

# Roy-Steiner-equation analysis of pion-nucleon scattering

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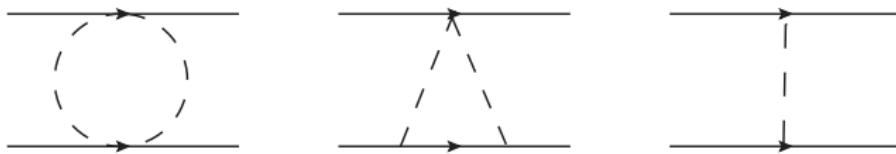
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# Motivation: Why $\pi N$ scattering?

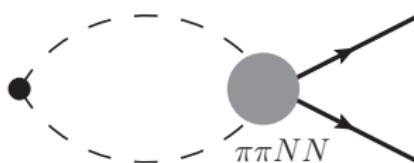
- **Low energies:** test chiral dynamics in the baryon sector  
⇒ low-energy theorems e.g. for the scattering lengths
- **Higher energies:** resonances, baryon spectrum
- **Input for  $NN$  scattering:** LECs  $c_i$ ,  $\pi NN$  coupling



- **Crossed channel  $\pi\pi \rightarrow \bar{N}N$ :** nucleon form factors

⇒ probe the structure of the nucleon

- ▷ scalar form factors (S-wave)
- ▷ electromagnetic form factors (P-waves)
- ▷ generalized PDF (D-waves)



# The pion-nucleon $\sigma$ -term

Scalar form factor of the nucleon:

$$\sigma(t) = \langle N(p') | \hat{m} (\bar{u}u + \bar{d}d) | N(p) \rangle \quad t = (p' - p)^2 \quad \sigma_{\pi N} = \sigma(0)$$

- $\sigma_{\pi N}$  measures the **light-quark contribution** to the nucleon mass
- Unfortunately, no direct experimental access to it
- Only very recent precise lattice results
- Linked to  $\pi N$  via the **Cheng-Dashen theorem**

[Cheng, Dashen 1971]

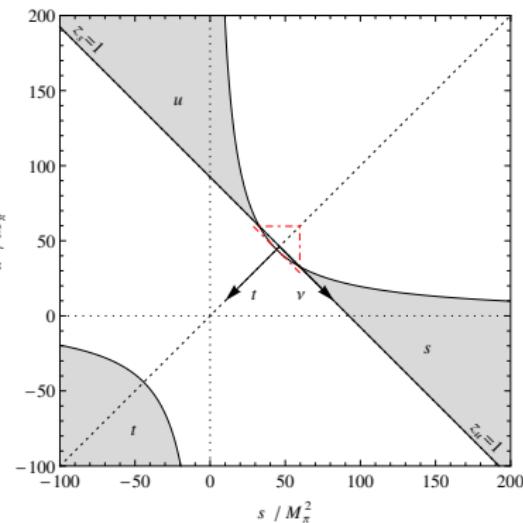
$$\underbrace{F_\pi^2 \bar{D}^+(\nu=0, t=2M_\pi^2)}_{F_\pi^2(d_{00}^+ + 2M_\pi^2 d_{01}^+) + \Delta_D} = \underbrace{\sigma(2M_\pi^2)}_{\sigma_{\pi N} + \Delta_\sigma} + \Delta_R$$

$$|\Delta_R| \lesssim 2 \text{ MeV}$$

[Bernard, Kaiser, Meißner 1996]

$$\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2) \text{ MeV}$$

[Hoferichter et al. 2012]



# Phenomenological status

- **Karlsruhe/Helsinki** partial-wave analysis KH80 Höhler et al. 1980s
  - ↪ comprehensive analyticity constraints, old data
- Formalism for the extraction of  $\sigma_{\pi N}$  via the **Cheng–Dashen low-energy theorem** Gasser, Leutwyler, Locher, Sainio 1988,1991
  - ↪ “canonical value”  $\sigma_{\pi N} \sim 45$  MeV, based on KH80 input
- **GWU/SAID** partial-wave analysis Pavan, Strakovsky, Workman, Arndt 2002
  - ↪ much larger value  $\sigma_{\pi N} = (64 \pm 8)$  MeV
- More recently: ChPT in different regularizations (w/ and w/o  $\Delta$ ) Alarcón et al. 2012
  - ↪ fit to PWAs,  $\sigma_{\pi N} = 59 \pm 7$  MeV

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- This talk: two new sources of information on low-energy  $\pi N$  scattering

- Precision extraction of  $\pi N$  **scattering lengths** from **hadronic atoms**

Baru et al. 2011

- **Roy-equation** constraints: analyticity, unitarity, crossing symmetry

# Hadronic atoms: constraints for $\pi N$

- $\pi H/\pi D$ : bound state of  $\pi^-$  and p/d spectrum sensitive to threshold  $\pi N$  amplitude
- Combined analysis of  $\pi H$  and  $\pi D$ :

$$\begin{aligned} \bar{a}_0^+ &\equiv a^+ = (7.5 \pm 3.1) \cdot 10^{-3} M_\pi^{-1} \\ \bar{a}_0^- &\equiv a^- = (86.0 \pm 0.9) \cdot 10^{-3} M_\pi^{-1} \end{aligned}$$

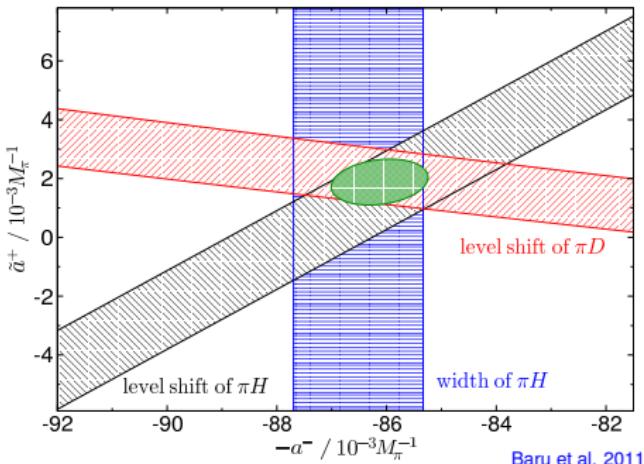
↪ Large  $a^+$  suggests a large  $\sigma_{\pi N}$ ,

- But:  $a^+$  very sensitive to isospin breaking, PWA based on  $\pi^\pm p$  channels

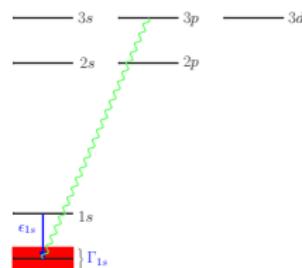
↪ use instead

$$\frac{\bar{a}_{\pi-p} + \bar{a}_{\pi+p}}{2} = (-0.9 \pm 1.4) \cdot 10^{-3} M_\pi^{-1}$$

- Isospin breaking in  $\sigma_{\pi N}$  could be important
- We revisit the Cheng-Dashen low-energy theorem



Baru et al. 2011



$$\bar{a}^+ = a^+ + \frac{1}{1 + M_\pi/m_p} \left\{ \frac{M_\pi^2 - m_0^2}{\pi F_\pi^2} c_1 - 2\alpha f_1 \right\}$$



# Motivation: Why Roy-Steiner equations?

**Roy(-Steiner) eqs.** = Partial-Wave (Hyperbolic) Dispersion Relations coupled by unitarity and crossing symmetry

- **Respect all symmetries:** analyticity, unitarity, crossing
- **Model independent**  $\Rightarrow$  the actual parametrization of the data is irrelevant once it is used in the integral.
- Framework allows for **systematic improvements** (subtractions, higher partial waves, ...)
- **PW(H)DRs** help to study processes with **high precision**:

- $\pi\pi$ -scattering:

Ananthanarayan et al. (2001), García-Martín et al. (2011)

- $\pi K$ -scattering:

Büttiker et al. (2004)

- $\gamma\gamma \rightarrow \pi\pi$  scattering:

Hoferichter et al. (2011)

## Warm up: Roy-equations for $\pi\pi$

- $\pi\pi \rightarrow \pi\pi \Rightarrow$  fully crossing symmetric in Mandelstam variables  $s$ ,  $t$ , and  $u = 4M_\pi - s - t$
- Start from twice-subtracted fixed-t DRs of the generic form

$$T^I(s, t) = c(t) + \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2} \left[ \frac{s^2}{(s' - s)} - \frac{u^2}{(s' - u)} \right] \text{Im } T^I(s', t)$$

- Subtraction functions  $c(t)$  are determined via crossing symmetry

↪ functions of the  $l=0,2$  scattering lengths:  $a_0^0$  and  $a_0^2$

- PW-expansion of these DRs yields the **Roy-equations**

Roy (1971)

$$t_J^I(s) = ST_J^I(s) + \sum_{J'=0}^{\infty} (2J' + 1) \sum_{I'=0,1,2} \int_{4m_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s', s) \text{Im } t_{J'}^{I'}(s')$$

- $K_{JJ'}^{II'}(s', s) \equiv$  kernels  $\Rightarrow$  analytically known

# Roy–Steiner equations for $\pi N$ : differences to $\pi\pi$ Roy equations

Key differences compared to  $\pi\pi$  Roy equations

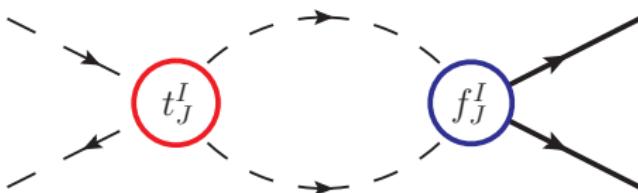
- **Crossing:** coupling between  $\pi N \rightarrow \pi N$  (s-channel) and  $\pi\pi \rightarrow \bar{N}N$  (t -channel)

⇒ need a different kind of dispersion relations

Hite, Steiner 1973, Büttiker et al. 2004

- **Unitarity** in t-channel, e.g. in single-channel approximation

$$\text{Im}f_{\pm}^J(t) = \sigma_t^\pi f_{\pm}^J(t) t_J^I(t)^*$$



⇒ **Watson's theorem:** phase of  $f_{\pm}^J(t)$  equals  $\delta_{IJ}$

Watson 1954

↪ solution in terms of Omnès function

[Muskhelishvili 1953, Omnès 1958]

- **Large pseudo-physical region** in t -channel

↪  $\bar{K}K$  intermediate states for s-wave in the region of the  $f_0(980)$

# Roy–Steiner equations for $\pi N$ : flow of information

## Limited range of validity

$$\sqrt{s} \leq \sqrt{s_m} = 1.38 \text{ GeV}$$

$$\sqrt{t} \leq \sqrt{t_m} = 2.00 \text{ GeV}$$

### Input/Constraints

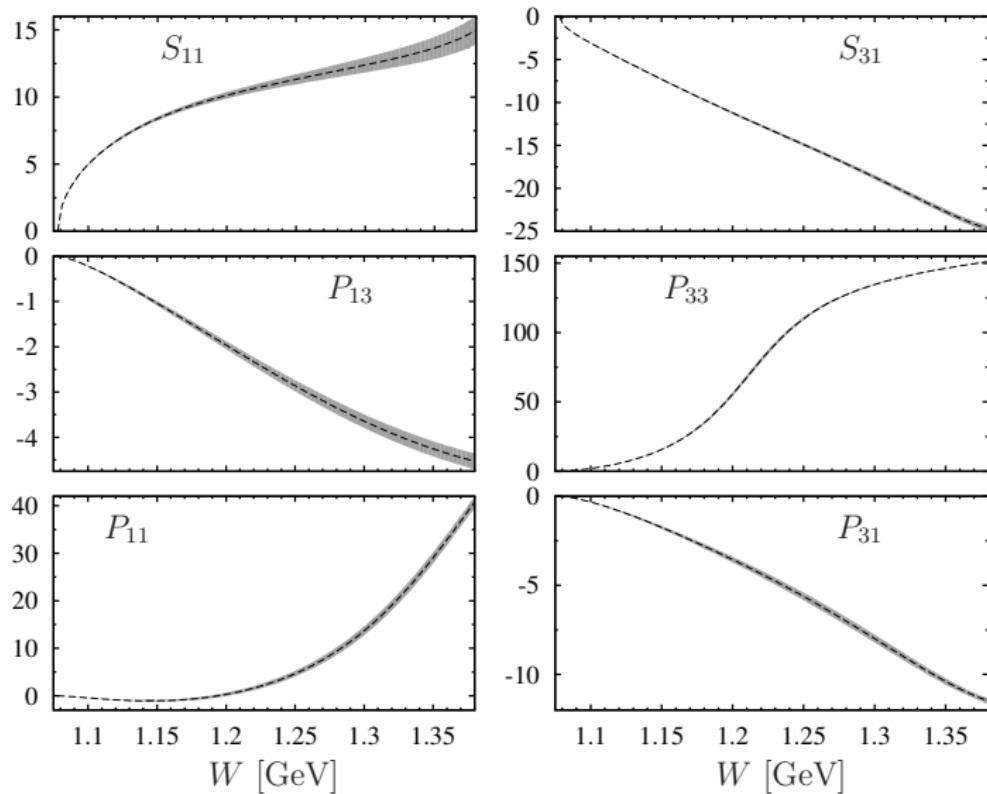
- S- and P-waves **above** matching point  
 $s > s_m$  ( $t > t_m$ )
- Inelasticities
- Higher waves (D-, F-,  $\dots$ )
- Scattering lengths from hadronic atoms

Baru et al. 2011

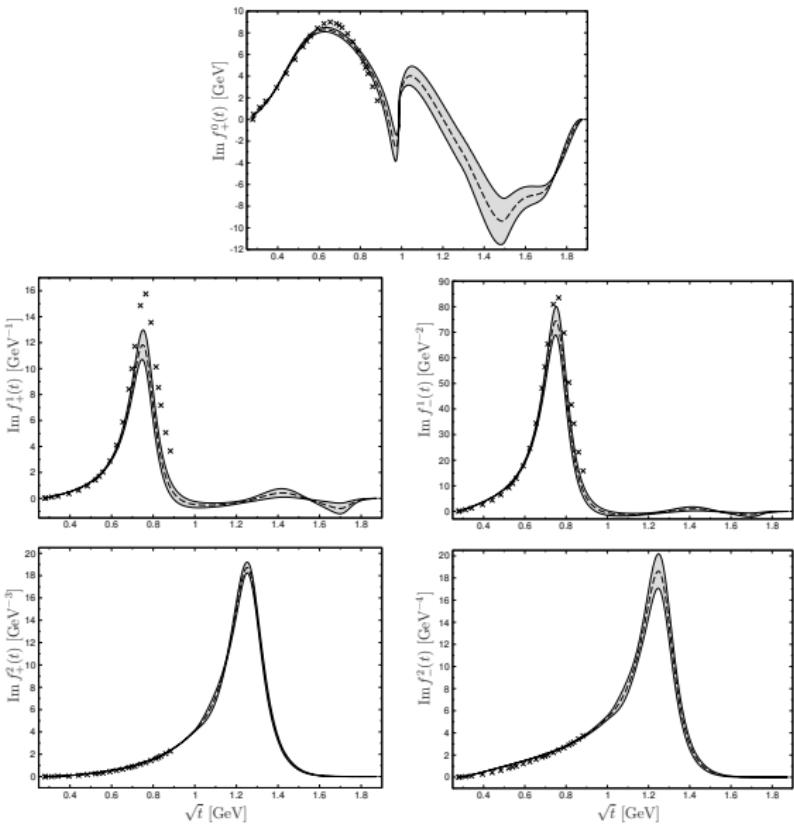
### Output

- S- and P-wave **phase-shifts** at low energies  
 $s < s_m$  ( $t < t_m$ )
- Subthreshold parameters
  - ▷ Pion-nucleon  $\sigma$ -term
  - ▷ Nucleon **form factor** spectral functions
  - ▷ ChPT **LECs**

