Hadronic matrix elements and distribution amplitudes from lattice QCD

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Outline

Benchmark quantity: $g_A$

Isovector nucleon form factors:

Electromagnetic: $G_E(Q^2), G_M(Q^2) \to$ large $Q^2$ behaviour.

Axial: $G_A(Q^2), \tilde{G}_P(Q^2) \to$ neutrino oscillation experiments.

Isoscalar nucleon form factors:

$G_A^s(Q^2), G_A^s(0) = \Delta s + \Delta \bar{s} \to$ contributions of strangeness to properties of nucleon.

Pion form factor: electromagnetic, $F_\pi(Q^2) \to$ large $Q^2$ behaviour to be studied at JLab Hall C to 6 GeV$^2$.

Distribution amplitudes:

Pion and nucleon $\to$ form factors at large $Q^2$.

Conventional (moments) + direct approaches.

Summary/Outlook. Many results are preliminary from Lattice 2016!
General considerations: $\langle N | \bar{q} \Gamma q | N \rangle$

(Isospin symmetric limit) Isovector combinations only connected. Isoscalar also disconnected.

Systematics:

- Excited state pollution.
- Renormalisation+ improvement: for $\vec{p} = \vec{p}' = 0$ / some operators/actions $c_\mathcal{O} = 0$ or $b_\mathcal{O} = 0$

$$\mathcal{O}^{\overline{\text{MS}}} (\mu) = Z^{\overline{\text{MS}}, \text{latt}} (a_\mu) \left[ (1 + b_\mathcal{O} am_q) \mathcal{O}^{\text{latt}} + ac_\mathcal{O} \mathcal{O}^{\text{latt}}_1 \right]$$

- Volume: exponentially suppressed $\sim e^{-Lm_\pi}, \quad Lm_\pi > 4$.
- Discretisation effects: $\mathcal{O}(a)$ or $\mathcal{O}(a^2)$.
- Physical point extrapolation: chiral pert. (inspired) $m_\pi \rightarrow m_\pi^{\text{phys}}$. 
Points were obtained by different collaborations, using different lattice actions and with different systematics.

Taking physical quark mass, infinite volume and continuum limits involves a limited subset.
Axial charge $g_A = \Delta u - \Delta d$

$\beta$-decay, $g_A/g_V = 1.2723(23)$ PDG 2015.

Benchmark quantity sensitive to systematics.

Status Lattice 2016
Isovector charges $g_A$

Several $m_\pi < 165$ MeV results.

Impose $Lm_\pi > 4$, $a < 0.1$ fm
Axial charge $g_A$

First study with continuum extrapolation. Volume, $m_\pi$ dependence and excited state contamination also investigated.

PNDME: 1606.07049

Final result: $g_A = 1.195(33)(20)$
Isovector charges $g_A$

**CLS:** $N_f = 2 + 1$ NP clover

Physical point along:

$2m_l + m_s = \text{const.}$ and $m_s = \text{const.}$

$a = 0.039 - 0.085$ fm
Nucleon isovector form factors

\[ \langle p(p_f)| V_{\mu}^{u-d} | p(p_i) \rangle = \bar{u}_p(p_f) \left[ F_1^\gamma(Q^2) \gamma_\mu + \frac{F_2^\gamma(Q^2)}{2m_N} \sigma_{\mu\nu} Q^\nu \right] u_p(p_i) \]

\[ \langle p(p_f)| A_{\mu}^{u-d} | p(p_i) \rangle = \bar{u}_p(p_f) \left[ G_A^\gamma(Q^2) \gamma_\mu - i \frac{\tilde{G}_P^\gamma(Q^2)}{2m_N} Q_\mu \right] \gamma_5 u_p(p_i) \]

Sachs ff.:

\[ G_E^\gamma(Q^2) = F_1^\gamma(Q^2) - \frac{Q^2}{4m_N^2} F_2^\gamma(Q^2), \quad G_M^\gamma(Q^2) = F_1^\gamma(Q^2) + F_2^\gamma(Q^2) \]

