \( J_{ij} = (U \partial_\mu U^\dagger)_{ij} \) and adjust dimensions of \( \chiPT_3 \) operators:

\[
q_8 = J_{13} J_{21} - J_{23} J_{11} \\
q_{27} = J_{13} J_{21} + \frac{3}{2} J_{23} J_{11}
\]

and

\[
q_{mw} = \text{Tr}(\lambda_6 - i\lambda_7)(g_M M U^\dagger + \bar{g}_M U M^\dagger)
\]

Vacuum alignment of the resulting effective Lagrangian [Crewther (86)]

\[
\mathcal{L}_w = g_8 q_8 e^{(2-\gamma_8)\sigma/F_\sigma} + g_{27} q_{27} e^{(2-\gamma_{27})\sigma/F_\sigma} + q_{mw} e^{(3-\gamma_{mw})\sigma/F_\sigma} + \text{h.c.}
\]

\( \Rightarrow K_S \sigma \) vertex from mismatch of mass and weak anomalous dimensions

\( g_8 \) and \( g_{27} \) allowed to have similar magnitude

Magnitude of \( g_{K_S \sigma} \) needed is consistent with \( K_S \rightarrow \gamma \gamma \) and \( \gamma \gamma \rightarrow \pi \pi \)
7 | Testing $\chi PT_\sigma$ on the lattice

The signal on the lattice for an IR fixed point in NG mode is freezing of the running coupling outside the conformal window [e.g. Horseley et al. (13)]

Scheme/definition dependence of $\alpha_s$ ⇒ hard to test $\chi PT_\sigma$ conclusively

Instead, revive lattice proposal to measure $K \rightarrow \pi$ on shell [Crewther (86)] and consider low-energy theorems for $K \rightarrow \pi\pi$:

1. Vacuum alignment ⇒ $K \rightarrow |\text{vac}\rangle = O(p^4)$ (still true in $\chi PT_\sigma$)
   ⇒ try lattice determination for $\langle \pi | L_{\text{eff}} | K \rangle$ with both $K$ and $\pi$ on shell

2. In $\chi PT_\sigma$ we have the following low-energy theorem
   \[
   \{K \rightarrow \pi\pi\} = \{K \rightarrow \pi \text{ on shell}\} + \{\sigma/f_0 \text{ pole}\} + O(p^4)
   \]

3. The on shell value of $\langle \pi | L_{\text{eff}} | K \rangle$ is not affected by $\sigma/f_0$ pole
   ⇒ a lattice determination of $K \rightarrow \pi$ measures $|g_8/g_{27}|$ directly which is what the $\Delta l=1/2$ puzzle has always been about