# Uncertainties: s-channel pw



# Uncertainties: Imaginary part t-channel pw



# Results for the sigma-term

$$\sigma_{\pi N} = F_\pi^2 \left( d_{00}^+ + 2M_\pi^2 d_{01}^+ \right) + \Delta_D - \Delta_\sigma - \Delta_R$$

- subthreshold parameters output of the Roy–Steiner equations

$$d_{00}^+ = -1.36(3) M_\pi^{-1} \quad [\text{KH}: -1.46(10) M_\pi^{-1}]$$

$$d_{01}^+ = 1.16(2) M_\pi^{-3} \quad [\text{KH}: 1.14(2) M_\pi^{-3}]$$

- $\Delta_D - \Delta_\sigma = -(1.8 \pm 0.2) \text{ MeV}$

[MH at al. 2012]

$$|\Delta_R| \lesssim 2 \text{ MeV}$$

[Bernard, Kaiser, Meißner 1996]

- Isospin breaking in the CD theorem shifts  $\sigma_{\pi N}$  by  $+3.0 \text{ MeV}$

- Final results:

$$\sigma_{\pi N} = (59.1 \pm 1.9_{\text{RS}} \pm 3.0_{\text{LET}}) \text{ MeV} = (59.1 \pm 3.5) \text{ MeV}$$

[MH, JRE, Kubis, Meißner]

- $\sigma_{\pi N}$  depends linearly on the scattering lengths

$$\sigma_{\pi N} = 59.1 + \sum_{l_s} c_{l_s} \Delta a_{0+}^{l_s}$$

# $\sigma_{\pi N}$ : comparison with KH80 and SAID

## Comparison with KH80

- RS eqs. with KH80 input  $\hookrightarrow \sigma_{\pi N} = 46$  MeV

$\hookrightarrow$  to be compared with  $\sigma_{\pi N} = 45$  MeV

Gasser, Leutwyler, Sainio 1988, Gasser, Leutwyler, Sainio 1991

$\hookrightarrow$  KH80 is internally **consistent** but at odd with the modern **SL** determinations

How are  $d_{00}^+$  and  $d_{01}^+$  extracted in KH80 and SAID?

- Standard approach:

Gasser, Leutwyler, Locher, Sainio 1988

replace  $d_{00}^+$  and  $d_{01}^+$  in favor of threshold parameters:  $a_{0+}^+$  and  $a_{1+}^+$

$\hookrightarrow$  corrections from PWA via DRs ( $D^+$  and  $E^+$ )

	Born	$a_{0+}^+$	$a_{1+}^+$	$D^+$	$E^+$	$\Sigma_d = F_\pi^2 \left( d_{00}^+ + 2M_\pi^2 d_{01}^+ \right)$
KH80	-133	-7	<b>+352</b>	-91	-72	50
SAID	-127	0	<b>+351</b>	-88	-69	67
diff	+6	7	-1	+3	+3	17

- difference with KH80 due the  $a_{0+}^+$
- large weight of  $a_{1+}^+$   $\Rightarrow$  **It has to be known extremely accurately!**  
 $\hookrightarrow$  the difference 132.7 (SAID)/131.2 (RS) translates in 5 MeV in the  $\sigma_{\pi N}$

# Comparison with lattice $\sigma_{\pi N}$ results

- Recent lattice determination of  $\sigma_{\pi N}$  at (almost) the physical point

▷ BMW  $\sigma_{\pi N} = 38(3)(3)$  MeV

Durr et al. 2015

▷  $\chi$ QCD  $\sigma_{\pi N} = 44.4(3.2)(4.5)$  MeV

Yang et al. 2015

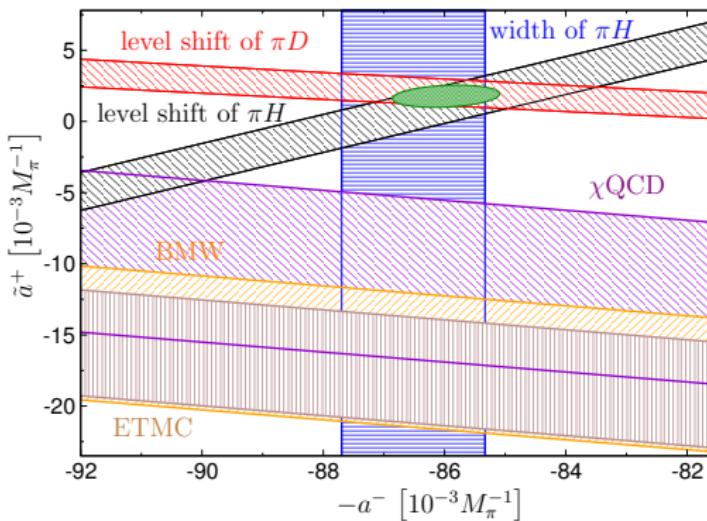
▷ ETMC  $\sigma_{\pi N} = 37.22(2.57)(1)$  MeV

Abdel-Rehim et al. 2015

▷ RQCD  $\sigma_{\pi N} = 35(6)$  MeV

Bali et al. 2016

- The linear dependence of  $\sigma_{\pi N}$  on the scattering lengths introduces an additional constraint



↪ Fully inconsistent with the hadronic atom phenomenology

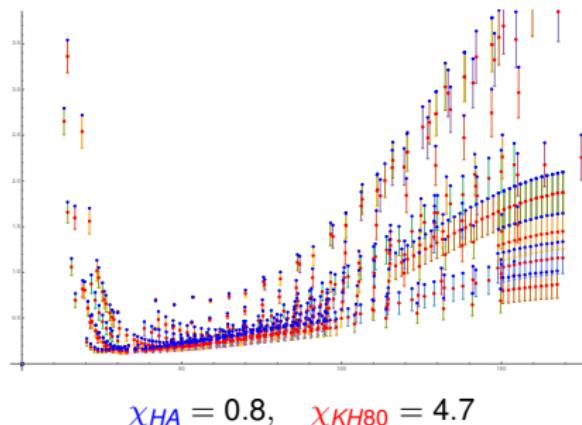
# Comparison with experimental cross-section data

- Discrepancies concentrated in the  $\pi^+ p \rightarrow \pi^+ p$  channel

	RS	KH80
$a_{0+}^{1/2} [10^{-3} M_\pi^{-1}]$	$169.8 \pm 2.0$	$173 \pm 3$
$a_{0+}^{3/2} [10^{-3} M_\pi^{-1}]$	$-86.3 \pm 1.8$	$-101 \pm 4$

- Study  $\pi^+ p \rightarrow \pi^+ p$  total cross-section

- Generate RS cross-section solutions for different scattering lengths
- below  $T_\pi = 50$  MeV uncertainties dominated by scattering length errors
- compare with the  $\pi N$  data base



Preliminary

→ Hadronic atom solution compatible with experimental results

# Matching to Chiral Perturbation Theory

Matching to ChPT at the subthreshold point:

- Chiral expansion expected to work best at **subthreshold point**
  - ▷ Maximal distance from threshold **singularities**
  - ▷  $\pi N$  amplitude can be expanded as **polynomial**
- Preferred choice for  $NN$  scattering due to proximity of relevant kinematic regions

Express the subthreshold parameters in terms of the LECs to  $\mathcal{O}(p^4)$

$$d_{00}^+ = -\frac{2M_\pi^2(2\tilde{c}_1 - \tilde{c}_3)}{F_\pi^2} + \frac{g_a^2(3 + 8g_a^2)M_\pi^3}{64\pi F_\pi^4} + M_\pi^4 \left\{ \frac{16\bar{e}_{14}}{F_\pi^2} - \frac{2c_1 - c_3}{16\pi^2 F_\pi^4} \right\}$$

- Chiral  $\pi N$  amplitude to  $\mathcal{O}(p^4)$  depends on **13** low-energy constants
- Roy–Steiner system contains **10 subtraction constants**
  - ▷ Calculate remaining **3** from **sum rules**
  - ▷ **Invert the system** to solve for LECs

# Chiral low-energy constants

	NLO	N <sup>2</sup> LO	N <sup>3</sup> LO
$c_1$ [GeV <sup>-1</sup> ]	$-0.74 \pm 0.02$	$-1.07 \pm 0.02$	$-1.11 \pm 0.03$
$c_2$ [GeV <sup>-1</sup> ]	$1.81 \pm 0.03$	$3.20 \pm 0.03$	$3.13 \pm 0.03$
$c_3$ [GeV <sup>-1</sup> ]	$-3.61 \pm 0.05$	$-5.32 \pm 0.05$	$-5.61 \pm 0.06$
$c_4$ [GeV <sup>-1</sup> ]	$2.17 \pm 0.03$	$3.56 \pm 0.03$	$4.26 \pm 0.04$
$\bar{d}_1 + \bar{d}_2$ [GeV <sup>-2</sup> ]	—	$1.04 \pm 0.06$	$7.42 \pm 0.08$
$\bar{d}_3$ [GeV <sup>-2</sup> ]	—	$-0.48 \pm 0.02$	$-10.46 \pm 0.10$
$\bar{d}_5$ [GeV <sup>-2</sup> ]	—	$0.14 \pm 0.05$	$0.59 \pm 0.05$
$\bar{d}_{14} - \bar{d}_{15}$ [GeV <sup>-2</sup> ]	—	$-1.90 \pm 0.06$	$-12.18 \pm 0.12$
$\bar{e}_{14}$ [GeV <sup>-3</sup> ]	—	—	$0.89 \pm 0.04$
$\bar{e}_{15}$ [GeV <sup>-3</sup> ]	—	—	$-0.97 \pm 0.06$
$\bar{e}_{16}$ [GeV <sup>-3</sup> ]	—	—	$-2.61 \pm 0.03$
$\bar{e}_{17}$ [GeV <sup>-3</sup> ]	—	—	$0.01 \pm 0.06$
$\bar{e}_{18}$ [GeV <sup>-3</sup> ]	—	—	$-4.20 \pm 0.05$

- Subthreshold errors tiny, chiral expansion dominates uncertainty
- $\bar{d}_i$  at N<sup>3</sup>LO increase by an order of magnitude
  - ↳ due to terms proportional to  $g_A^2(c_3 - c_4) = -16$  GeV<sup>-1</sup>
  - ↳ mimic loop diagrams with  $\Delta$  degrees of freedom
- What's going on with chiral convergence?
  - ↳ look at convergence of threshold parameters with LECs fixed at subthreshold point

# Convergence of the chiral series

	NLO	N <sup>2</sup> LO	N <sup>3</sup> LO	RS
$a_{0+}^+ [10^{-3} M_\pi^{-1}]$	-23.8	0.2	-7.9	$-0.9 \pm 1.4$
$a_{0+}^- [10^{-3} M_\pi^{-1}]$	79.4	92.9	59.4	$85.4 \pm 0.9$
$a_{1+}^+ [10^{-3} M_\pi^{-3}]$	102.6	121.2	131.8	$131.2 \pm 1.7$
$a_{1+}^- [10^{-3} M_\pi^{-3}]$	-65.2	-75.3	-89.0	$-80.3 \pm 1.1$
$a_{1-}^+ [10^{-3} M_\pi^{-3}]$	-45.0	-47.0	-72.7	$-50.9 \pm 1.9$
$a_{1-}^- [10^{-3} M_\pi^{-3}]$	-11.2	-2.8	-22.6	$-9.9 \pm 1.2$
$b_{0+}^+ [10^{-3} M_\pi^{-3}]$	-70.4	-23.3	-44.9	$-45.0 \pm 1.0$
$b_{0+}^- [10^{-3} M_\pi^{-3}]$	20.6	23.3	-64.7	$4.9 \pm 0.8$

- N<sup>3</sup>LO results bad due to large Delta loops
- Conclusion: lessons for few-nucleon applications

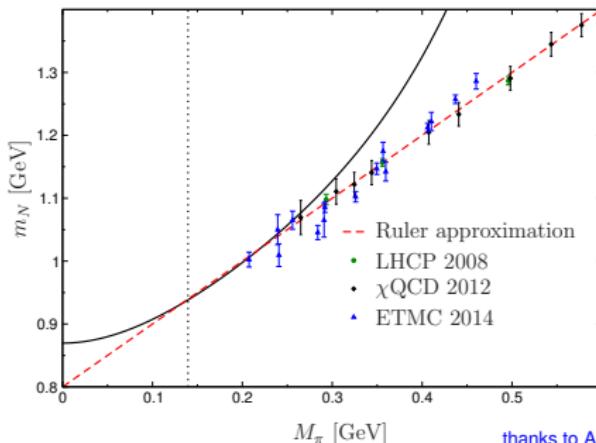
- 1- either include the  $\Delta$  to reduce the size of the loop corrections or use LECs from subthreshold kinematics work in progress
- 2- error estimates: consider chiral convergence of a given observable, difficult to assign a global chiral error to LECs

# The “ruler plot” vs. ChPT

Lattice QCD simulations can be performed at different quark/pion masses

Pion mass dependence of  $m_N$  up to NNNLO in ChPT, using

- Input from Roy–Steiner solution



thanks to A. Walker-Loud for providing the lattice data

- range of convergence of the chiral expansion is very limited
- huge cancellation amongst terms to produce a linear behavior

# Summary & Outlook

## Summary

- Derived a closed system of Roy–Steiner equations for  $\pi N$
- Numerical solution and error analysis of the full system of RS eqs.
- Precise determination of the  $\sigma_{\pi N}$ 
  - ▷ Roy–Steiner formalism reproduces KH80 result with KH80 input
  - ▷ With modern input for scattering lengths and coupling constant  $\sigma_{\pi N}$  increases
- Precise determination of threshold parameters
- Extraction of the ChPT LECs
- Study of the chiral convergence