Forward limit:

\[ G_E^\gamma(0) = F_1^\gamma(0) = 1, \quad G_A^\gamma(0) = g_A, \quad G_M^\gamma(0) = 1 + F_2^\gamma(0) = \mu_{p-n} = 1 + \kappa_{p-n} = 4.79 \]

Shape at low \( Q^2 \), \( \langle r_X^2 \rangle = -6 \frac{dG_X(Q^2)}{dQ^2} \): different probe \( \rightarrow \) different radius.

\[ G_X(Q^2) = G_X(0) \left[ 1 - \frac{1}{6} \langle r_X^2 \rangle Q^2 + \ldots \right] \]
Challenges for the lattice:

Low $Q^2$

Achieving low $Q^2$, (conventionally) $\rightarrow$ large $L$, $ap = (2\pi n/L)$.

Very sensitive to $m_\pi$: radii diverge as $m_\pi \rightarrow 0$.

Parameterising $\rightarrow$ dipole form, $z$ expansion etc $\rightarrow \langle r^2_E \rangle, \langle r^2_A \rangle$.

Extrapolation $\rightarrow \tilde{G}_P^\gamma(0), G_M^\gamma(0), \langle r^2_M \rangle$.

High $Q^2$

Achieving high $Q^2$, signal/noise deteriorates.

Discretisation effects: $O(ap)$ and $O(a^2 p^2)$.

However, less sensitive to $m_\pi$
Electromagnetic form factors

[Preprint,1503.01452]

Proton charge radius:

Large $Q^2$:

Radius: would need $< 2\%$ error with all systematics included.

Compute isovector form factors: comparing with $\langle r_E^2 \rangle^v = \langle r_E^2 \rangle^{p-n}$.
$G_E^V(Q^2)$ and $G_M^V(Q^2)$

**LHPC [Green,1404.4029]:** $N_f = 2 + 1$, $a = 0.12$ fm, $L = 5.6$ fm, $Lm_\pi = 4.2$, $m_\pi = 0.149$ MeV.

<table>
<thead>
<tr>
<th>$m_\pi$</th>
<th>$\langle r_1^2 \rangle / \text{fm}^2$</th>
<th>$\kappa^{p-n}$</th>
<th>Extrap</th>
<th>Expt</th>
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<tbody>
<tr>
<td></td>
<td>149 MeV</td>
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<td>0.498(55)</td>
<td>3.76(38)</td>
<td>3.68(38)</td>
<td>0.605(27)</td>
<td>0.640(9) or 0.578(2)</td>
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$Q^2 [\text{GeV}^2]$
$G_E(Q^2)$ and $G_M(Q^2)$

**Lattice 2016:** $a \sim 0.09$ fm, $m_\pi = 131 - 145$ MeV

- $N_f = 2$ ETMC $Lm_\pi = 3$ $L = 4.5$ fm
- $N_f = 2 + 1$ PNDME $Lm_\pi = 4$ $L = 5.6$ fm
- $N_f = 2 + 1$ PACS $Lm_\pi = 6$ $L = 8.1$ fm
$G^v_E(Q^2)$ and $G^v_M(Q^2)$ at large $Q^2$

CSSM/QCDSF/UKQCD [Chambers,1511.07090]: Feynman-Hellmann approach.

For connected matrix elements:

Add term to Lagrangian when computing quark propagators (ensemble generation unchanged).

\[ \mathcal{L}(x) \to \mathcal{L}(x) + \lambda \mathcal{O}, \quad \mathcal{O} = 2(\cos \bar{q} \cdot \bar{x}) V_\mu(x) \quad V_\mu = \bar{q} \gamma_\mu q. \]

Determine $E_N$ as a function of $\lambda$ from the two-point function:

\[ C_{2\text{pt}}(\bar{p}, t) = \sum_{\bar{x}} e^{-i\bar{p} \cdot \bar{x}} \langle N(\bar{x}, t) \bar{N}(\bar{0}, 0) \rangle \propto e^{-E_N(\lambda, \bar{p}) t} \]