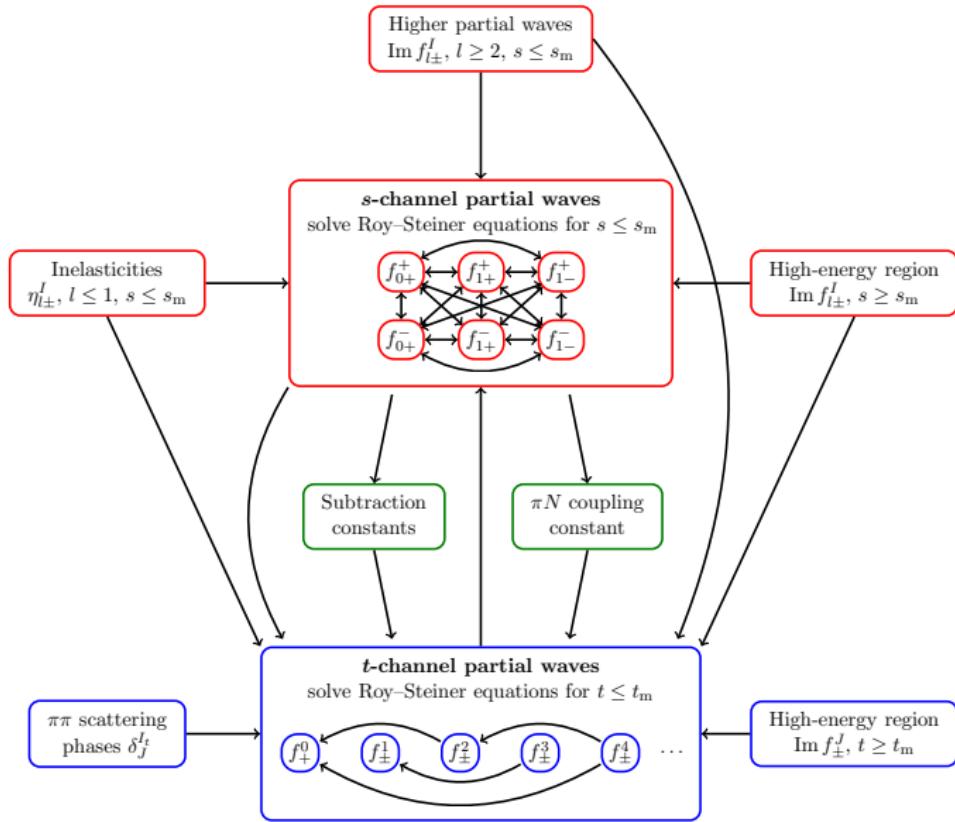
## Outlook

- Dispersive determination of  $\Delta$  pole parameters
- Matching to ChPT with explicit  $\Delta$ 's
  - ▷ LECs extraction and study of the chiral convergence
  - ▷ Large- $N_c$  constraints on  $\Delta$  LECs
- $\pi\pi$ -continuum contribution to the nucleon form factors
  - ▷ update of Höhler spectral functions, including also isospin breaking

criticism by Lee et al. 2015

# Spare slides

# Roy-Steiner equations for $\pi N$ : flow of information



- **Roy-equations** rigorously valid for a finite energy range

⇒ introduce a matching point  $s_m$

- only partial waves with  $J \leq J_{\max}$  are solved

- assume isospin limit

- **Input**

- High-energy region:  $\text{Im}t_{IJ}(s)$  for  $s \geq s_m$  and for all  $J$
- Higher partial waves:  $\text{Im}t_{IJ}(s)$  for  $J > J_{\max}$  and for all  $s$

- **Output**

- Self-consistent solution for  $\delta_{IJ}(s)$  for  $J \leq J_{\max}$  and  $s_{\text{th}} \leq s \leq s_m$
- Constraints on subtraction constants

# $\pi N$ -scattering basics

$$\pi^a(q) + N(p) \rightarrow \pi^b(q') + N(p')$$

- **Isospin Structure:**

$$T^{ba} = \delta^{ba} T^+ + \epsilon^{ab} T^-$$

- **Lorentz Structure:**  $I \in \{+, -\}$

$$T^I = \bar{u}(p') \left( A^I + \frac{\not{q} + \not{q}'}{2} B^I \right) u(p)$$

$$D^I = A^I + \nu B^I, \quad \nu = \frac{s-u}{4m}$$

- **Isospin basis:**  $I_s \in \{1/2, 3/2\}$

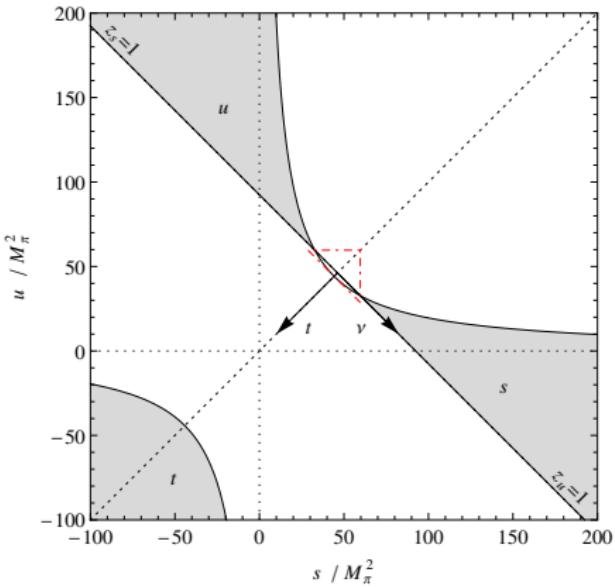
$$\{T^+, T^-\} \Leftrightarrow T^{1/2}, T^{3/2}$$

- **PW projection:**

$$s\text{-channel pw: } f_{I\pm}^I$$

$$t\text{-channel pw: } f_{\pm}^J$$

**Bose symmetry**  $\Rightarrow$  even/odd  $J \Leftrightarrow I = +/-$



# $\pi N$ -scattering basics: partial waves

- **s-channel** projection:

$$f_{l\pm}^I(W) = \frac{1}{16\pi W_1} \left\{ (E+m)[A_l^I(s) + (W-m)B_l^I(s)] + (E-m)[-A_{l\pm 1}^I(s) + (W+m)B_{l\pm 1}^I(s)] \right\}$$
$$X_l^I(s) = \int_{-1}^1 dz_s P_l(z_s) X^I(s, t) \Big|_{t=t(s, z_s)=-2q^2(1-z_s)} \quad \text{for } X \in \{A, B\} \text{ and } W = \sqrt{s}$$

- **McDowell symmetry:**  $f_{l+}^I(W) = -f_{(l+1)-}^I(-W) \quad \forall I \geq 0$

- **t-channel** projection:

$$f_+^J(t) = -\frac{1}{4\pi} \int_0^1 dz_t P_J(z_t) \left\{ \frac{p_t^2}{(p_t q_t)^J} A^I(s, t) \Big|_{s=s(t, z_t)} - \frac{m}{(p_t q_t)^{J-1}} z_t B^I(s, t) \Big|_{s=s(t, z_t)} \right\} \quad \forall J \geq 0$$
$$f_-^J(t) = \frac{1}{4\pi} \frac{\sqrt{J(J+1)}}{2J+1} \frac{1}{(p_t q_t)^{J-1}} \int_0^1 dz_t [P_{J-1}(z_t) - P_{J+1}(z_t)] B^I(s, t) \Big|_{s=s(t, z_t)} \quad \forall J \geq 1$$

- **Bose symmetry**  $\Rightarrow$  even/odd  $J \Leftrightarrow I = +/-$

# $\pi N$ -scattering basics: Unitarity relations

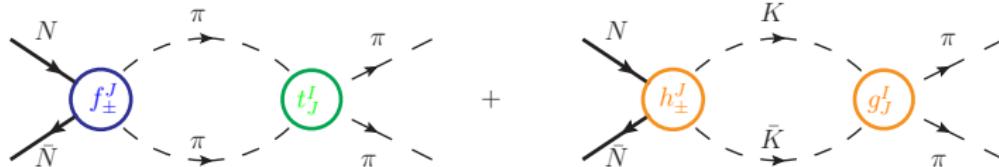
- **s-channel** unitarity relations ( $I_s \in \{1/2, 3/2\}$ ):

$$\text{Im } f_{I\pm}^{Is}(W) = q |f_{I\pm}^{Is}(W)|^2 \theta(W - W_+) + \frac{1 - (\eta_{I\pm}^{Is}(W))^2}{4q} \theta(W - W_{\text{inel}})$$



- **t-channel** unitarity relations: 2-body intermediate states:  $\pi\pi + \bar{K}K + \dots$

$$\text{Im } f_{\pm}^J(t) = \sigma_t^\pi (t_J^J(t))^* f_{\pm}^J(t) \theta(t - t_\pi) + 2c_J \sqrt{2} k_t^{2J} \sigma_t^K (g_J^J(t))^* h_{\pm}^J(t) \theta(t - t_K)$$



- Only linear in  $f_{\pm}^J(t) \Rightarrow$  less restrictive

# Roy-Steiner equations for $\pi N$ : HDR's

- **Hyperbolic DRs:**  $(s - a)(u - a) = b = (s' - a)(u' - a)$  with  $a, b \in \mathbb{R}$

$$A^+(s, t; a) = \frac{1}{\pi} \int_{s_+}^{\infty} \textcolor{red}{ds'} \left[ \frac{1}{s' - s} + \frac{1}{s' - u} - \frac{1}{s' - a} \right] \text{Im } A^+(s', t') + \frac{1}{\pi} \int_{t_\pi}^{\infty} \textcolor{blue}{dt'} \frac{\text{Im } A^+(s', t')}{t' - t}$$
$$B^+(s, t; a) = N^+(s, t) + \frac{1}{\pi} \int_{s_+}^{\infty} \textcolor{red}{ds'} \left[ \frac{1}{s' - s} - \frac{1}{s' - u} \right] \text{Im } B^+(s', t') + \frac{1}{\pi} \int_{t_\pi}^{\infty} \textcolor{blue}{dt'} \frac{\nu}{\nu'} \frac{\text{Im } B^+(s', t')}{t' - t}$$
$$N^+(s, t) = g^2 \left( \frac{1}{m^2 - s} - \frac{1}{m^2 - u} \right) \quad \text{similar for } A^-, B^- \text{ and } N^- \quad [\text{Hite/Steiner (1973)}]$$

- Why **HDR?**

- Combine all physical regions  $\Rightarrow$  crucial for t-channel projection
- Evade double-spectral regions  $\Rightarrow$  the PW decompositions converge
- Range of convergence can be maximized by tuning the free hyperbola parameter  $a$
- No kinematical cuts, manageable kernel functions

# Roy-Steiner equations for $\pi N$ : derivation

- Recipe to derive **Roy-Steiner** equations:
  - **Expand** imaginary parts in terms of s- and t-channel partial waves
  - **Project** onto s- and t-channel partial waves
  - **Combine** the resulting equations using s- and t-channel **PW unitarity relations**
- Similar structure to  $\pi\pi$  **Roy equations**
- **Validity:** assuming Mandelstam analyticity

- s-channel  $\Rightarrow$  optimal for  $a = -23.2M_\pi^2$

$$s \in [s_+ = (m + M_\pi)^2, 97.30 M_\pi^2] \quad \Leftrightarrow \quad W \in [W_+ = 1.08 \text{ GeV}, 1.38 \text{ GeV}]$$

- t-channel  $\Rightarrow$  optimal for  $a = -2.71M_\pi^2$

$$t \in [t_\pi = 4M_\pi^2, 205.45 M_\pi^2] \quad \Leftrightarrow \quad \sqrt{t} \in [\sqrt{t_\pi} = 0.28 \text{ GeV}, 2.00 \text{ GeV}] .$$

# Roy-Steiner equations for $\pi N$ : subtractions

- **Subtractions** are necessary to ensure the convergence of DR integrals  
⇒ asymptotic behavior
- Can be introduced to lessen the dependence of the low-energy solution on the high-energy behavior
- Parametrize high-energy information in (a priori unknown) subtraction constants  
⇒ matching to ChPT
- Subthreshold expansion around  $\nu = t = 0$

$$\begin{aligned}\bar{A}^+(\nu, t) &= \sum_{m,n=0}^{\infty} a_{mn}^+ \nu^{2m} t^n & \bar{B}^+(\nu, t) &= \sum_{m,n=0}^{\infty} b_{mn}^+ \nu^{2m+1} t^n, \\ \bar{A}^-(\nu, t) &= \sum_{m,n=0}^{\infty} a_{mn}^- \nu^{2m+1} t^n & \bar{B}^-(\nu, t) &= \sum_{m,n=0}^{\infty} b_{mn}^- \nu^{2m} t^n,\end{aligned}$$

where

$$\begin{aligned}\bar{A}^+(s, t) &= A^+(s, t) - \frac{g^2}{m} & \bar{B}^+(s, t) &= B^+(s, t) - g^2 \left[ \frac{1}{m^2 - s} - \frac{1}{m^2 - u} \right], \\ \bar{A}^-(s, t) &= A^-(s, t), & \bar{B}^-(s, t) &= B^-(s, t) - g^2 \left[ \frac{1}{m^2 - s} + \frac{1}{m^2 - u} \right] + \frac{g^2}{2m^2},\end{aligned}$$

# RS-eqs for $\pi N$ : subthreshold expansion

- Subthreshold expansion around  $\nu = t = 0$

$$A^+(\nu, t) = \frac{g^2}{m} + d_{00}^+ + d_{01}^+ t + d_{10}^+ \nu^2 + \mathcal{O}(\nu^2 t, t^2)$$

$$A^-(\nu, t) = \nu a_{00}^- + a_{01}^- \nu t + a_{10}^- \nu^3 + \mathcal{O}(\nu^5, \nu t^2, \nu^3 t)$$

$$B^+(\nu, t) = g^2 \frac{4m\nu}{(m^2 - s_0)^2} + \nu b_{00}^+ + \mathcal{O}(\nu^3, \nu t) ,$$

$$B^-(\nu, t) = g^2 \left[ \frac{2}{m^2 - s_0} - \frac{t}{(m^2 - s_0)^2} \right] - \frac{g^2}{2m^2} + b_{00}^- + b_{01}^- t + b_{10}^- \nu^2 + \mathcal{O}(\nu^2, \nu^2 t, t^2)$$

- pseudovector Born terms:  $D^I = A^I + \nu B^I$

$$\bar{D}^+ = d_{00}^+ + d_{01}^+ t + d_{10}^+ \nu^2$$

$$d_{mn}^+ = a_{mn}^+ + b_{m-1,n}^+, \quad d_{mn}^- = a_{mn}^- + b_{mn}^- .$$

- Sum rules for subthreshold parameters:

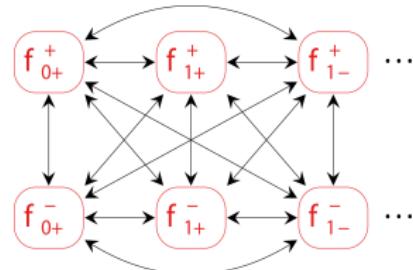
$$d_{00}^+ = -\frac{g^2}{m} + \frac{1}{\pi} \int_{s_+}^{\infty} ds' h_0(s') [\text{Im } A^+(s', z_s')]_{(0,0)} + \frac{1}{\pi} \int_{t_\pi}^{\infty} \frac{dt'}{t'} [\text{Im } A^+(t', z_t')]_{(0,0)}$$

$$h_0(s') = \frac{2}{s' - s_0} - \frac{1}{s' - a}$$

# Solving Roy-Steiner equations for $\pi N$ : Recoupling schemes

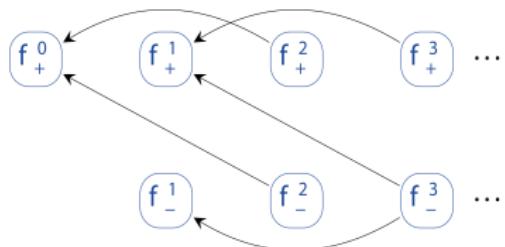
- **s-channel** subproblem:

- Kernels are diagonal for  $I \in \{+, -\}$ , but unitarity relations are diagonal for  $I_s \in \{1/2, 3/2\} \Rightarrow$  all partial-waves are interrelated
- Once the t-channel PWs are known  
 $\Rightarrow$  Structure similar to  $\pi\pi$  Roy-equations



- **t-channel** subproblem:

- Only higher PWs couple to lower ones
- Only PWs with even or odd J are coupled
- No contribution from  $f_+^J$  to  $f_-^{J+1}$   
 $\Rightarrow$  Leads to Muskhelishvili-Omnès problem



## s-channel RS equations

$$\begin{aligned}
 f_{l+}^l(W) &= N_{l+}^l(W) + \frac{1}{\pi} \int_{W_+}^{\infty} dW' \sum_{l'=0}^{\infty} \left\{ K_{ll'}^l(W, W') \operatorname{Im} f_{l'+}^{l'}(W') + K_{ll'}^l(W, -W') \operatorname{Im} f_{(l'+1)-}^{l'}(-W') \right\} \\
 &\quad + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \sum_J \left\{ G_{lJ}(W, t') \operatorname{Im} f_+^J(t') + H_{lJ}(W, t') \operatorname{Im} f_-^J(t') \right\} \\
 &= -f_{(l+1)-}^l(-W) \quad \forall l \geq 0 , \quad [\text{Hite/Steiner (1973)}]
 \end{aligned}$$

- e  $K_{ll'}^l(W, W')$ ,  $G_{lJ}(W, t')$  and  $H_{lJ}(W, t')$ -Kernels: analytically known,  
e.g.  $K_{ll'}^l(W, W') = \frac{\delta_{ll'}}{W' - W} + \dots \quad \forall l, l' \geq 0$ ,
- **Validity**: assuming Mandelstam analyticity  
 $\Rightarrow$  optimal for  $a = -23.2M_\pi^2$

$$s \in [s_+ = (m + M_\pi)^2, 97.30 M_\pi^2] \quad \Leftrightarrow \quad W \in [W_+ = 1.08 \text{ GeV}, 1.38 \text{ GeV}]$$

# Roy-Steiner equations for $\pi N$ : t-channel

## t-channel RS equations

$$\begin{aligned} f_+^J(t) &= \tilde{N}_+^J(t) + \frac{1}{\pi} \int_{W_+}^{\infty} dW' \sum_{l=0}^{\infty} \left\{ \tilde{G}_{Jl}(t, W') \operatorname{Im} f_{l+}^l(W') + \tilde{G}_{Jl}(t, -W') \operatorname{Im} f_{(l+1)-}^l(W') \right\} \\ &\quad + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \sum_{J'} \left\{ \tilde{K}_{JJ'}^1(t, t') \operatorname{Im} f_+^{J'}(t') + \tilde{K}_{JJ'}^2(t, t') \operatorname{Im} f_-^{J'}(t') \right\} \quad \forall J \geq 0 , \\ f_-^J(t) &= \tilde{N}_-^J(t) + \frac{1}{\pi} \int_{W_+}^{\infty} dW' \sum_{l=0}^{\infty} \left\{ \tilde{H}_{Jl}(t, W') \operatorname{Im} f_{l+}^l(W') + \tilde{H}_{Jl}(t, -W') \operatorname{Im} f_{(l+1)-}^l(W') \right\} \\ &\quad + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \sum_{J'} \tilde{K}_{JJ'}^3(t, t') \operatorname{Im} f_-^{J'}(t') \quad \forall J \geq 1 , \end{aligned}$$

- **Validity:** assuming Mandelstam analyticity  
⇒ optimal for  $a = -2.71 M_\pi^2$

$$t \in [t_\pi = 4M_\pi^2, 205.45 M_\pi^2] \Leftrightarrow \sqrt{t} \in [\sqrt{t_\pi} = 0.28 \text{ GeV}, 2.00 \text{ GeV}] .$$

# Solving t-channel: single channel

- Elastic-channel approximation: generic form of the integral equation

$$f(t) = \Delta(t) + (a + bt)(t - 4m^2) + \frac{t^2(t - 4m^2)}{\pi} \int_{t_\pi}^{\infty} dt' \frac{\text{Im } f(t')}{t'(t'^2 - 4m^2)(t' - t)}$$

- $\Delta(t)$ : Born terms, s-channel integrals, higher t -channel partial waves  
⇒ left-hand cut
- Introduce subtractions at  $\nu = t = 0$  ⇒ subthreshold parameters  $a, b$
- Solution in terms of Omnès function:

$$\begin{aligned} f(t) &= \Delta(t) + (t - 4m^2)\Omega(t)(1 - t\dot{\Omega}(0))a + t(t - 4m^2)\Omega(t)b \\ &\quad - \Omega(t) \frac{t^2(t - 4m^2)}{\pi} \left\{ \int_{4M_\pi^2}^{t_m} dt' \frac{\Delta(t') \text{Im } \Omega(t')^{-1}}{t'^2(t' - 4m^2)(t' - t)} + \int_{t_m}^{\infty} dt' \frac{\Omega(t')^{-1} \text{Im } f(t')}{t'(t' - 4m^2)(t' - t)} \right\} \\ \Omega(t) &= \exp \left\{ \frac{t}{\pi} \int_{t_\pi}^{t_m} \frac{dt'}{t'} \frac{\delta(t')}{t' - t} \right\} \end{aligned}$$

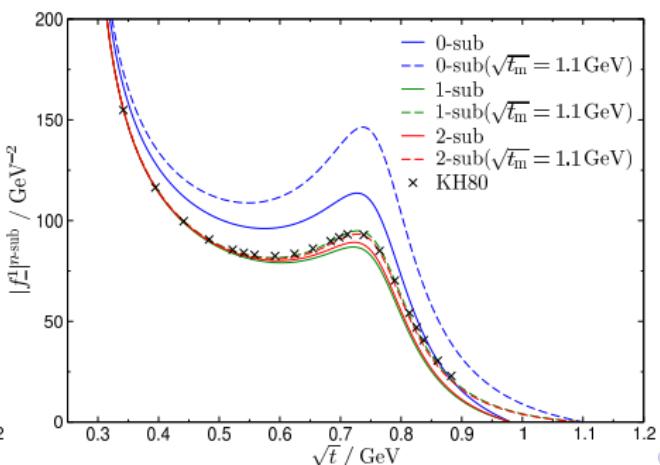
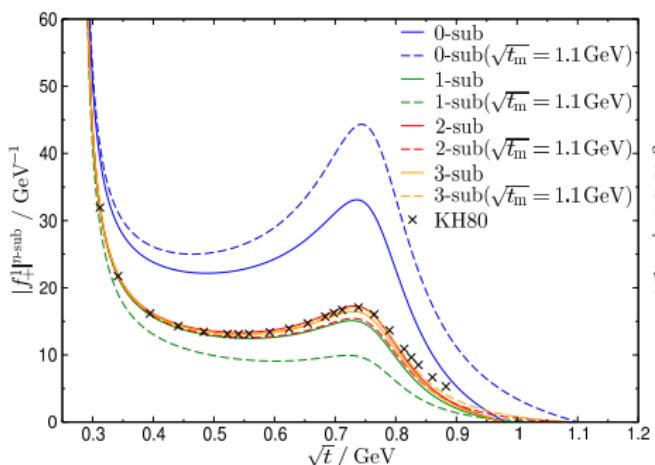
# Solving t-channel: input and subtractions

- elastic channel approximation:  $\sqrt{t_m} = 0.98 - 1.1 \text{ GeV}$ , for  $t > t_m \operatorname{Im} f_{\pm}^J(t) = 0$
- First step: check **consistency** with KH80 Höhler 1983
- Input needed:
  - $\pi\pi$  phase shifts:
  - $\pi N$  phase shifts: SAID, KH80
  - $\pi N$  at high energies: Regge model
  - $\pi N$  parameters: KH80

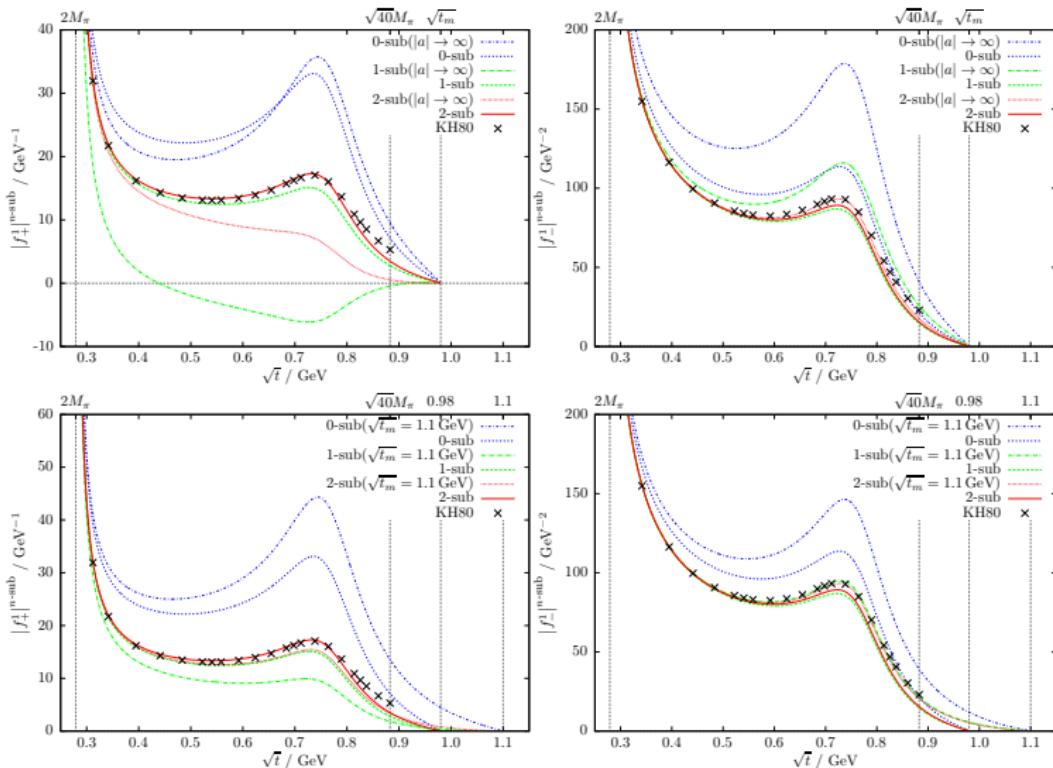
Caprini, Colangelo, Leutwyler, (in preparation), Madrid group

Arndt et al. 2008, Höhler 1983

Huang et al. 2010

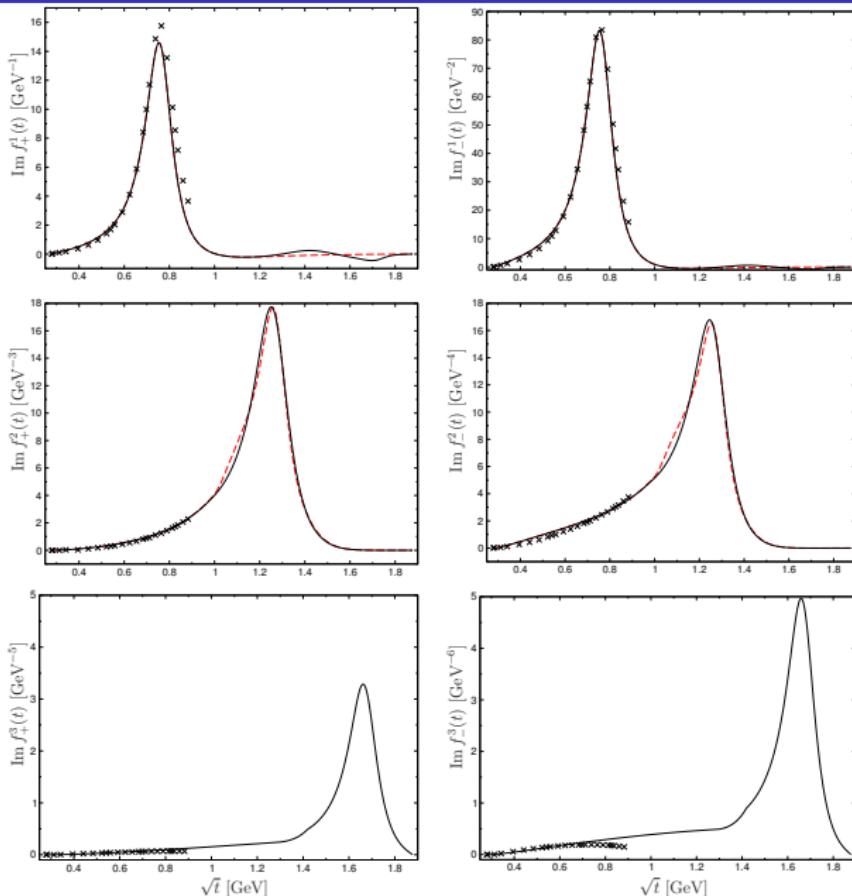


# Solving t-channel: P-wave results



**MO solutions in general consistent with KH80 results**

# Solving t-channel: P, D and F waves up to $\bar{N}N$



# Solving t-channel: coupled channels

- Generic coupled-channel integral equation

$$\mathbf{f}(t) = \Delta(t) + \frac{1}{\pi} \int_{t_\pi}^{t_m} dt' \frac{T^*(t') \Sigma(t') \mathbf{f}(t')}{t' - t} + \frac{1}{\pi} \int_{t_m}^{\infty} dt' \frac{\text{Im } \mathbf{f}(t')}{t' - t}$$

- Formal solution as in the single-channel case (now with Omnès matrix  $\Omega(t)$ )

⇒ Two-channel Muskhelishvili-Omnès problem

$$\mathbf{f}(t) = \begin{pmatrix} f_+^0(t) \\ h_+^0(t) \end{pmatrix} \quad \text{Im } \Omega(t) = (T(t))^* \Sigma(t) \Omega(t)$$

- Two linearly independent solutions  $\Omega_1, \Omega_2$

Muskhelishvili 1953

- In general no analytical solution for the Omnès matrix  
but for its determinant

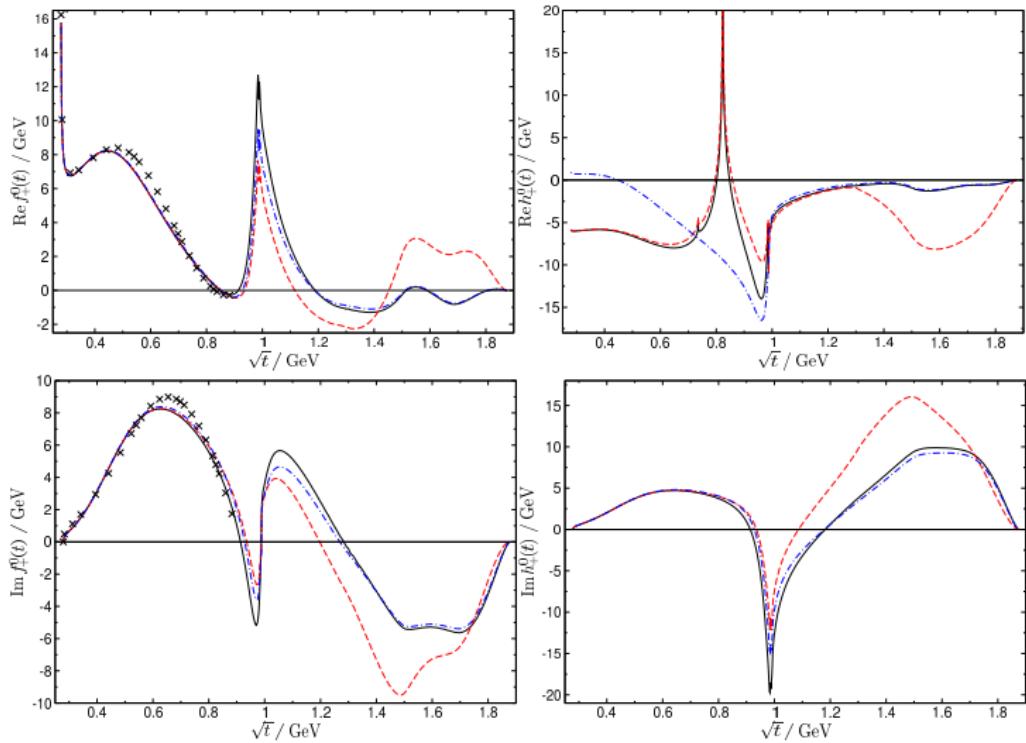
Moussallam 2000

$$\det \Omega(t) = \exp \left\{ \frac{t}{\pi} \int_{t_\pi}^{t_m} dt' \frac{\psi(t')}{t'(t' - t)} \right\}.$$

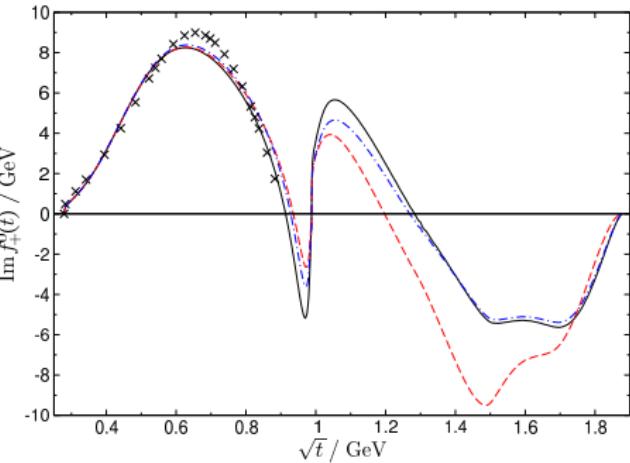
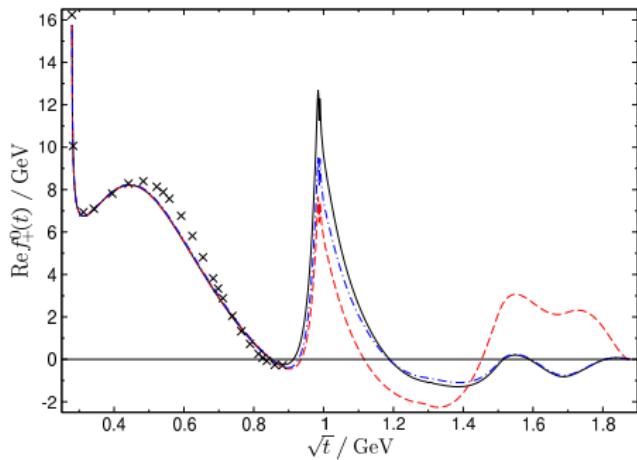
# Solving t-channel S-wave equations: input

- Input needed:
  - $\pi\pi$  s-wave partial waves: Caprini, Colangelo, Leutwyler, (in preparation)
  - $K\bar{K}$  s-wave partial waves: Büttiker. (2004)
  - $\pi N$  and  $KN$  s-wave pw: SAID, KH80 Arndt et al. 2008, Höhler 1983,
  - $\pi N$  at high energies: Regge model Huang et al. 2010
  - $\pi N$  parameters: KH80
  - Hyperon couplings from Jülich model 1989
  - KN subthreshold parameters neglected
- Two-channel approximation breaks down at  $\sqrt{t_0} = 1.3$  GeV  $\Rightarrow 4\pi$  channel
- From  $t_0$  to  $t = 2$  GeV, different approximations considered

# Solving t-channel: S-wave results



# Solving t-channel: S-wave results



MO solutions in general consistent with KH80 results

# Solving s-channel: S-wave results

- General form of the s-channel integral equation

$$f_{I+}^I(W) = \Delta_{I+}^I(W) + \frac{1}{\pi} \int_{W_+}^{\infty} dW' \sum_{I'=0}^{\infty} \left\{ K_{II'}^I(W, W') \operatorname{Im} f_{I'+}^I(W') + K_{II'}^I(W, -W') \operatorname{Im} f_{(I'+1)-}^I(W') \right\}$$

⇒ form of  $\pi\pi$  Roy-Equations

- $\Delta_{I+}^I(W)$  ≡ t-channel contribution and pole term

- valid up to  $W_m = 1.38$  GeV

- Input:**

- RS t-channel solutions for S and P waves
- s-channel partial waves for  $J > 1$
- s-channel partial waves for  $W_m < W < 2.5$  GeV
- high energy contribution for  $W > 2.5$  GeV: Regge model

SAID analysis

SAID analysis

Huang et al. 2010

- Output:**

- Self-consistent solution for S and P waves for  $J \leq J_{\max}$  and  $s_{\text{th}} \leq s \leq s_m$
- Constraints on subtraction constants ⇒ subthreshold parameters

# Solving s-channel: subtractions

- Existence and uniqueness of solutions

Gasser, Wanders 1999

⇒ no-cusp condition for each pw + 2 additional constraints are needed

- Take advantage of the precise data for pionic atoms

Gotta et al. 2005, 2010

⇒ Impose as a **constraint** scattering lengths from a combined analysis of pionic hydrogen and deuterium

Baru et al. 2011

$$a_{0+}^{1/2} = (169.8 \pm 2.0) 10^{-3} M_\pi^{-1} \quad a_{0+}^{3/2} = (-86.3 \pm 1.8) 10^{-3} M_\pi^{-1}$$

$$\text{Re } f_{l\pm}^I(s) = \mathbf{q}^{2l} \left( a_{l\pm}^I + b_{l\pm}^I \mathbf{q}^2 + \dots \right)$$

**10 subthreshold parameters** are needed to match **d.o.f**  
⇒ **three subtractions**

# Solving s-channel: strategy

- Parameterize S and P waves up to  $W < W_m$ 
  - Using SAID partial waves as starting point
- Impose as **constraints** the hadronic atom **scattering lengths**
- Introduce as many **subtractions** as necessary to **match d.o.f**
- Minimize difference between **LHS** and the **RHS** on a grid of points  $W_j$

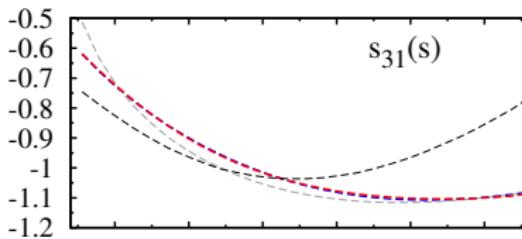
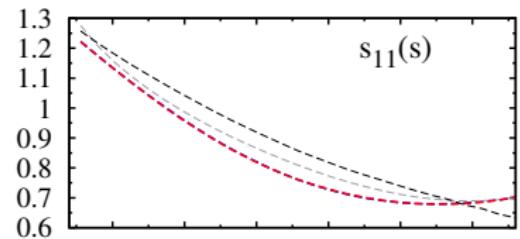
[Gasser, Wanders 1999]

$$\chi^2 = \sum_{l,l_s,\pm} \sum_{j=1}^N \frac{\left( \text{Re } f_{l\pm}^{ls}(W_j) - F[f_{l\pm}^{ls}](W_j) \right)^2}{\text{Re } f_{l\pm}^{ls}(W_j)}$$

$F[f_{l\pm}^{ls}](W_j)$   $\equiv$  right hand side of RS-equations

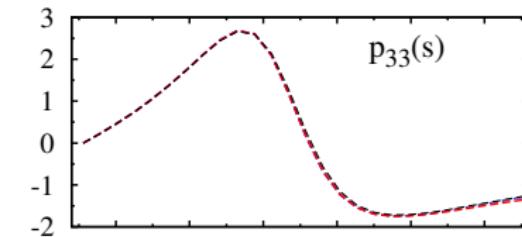
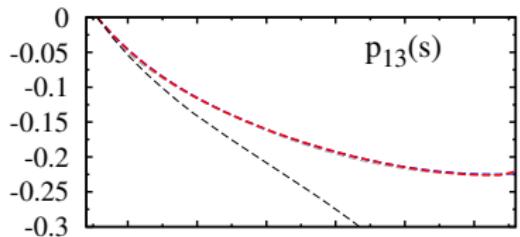
- Parametrization and subthreshold parameters are the fitting parameters

# Solving s-channel: results

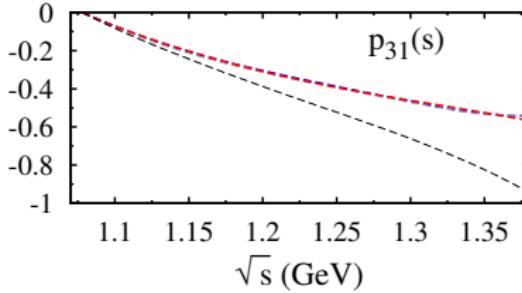
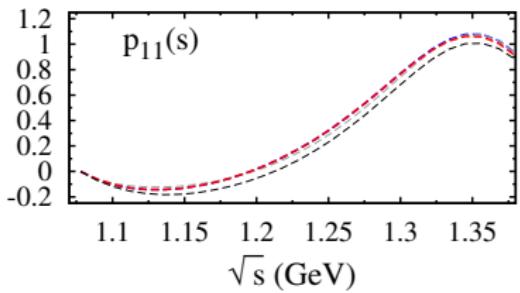


blue/red  
 $\Updownarrow$   
 LHS/RHS

after the fit

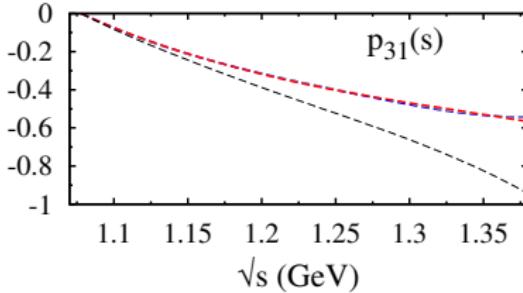
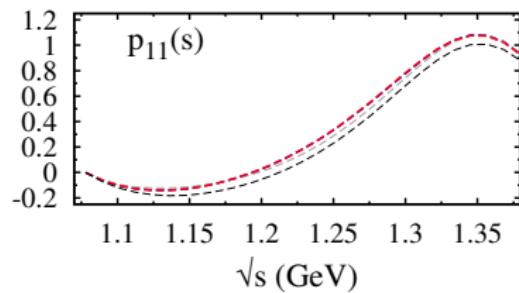
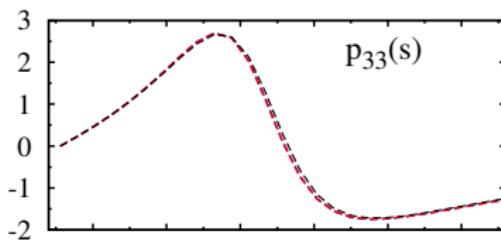
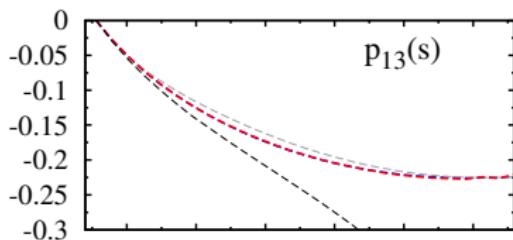
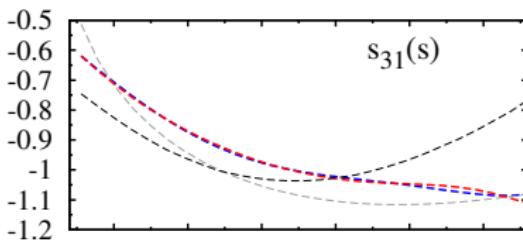
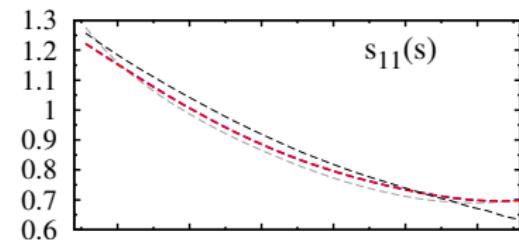


gray/black  
 $\Updownarrow$   
 LHS/RHS  
 before the fit



Notation:  $L_2 l_5 l_2 J$

# Results: s-channel PWs



blue/red  
 $\Updownarrow$   
 LHS/RHS

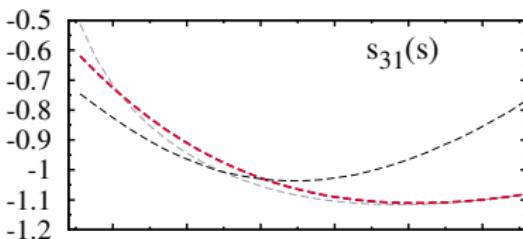
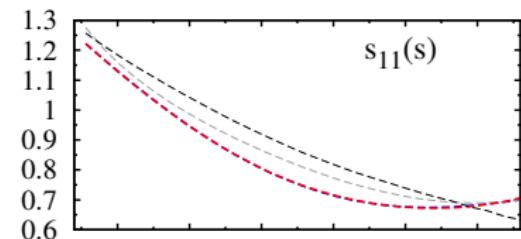
after the fit

gray/black  
 $\Updownarrow$   
 LHS/RHS

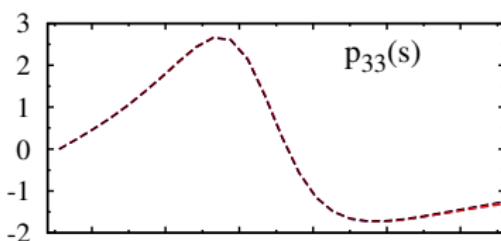
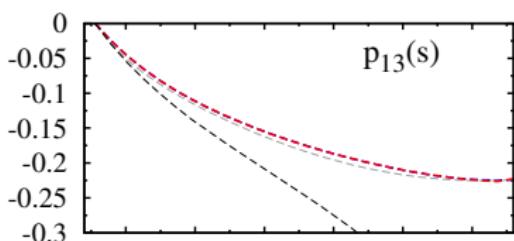
before the fit

Notation:  $L_2 I_s 2J$

# Results: s-channel PWs

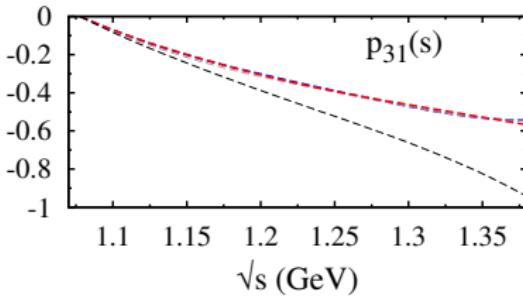
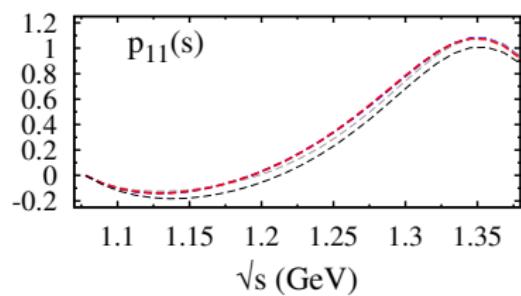


sizable  
 $f_2(1275)$  effect  
 blue/red



$\Updownarrow$   
**LHS/RHS**

after the fit

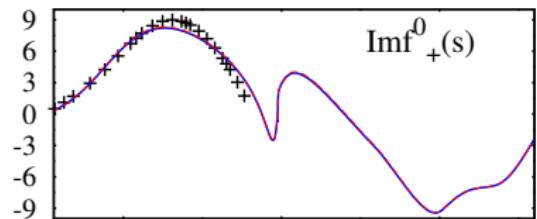


gray/black  
 $\Updownarrow$   
**LHS/RHS**

before the fit

Notation:  $L_2 l_5 2J$

# Results: t-channel PWs



blue

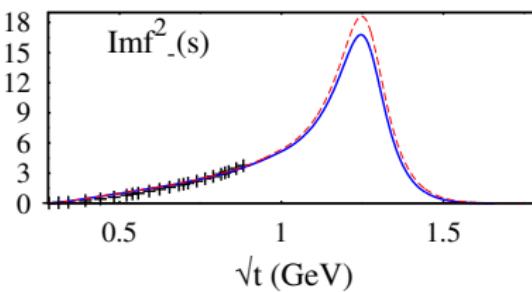
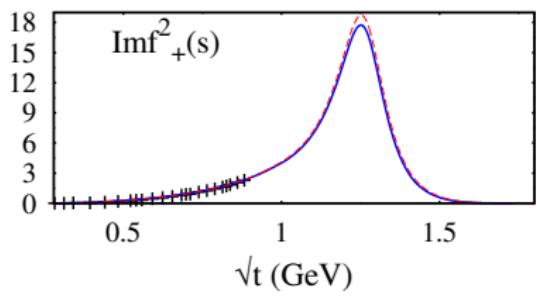
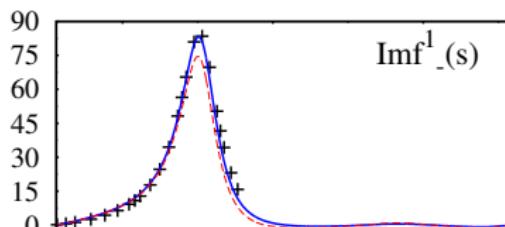
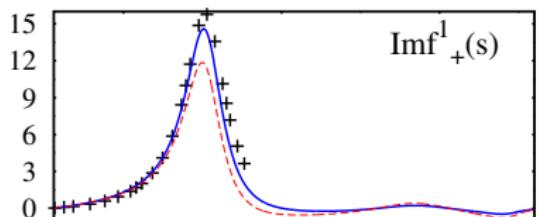


before the fit

red



after the fit



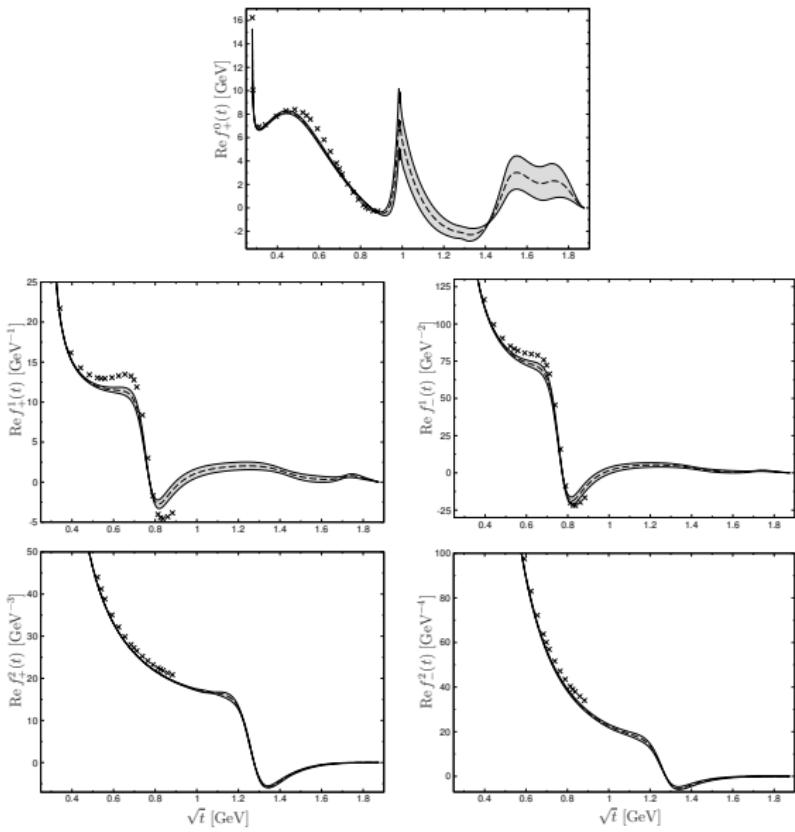
# Solving the full RS system: strategy

- Full solution: self-consistent, **iterative** solution of the **full RS** system  
⇒ consistent set of **s**- and **t**-channel PWs & **low-energy parameters**
- However:
  - t-channel RS eqs. depend only weakly on s-channel PWs
  - resulting s-channel PW change little from **SAID**

A **full solution** can be achieved including in the **s-channel** RS eqs. the **t-channel** dependence on the **subthreshold parameters**

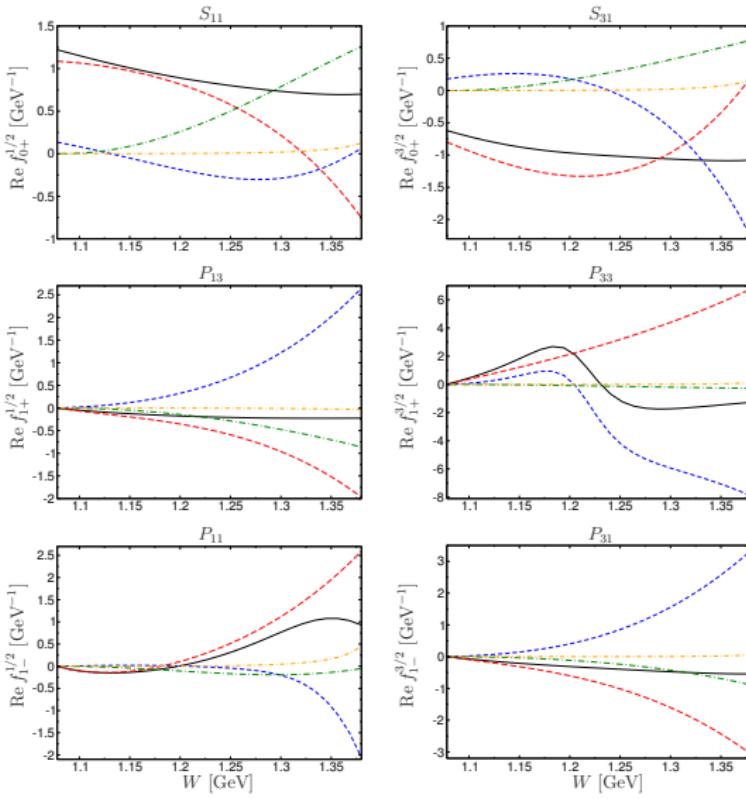
- Statistical errors (at intermediate energies)
  - ▷ important correlations between subthreshold parameters
  - ▷ shallow fit minima
  - ⇒ Sum rules for subthreshold parameters become essential to reduce the errors
- Input variation (small)
  - ▷ small effect for considering s-channel KH80 input
  - ▷ very small effects from  $L > 5$  s-channel PWs
  - ▷ small effect from the different S-wave extrapolation for  $t > 1.3$  GeV
  - ▷ negligible effect of  $\rho'$  and  $\rho''$
  - ▷ very significant effects of the D-waves ( $f_2(1275)$ )
  - ▷ F-waves shown to be negligible
- matching conditions (close to  $W_m$ )
- scattering length (SL) errors (on S-waves and subthreshold parameters)
  - ▷ very important for the  $\sigma_{\pi N}$

# Uncertainties: Real part t-channel pw



# Decomposition of s-channel RS pw

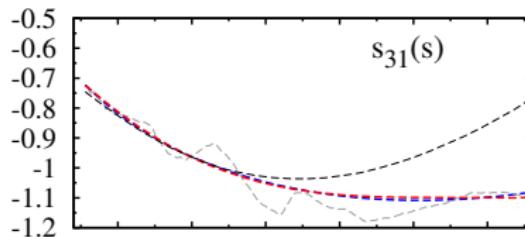
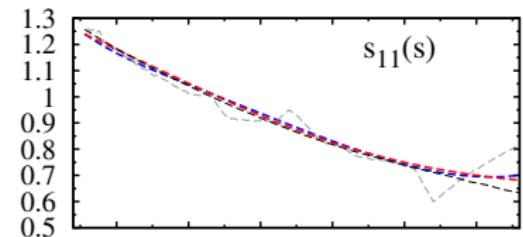
$$f_{I\pm}^{Is}(W) = N_{I\pm}^{Is}(W) + K_S T(W)_{I\pm}^{Is} + K_t T(W)_{I\pm}^{Is} + D T(W)_{I\pm}^{Is}$$



# Comparison with KH80

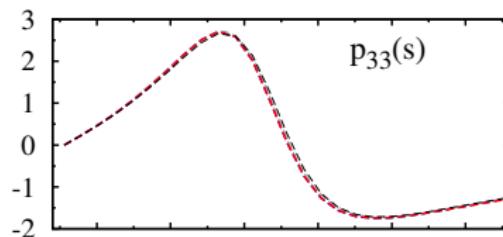
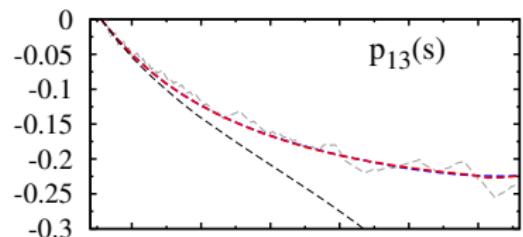
- Karlsruhe-Helsinki analysis **KH80**
  - comprehensive analyticity constraints based on fixed-t dispersion relations
  - old experimental data
- Here, an update of **KH80** results with modern input
  - HDR increase the range of validity of the equations
  - $\pi N$  scattering length extracted from hadronic atoms  $\Rightarrow$  essential for the  $\sigma_{\pi N}$
  - Goldberger-Miyazawa-Oehme sum rule:  
$$g_{\pi N}^2 / 4\pi = 13.7 \pm 0.2$$
Baru et al. 2011  
compare: 
$$g_{\pi N}^2 / 4\pi = 14.28$$
Höhler et al. 1983
- s-channel PWs from **SAID**
  - $f_2(1275)$  included  $\Rightarrow$  sizable effect
- **KH80** is internally consistent  $\Rightarrow$  RS reproduces **KH80** results with **KH80** input

# Results: s-channel PWs with KH80 input

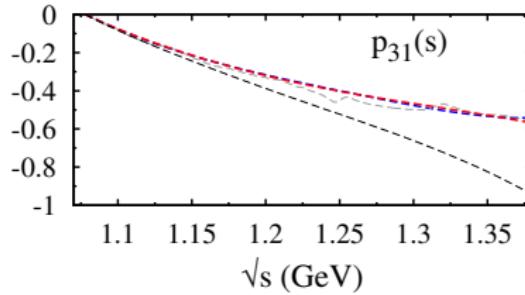
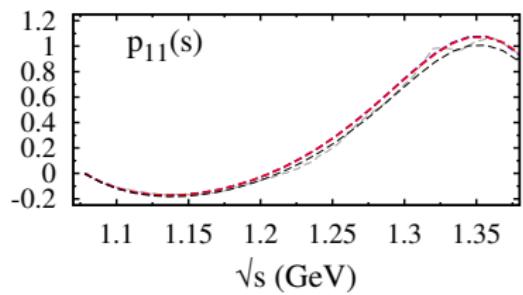


blue/red  
 $\Updownarrow$   
 LHS/RHS

after the fit

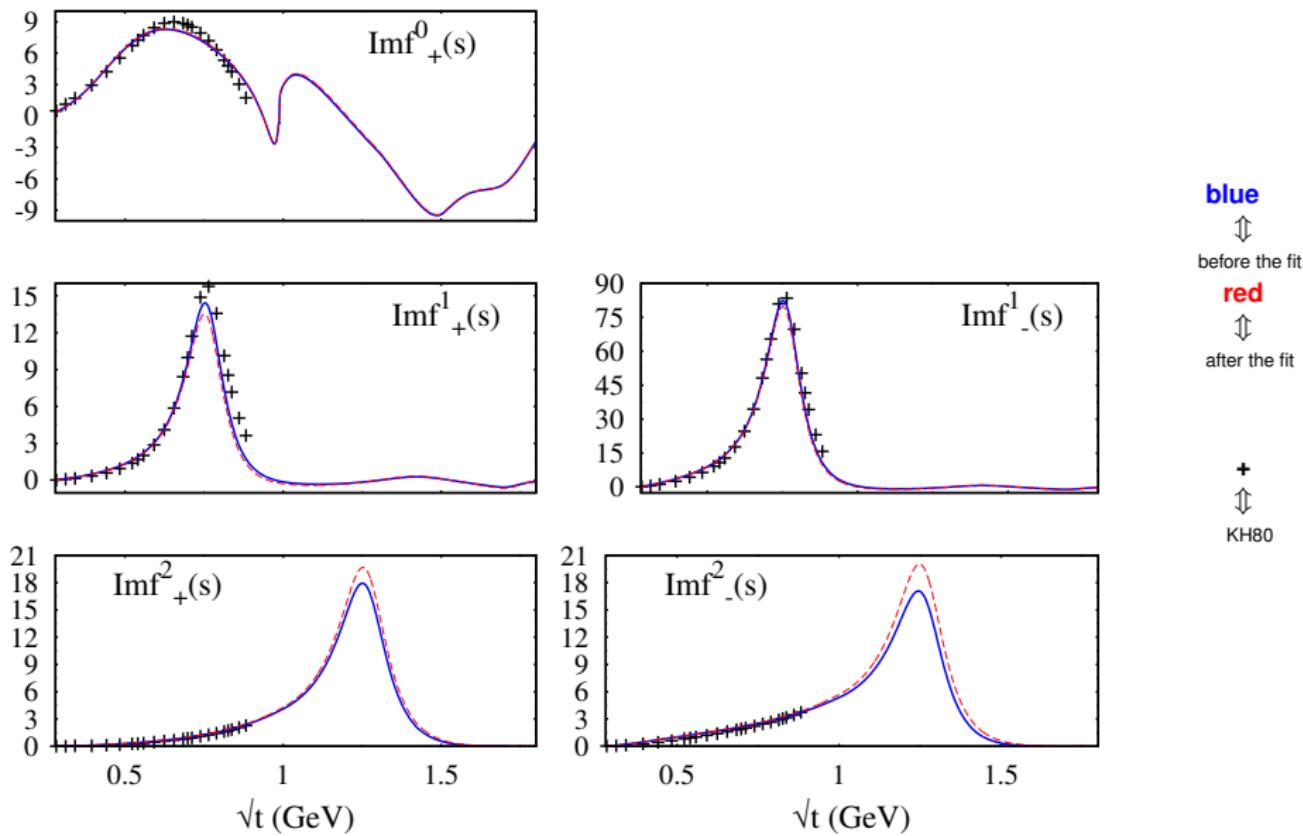


gray/black  
 $\Updownarrow$   
 LHS/RHS  
 before the fit



Notation:  $L_2 l_5 l_2 J$

# Results: t-channel PWs with KH80 input



# Threshold parameters

- Threshold parameters defined as:  $\text{Re } f_{I\pm}^I(s) = q^{2I} \{ a_{I\pm}^I + b_{I\pm}^I q^2 + \dots \}$
- Extracted from hyperbolic sum rules

	RS	KH80
$a_{0+}^+ [10^{-3} M_\pi^{-1}]$	$-0.9 \pm 1.4$	$-9.7 \pm 1.7$
$a_{0+}^- [10^{-3} M_\pi^{-1}]$	$85.4 \pm 0.9$	$91.3 \pm 1.7$
$a_{1+}^+ [10^{-3} M_\pi^{-3}]$	$131.2 \pm 1.7$	$132.7 \pm 1.3$
$a_{1+}^- [10^{-3} M_\pi^{-3}]$	$-80.3 \pm 1.1$	$-81.3 \pm 1.0$
$a_{1-}^+ [10^{-3} M_\pi^{-3}]$	$-50.9 \pm 1.9$	$-56.7 \pm 1.3$
$a_{1-}^- [10^{-3} M_\pi^{-3}]$	$-9.9 \pm 1.2$	$-11.7 \pm 1.0$
$b_{0+}^+ [10^{-3} M_\pi^{-3}]$	$-45.0 \pm 1.0$	$-44.3 \pm 6.7$
$b_{0+}^- [10^{-3} M_\pi^{-3}]$	$4.9 \pm 0.8$	$13.3 \pm 6.0$

- Reasonable agreement with KH80 but for the scattering lengths
- Disagreement in the scattering lengths in  $\sim 4\sigma$

# RS-eqs for $\pi N$ : Range of convergence

- Assumption: Mandelstam analyticity

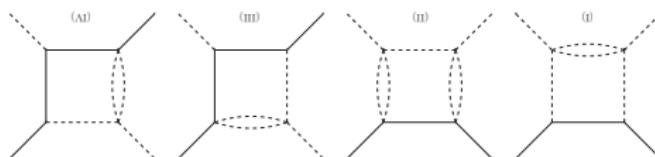
Mandelstam (1958,1959)

$\Rightarrow T(s,t)$  can be written in terms double spectral densities:  $\rho_{st}$ ,  $\rho_{su}$ ,  $\rho_{ut}$

$$T(s, t) = \frac{1}{\pi^2} \iint ds' du' \frac{\rho_{su}(s', u')}{(s' - s)(u' - u)} + \frac{1}{\pi^2} \iint dt' du' \frac{\rho_{tu}(t', u')}{(t' - t)(u' - u)} + \frac{1}{\pi^2} \iint ds' dt' \frac{\rho_{st}(s', t')}{(s' - s)(t' - t)}$$

↪ integration ranges defined by the support of the double spectral densities  $\rho$

- Boundaries of  $\rho$  are given lowest lying intermediate states



- They limit the range of validity of the HDRS:

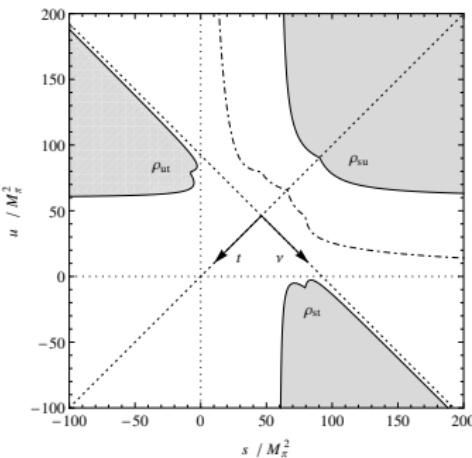
- Pw expansion converge

$\Rightarrow z = \cos \theta \in$  Lehmann ellipses

Lehmann (1958)

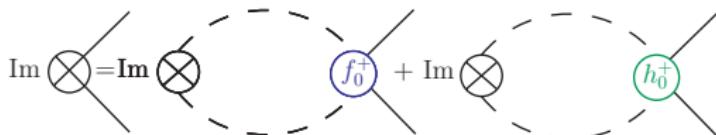
- the hyperbolae  $(s - a)(u - a) = b$  does not enter any double spectral region

$\Rightarrow$  for a value of  $a$ , constraints on  $b$  yield ranges in  $s$  &  $t$

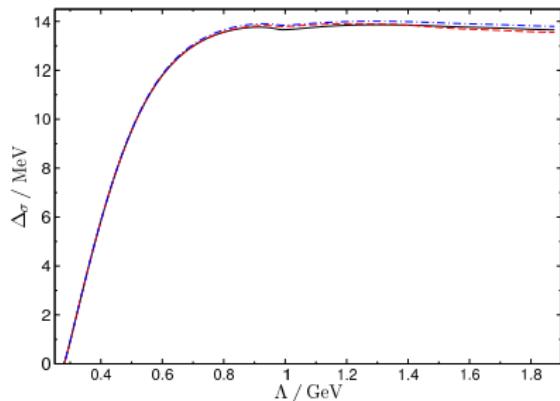


# Dispersion relation for the scalar form factor of the nucleon

- Unitarity relation:  $\text{Im } \sigma(t) = \frac{2}{4m^2-t} \left\{ \frac{3}{4} \sigma_t^\pi (F_\pi^S(t))^* f_+^0(t) + \sigma_t^K (F_K^S(t))^* h_+^0(t) \right\}$



- Once subtracted dispersion relation:  $\sigma(t) = \sigma_{\pi N} + \frac{t}{\pi} \int_{t_\pi}^{\infty} dt' \frac{\text{Im}\sigma(t')}{t'(t'-t)}$



- $\Delta_\sigma = \sigma(2M_\pi^2) - \sigma_{\pi N}$

# Dispersion relation for the $\pi N$ amplitude

- t-channel expansion of the subtracted pseudo-Born amplitude

$$\bar{D}(\nu = 0, t) = 4\pi \left\{ \frac{1}{p_t^2} \bar{f}_0^+(t) + \frac{5}{2} q_t^2 \bar{f}_2^+(t) + \frac{27}{8} p_t^2 q_t^4 \bar{f}_4^+(t) + \frac{56}{16} p_t^4 q_t^6 \bar{f}_6^+(t) + \dots \right\}$$

- Insert t-channel RS equations for Born-term-subtracted amplitudes  $\bar{f}_J^+(t)$

$$\bar{D}(\nu = 0, t) = d_{00}^+ + d_{01}^+ t - 16t^2 \int_{t_\pi}^{\infty} dt' \frac{\text{Im}\bar{f}_0^+(t')}{t'^2(t' - 4m^2)(t' - t)} + \{J \geq 2\} + \{\text{s-channel integral}\}$$

- $\Delta_D = F_\pi^2 (\bar{D}(\nu = 0, t) - d_{00}^+ + d_{01}^+ t)$  from evaluation at  $t = 2M_\pi^2$

# Summary: $\sigma$ -term corrections

- Nucleon scalar form factor

$$\Delta_{\sigma} = (13.9 \pm 0.3) \text{ MeV}$$

$$+ Z_1 \left( \frac{g^2}{4\pi} - 14.28 \right) + Z_2 \left( d_{00}^+ M_\pi + 1.46 \right) + Z_3 \left( d_{01}^+ M_\pi^3 - 1.14 \right) + Z_4 \left( b_{00}^+ M_\pi^3 + 3.54 \right)$$

$$Z_1 = 0.36 \text{ MeV}, \quad Z_2 = 0.57 \text{ MeV}, \quad Z_3 = 12.0 \text{ MeV}, \quad Z_4 = -0.81 \text{ MeV}$$

- $\pi N$  amplitude

$$\Delta_D = (12.1 \pm 0.3) \text{ MeV}$$

$$+ \hat{Z}_1 \left( \frac{g^2}{4\pi} - 14.28 \right) + \hat{Z}_2 \left( d_{00}^+ M_\pi + 1.46 \right) + \hat{Z}_3 \left( d_{01}^+ M_\pi^3 - 1.14 \right) + \hat{Z}_4 \left( b_{00}^+ M_\pi^3 + 3.54 \right)$$

$$\hat{Z}_1 = 0.42 \text{ MeV}, \quad \hat{Z}_2 = 0.67 \text{ MeV}, \quad \hat{Z}_3 = 12.0 \text{ MeV}, \quad \hat{Z}_4 = -0.77 \text{ MeV}$$

→ most of the dependence on the  $\pi N$  parameters cancels in the difference

## Full Correction

$$\Delta_D - \Delta_{\sigma} = (-1.8 \pm 0.2) \text{ MeV}$$

# Cheng-Dashen theorem in the presence of isospin breaking

- Define as **isoscalar** as

$$X^+ \rightarrow X^P = \frac{1}{2}(X_{\pi^+ p \rightarrow \pi^+ p} + X_{\pi^- p \rightarrow \pi^- p}), \quad X \in \{D, d_{00}, d_{01}, a_0, \dots\}$$

and “**isospin limit**” by proton and charged pion

- Assume virtual photons to be removed

↪ scenario closest to actual  $\pi N$  PWA

- Calculate **IV corrections** in SU(2) ChPT, mainly due to  $\Delta_\pi = M_\pi^2 - M_{\pi^0}^2$

- For the  $\sigma$  term no differences at  $\mathcal{O}(p^3)$

$$\sigma_{\pi N} = \sigma_p = \sigma_N = -4c_1 M_{\pi^0}^2 - \frac{3g_A^2 M_{\pi^0}^2}{64\pi F_\pi^2} (2M_\pi + M_{\pi^0}) + \mathcal{O}(M_\pi^4)$$

- Slope of the scalar form factor

$$\Delta_\sigma^p = \sigma_p(2M_\pi^2) - \sigma_p = \frac{3g_A^2 M_\pi^3}{64\pi F_\pi^2} + \frac{g_A^2 M_\pi \Delta_\pi}{128\pi F_\pi^2} (-7 + \sqrt{2} \log(3 + 2\sqrt{2})) + \mathcal{O}(M_\pi^4)$$

- Similarly for  $\Delta_D^p$

$$\Delta_D^p = F_\pi^2 \left\{ \bar{D}_p(0, 2M_\pi^2) - d_{00}^p - 2M_\pi^2 d_{01}^p \right\} = \frac{23g_A^2 M_\pi^3}{384\pi F_\pi^2} + \frac{g_A^2 M_\pi \Delta_\pi}{256\pi F_\pi^2} (3 + 4\sqrt{2} \log(1 + \sqrt{2})) + \mathcal{O}(M_\pi^4)$$

# Cheng-Dashen theorem in the presence of isospin breaking

- Taking everything together

$$\begin{aligned}\sigma_{\pi N} &= F_\pi^2 (d_{00}^p + 2M_\pi^2 d_{01}^p) - \Delta_R + \Delta_D - \Delta_\sigma + (\Delta_D^p - \Delta_D) - (\Delta_\sigma^p - \Delta_\sigma) \\ &\quad + \sigma_p(2M_\pi^2) + F_\pi^2 \bar{D}(0, 2M_\pi^2) \\ &= F_\pi^2 (d_{00}^p + 2M_\pi^2 d_{01}^p) - \underbrace{\Delta_R}_{\lesssim 2 \text{ MeV}} + \underbrace{\Delta_D - \Delta_\sigma}_{(-1.8 \pm 0.2) \text{ MeV}} + \underbrace{\frac{81g_a^2 M_\pi \Delta_\pi}{256\pi F_\pi^2}}_{3.4 \text{ MeV}} + \underbrace{\frac{e^2}{2} F_\pi^2 (4f_1 + f_2)}_{(-0.4 \pm 2.2) \text{ MeV}}\end{aligned}$$

↪ sizable corrections from  $\Delta_\pi$  increasing the value of the  $\sigma_{\pi N}$

- relate  $\sigma_{\pi N}$  to strangeness content of the nucleon:

$$\sigma_{\pi N} = \frac{\hat{m}}{2m_N} \frac{\langle N|\bar{u}u + \bar{d}d - 2\bar{s}s|N\rangle}{1 - y} = \frac{\sigma_0}{1 - y}, \quad y \equiv \frac{2\langle N|\bar{s}s|N\rangle}{\langle N|\bar{u}u + \bar{d}d|N\rangle}$$

$(m_s - m)(\bar{u}u + \bar{d}d - 2\bar{s}s) \subset$  LQCD produces SU(3) mass splittings:

$$\sigma_{\pi N} = \frac{\sigma_0}{1 - y}, \quad \sigma_0 = \frac{\hat{m}}{m_s - \hat{m}} (m_{\Xi} + m_{\Sigma} - 2m_N) \sim 26 \text{ MeV}$$

higher-order corrections:  $\sigma_0 \rightarrow (36 \pm 7) \text{ MeV}$

Borasoy, Meißner 1997

- potentially large effects

- ▷ from the decuplet
- ▷ from relativistic corrections (EOMS vs. heavy-baryon)
- ↪ may increase to  $\sigma_0 = (58 \pm 8) \text{ MeV}$

Alarcon et al. 2013, Siemens et al. 2016

- Conclusion:**

- ▷  $\sigma_{\pi N} = (59.1 \pm 3.5) \text{ MeV}$  not incompatible with small  $y$
- ▷ chiral convergence of  $\sigma_0$  (hence  $\langle N|\bar{s}s|N\rangle$ ) very doubtful

# Chiral Low Energy Constants with $\Delta$ 's

	HB- $NN$		HB- $\pi N$		covariant	
$N^2LO$	$Q^3$	$\varepsilon^3$	$Q^3$	$\varepsilon^3$	$Q^3$	$\varepsilon^3$
$c_1$	-1.08(2)	-1.25(3)	-1.08(2)	-1.24(3)	-1.00(2)	-1.19(4)
$c_2$	3.26(3)	1.71(1.01)	3.26(3)	1.13(1.02)	2.55(3)	1.14(19)
$c_3$	-5.39(5)	-2.68(84)	-5.39(5)	-2.75(84)	-4.90(5)	-2.56(40)
$c_4$	3.62(3)	1.57(16)	3.62(3)	1.58(16)	3.08(3)	1.33(20)
$d_{1+2}$	1.02(6)	0.14(17)	1.02(6)	-0.07(18)	1.78(6)	0.62(16)
$d_3$	-0.46(2)	-0.84(14)	-0.46(2)	-0.48(15)	-1.12(2)	-1.45(5)
$d_5$	0.15(5)	0.80(7)	0.15(5)	0.47(6)	-0.05(5)	0.29(6)
$d_{14-15}$	-1.85(6)	-1.09(30)	-1.85(6)	-0.72(31)	-2.27(6)	-0.98(13)
$N^3LO$	$Q^4$	$\varepsilon^4$	$Q^4$	$\varepsilon^4$	$Q^4$	$\varepsilon^4$
$c_1$	-1.11(3)	-1.11(3)	-1.11(3)	-1.11(3)	-1.12(3)	-1.10(3)
$c_2$	3.61(4)	1.41(38)	3.17(3)	1.28(20)	3.35(3)	1.16(20)
$c_3$	-5.60(6)	-1.88(45)	-5.67(6)	-2.04(39)	-5.70(6)	-2.10(39)
$c_4$	4.26(4)	2.03(28)	4.35(4)	2.07(29)	3.97(3)	1.91(27)
$d_{1+2}$	6.37(9)	1.78(31)	7.66(9)	2.90(30)	4.70(7)	1.78(24)
$d_3$	-9.18(9)	-3.64(36)	-10.77(10)	-5.91(50)	-5.26(5)	-3.25(14)
$d_5$	0.87(5)	1.52(7)	0.59(5)	1.03(7)	0.31(5)	0.66(6)
$d_{14-15}$	-12.56(12)	-4.38(54)	-13.44(12)	-5.17(55)	-8.84(10)	-3.41(41)
$e_{14}$	1.16(4)	1.64(10)	0.85(4)	1.12(16)	1.17(4)	1.28(11)
$e_{15}$	-2.26(6)	-4.95(15)	-0.83(6)	-3.30(25)	-2.58(7)	-3.07(13)
$e_{16}$	-0.29(3)	4.21(16)	-2.75(3)	1.92(43)	-1.77(3)	1.71(17)
$e_{17}$	-0.17(6)	-0.44(6)	0.03(6)	-0.39(7)	-0.45(6)	-0.51(7)
$e_{18}$	-3.47(5)	1.34(29)	-4.48(5)	0.67(31)	-1.68(5)	1.30(17)

# Threshold kinematics from subthreshold with $\Delta$ 's

		HB- $NN$		HB- $\pi N$		covariant		RS
N <sup>2</sup> LO	$Q^3$	$\varepsilon^3$	$Q^3$	$\varepsilon^3$	$Q^3$	$\varepsilon^3$		
$a_{0+}^+[10^{-3}M_\pi^{-1}]$	0.5	-9.8(10.9)	0.5	-0.4(9.2)	-14.8	1.0(17.3)	-0.9(1.4)	
$a_{0+}^-[10^{-3}M_\pi^{-1}]$	92.2	92.7(1.0)	92.9	90.5(9)	89.9	81.7(1.6)	85.4(9)	
$a_{1+}^+[10^{-3}M_\pi^{-3}]$	113.8	125.8(16.7)	121.7	127.2(18.4)	116.4	128.5(9.6)	131.2(1.7)	
$a_{1+}^-[10^{-3}M_\pi^{-3}]$	-74.8	-77.4(2.5)	-75.5	-78.4(2.6)	-75.1	-79.7(3.0)	-80.3(1.1)	
$a_{1-}^+[10^{-3}M_\pi^{-3}]$	-54.1	-53.4(14.1)	-47.0	-52.5(15.8)	-55.5	-52.5(8.5)	-50.9(1.9)	
$a_{1-}^-[10^{-3}M_\pi^{-3}]$	-14.1	-13.1(2.7)	-2.5	-7.8(3.0)	-10.4	-9.7(4.1)	-9.9(1.2)	
$b_{0+}^+[10^{-3}M_\pi^{-3}]$	-45.7	-38.1(9.6)	-22.1	-23.7(14.4)	-50.9	-34.7(12.1)	-45.0(1.0)	
$b_{0+}^-[10^{-3}M_\pi^{-3}]$	35.9	26.4(1.0)	22.6	17.6(8)	21.6	14.2(2.0)	4.9(8)	
N <sup>3</sup> LO	$Q^4$	$\varepsilon^4$	$Q^4$	$\varepsilon^4$	$Q^4$	$\varepsilon^4$		
$a_{0+}^+[10^{-3}M_\pi^{-1}]$	-1.5	-1.5(8.5)	-8.0	1.2(20.4)	-5.7	-0.8(10.3)	-0.9(1.4)	
$a_{0+}^-[10^{-3}M_\pi^{-1}]$	68.5	96.3(2.0)	58.6	70.0(3.3)	83.8	83.6(1.9)	85.4(9)	
$a_{1+}^+[10^{-3}M_\pi^{-3}]$	134.3	136.0(9.7)	132.1	135.2(8.7)	128.0	132.7(9.0)	131.2(1.7)	
$a_{1+}^-[10^{-3}M_\pi^{-3}]$	-80.9	-80.0(3.4)	-90.1	-86.4(2.7)	-78.1	-81.1(3.6)	-80.3(1.1)	
$a_{1-}^+[10^{-3}M_\pi^{-3}]$	-55.7	-47.5(10.5)	-73.7	-56.9(7.1)	-53.5	-51.4(7.9)	-50.9(1.9)	
$a_{1-}^-[10^{-3}M_\pi^{-3}]$	-10.0	-5.6(4.9)	-23.7	-14.4(6.5)	-11.8	-10.4(5.7)	-9.9(1.2)	
$b_{0+}^+[10^{-3}M_\pi^{-3}]$	-42.2	-31.4(8.1)	-44.5	-32.6(21.3)	-54.7	-33.9(8.5)	-45.0(1.0)	
$b_{0+}^-[10^{-3}M_\pi^{-3}]$	-31.6	7.1(2.3)	-65.2	-34.1(5.7)	2.3	2.9(2.1)	4.9(8)	

# Goldberger-Miyazawa-Oehme sum rule

- Fixed- $t$  dispersion relations at threshold  $\hookrightarrow$  **GMO sum rule**

$$\frac{g^2}{4\pi} = \left( \left( \frac{m_p + m_n}{M_\pi} \right)^2 - 1 \right) \left\{ \left( 1 + \frac{M_\pi}{m_p} \right) \frac{M_\pi}{4} (a_{\pi^- p} - a_{\pi^+ p}) - \frac{M_\pi^2}{2} J^- \right\}$$
$$= 13.69 \pm 0.12 \pm 0.15$$

$$J^- = \frac{1}{4\pi^2} \int_0^\infty dk \frac{\sigma_{\pi^- p}^{\text{tot}}(k) - \sigma_{\pi^+ p}^{\text{tot}}(k)}{\sqrt{M_\pi^2 + k^2}}$$

- $J^-$  known very accurately

Ericson et al. 2002, Abaev et al. 2007

- other determinations

	de Swart et al. 97	Arndt et al. 94	Ericson et al. 02	Bugg et al. 73	KH80
method	NN	$\pi N$	GM0	$\pi N$	$\pi N$
$g^2/4\pi$	$13.54 \pm 0.05$	$13.75 \pm 0.15$	$14.11 \pm 0.20$	$14.30 \pm 0.18$	14.28

- With KH80 scattering lengths  $g^2/4\pi = 14.28$  is reproduced exactly

$\hookrightarrow$  discrepancy related to old scattering length values

# Spin-independent WIMP–nucleon scattering

- Effective Lagrangian

$$\mathcal{L} = C_{qq}^{SS} \frac{m_q}{\Lambda^3} \bar{\chi} \chi \bar{q} q + C_{qq}^{VV} \frac{m_q}{\Lambda^2} \bar{\chi} \gamma^\mu \chi \bar{q} \gamma_\mu q + \bar{C}_{gg}^S \frac{\alpha_S}{\Lambda^3} \bar{\chi} \chi G_{\mu\nu} G^{\mu\nu}$$

- WIMP  $\chi$  **Dirac fermion** and **SM singlet**
- Spin-independent cross section at vanishing momentum transfer

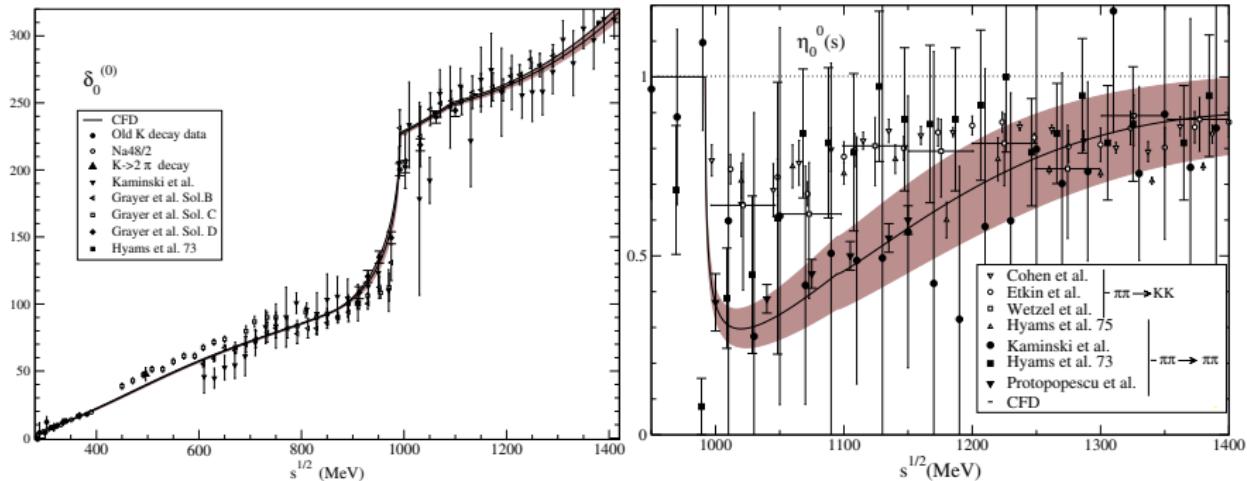
$$\sigma_N^{SI} = \frac{\mu_\chi^2}{\Lambda^4} \left| \left( \frac{m_N}{\Lambda} C_{qq}^{SS} f_q^N - 12\pi C_{gg}^S f_Q^N \right) + C_{qq}^{VV} f_V^N \right|^2$$

$$\mu_\chi = \frac{m_\chi m_N}{m_\chi + m_N} \quad f_q^N = 2 \quad f_q^N = \frac{\sigma_{\pi N}(1-\xi)}{m_N} + \Delta f_q^N$$

- nucleon-matrix elements dominated by  $\sigma_{\pi N}$

# Roy-equations: $\pi\pi$ results

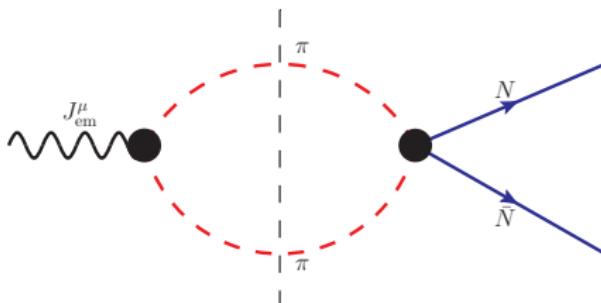
Solution for the  $\pi\pi$  S0-wave



Garcia-Martin, Kaminski, Pelaez, JRE (2011)

- Proton Radius Puzzle

↳ strong constraints from analyticity and unitarity



- first inelastic correction  $\hookrightarrow \pi\pi$  continuum, rigorous constraint fixed from:

- ▷ RS t-channel partial waves
  - ▷ pion form factor

- update of Höhler spectral functions, including also isospin breaking

criticism by Lee et al. 2015