Feynman-Hellmann theorem then gives:

\[ \frac{\partial E_N}{\partial \lambda} \bigg|_{\lambda=0} \propto \langle N(-\bar{p})|V_\mu|N(\bar{p})\rangle \propto G^q_E(Q^2), G^q_M(Q^2) \]
$G_E^V(Q^2)$ and $G_M^V(Q^2)$ at large $Q^2$

Lattice 2016 Chambers

$m_\pi = 470$ MeV, $a = 0.07$ fm.
Nucleon axial form factor $G_A(Q^2)$

Previously, [Lin,0802.0863], [Yamazaki,0904.2039], [Bratt,1001.3620], [Bali,1412.7336]

Needed for neutrino oscillation experiments:

Charged current quasielastic (CCQE) neutrino-nucleus interaction must be known to high precision.

Connecting quark - nucleon level: $G_A(Q^2)$ form factor.

nucleon - nucleus level: nuclear model.

Traditionally: information on $G_A(Q^2)$ extracted from expt. using dipole fit:

$$G_A(Q^2) = \frac{g_A}{(1 + \frac{Q^2}{M_A^2})^2} \quad \langle r_A^2 \rangle = \frac{12}{M_A^2}$$

World average (pre 1990) from $\nu$ scattering $M_A = 1.026(21)$ GeV.

Overconstrained form: different measurements, different $M_A$.

Lower energy expts: e.g. MiniBooNE: $M_A = 1.35(17)$ GeV

Systematics being explored including new analysis of old expt data:

$\langle r_A^2 \rangle = 0.46(22)$ fm$^2 \rightarrow M_A = 1.01(24)$ GeV from z-expansion [Meyer,1603.03048].
Nucleon axial form factor $G_A(Q^2)$

Lattice 2016: Mainz, $N_f = 2$ $m_\pi = 195 - 450$ MeV, $Lm_\pi \gtrsim 4$

\[ \langle r_A^2 \rangle \] extracted using z-expansion.

Expt: dipole form with $M_A = 1.02$. 
Nucleon axial form factor $G_A(Q^2)$

Lattice 2016: PNDME, $N_f = 2 + 1 + 1$, $a = 0.06 - 0.12$ fm, $m_\pi = 135 - 315$ MeV

Small dependence of $\langle r_A^2 \rangle$ on lattice spacing (dipole fit).
Nucleon induced pseudoscalar form factor $\tilde{G}_P(Q^2)$

Not relevant for neutrino oscillations: enters cross-section with $m_l^2/m_N^2$.

Not well known from expt: muon capture $\mu^- p \rightarrow \nu_\mu n$ gives

$$g^*_P = \frac{m_\mu}{m_N} \tilde{G}_P(Q^2 = 0.88 \ m_\mu^2) = 8.06(55) \ [\text{MuCap},1210.6545]$$

PCAC relation: $m_{ud} G_P(Q^2) = m_N G_A(Q^2) - \frac{Q^2}{4m_N} \tilde{G}_P(Q^2)$

where $\langle p(p_f) | P^{u-d} | p(p_i) \rangle = \bar{u}_p(p_f) G_P(Q^2) \gamma_5 u_p(p_i)$

Pion pole: $\tilde{G}_P(Q^2) = \frac{4c_N^2}{m_\pi^2 + Q^2} G_A(Q^2) + \cdots$, $c_N \rightarrow m_N$ as $m_\pi \rightarrow 0$. 
Nucleon induced pseudoscalar form factor $\tilde{G}_P(Q^2)$

RQCD [Bali,1412.7336]

Lattice 2016 PNDME

Left: pion pole

Right: $\tilde{G}_P(Q^2)/g_A = \frac{4m_N^2}{m^2 + Q^2} \left(1 - \frac{1}{1 + Q^2/M_A^2}\right)$
Isoscalar axial form factor

Lattice 2016 LHPC: \( N_f = 2 + 1, \ m_\pi = 317 \) MeV, \( a = 0.11 \) fm.

- Data points: statistical errors only. Fits using z-expansion, stat.+sys. error shown.
- \( G_A^s \) and \( G_P^s \) are small.
- \( G_P^{u+d} \): cancellation between connected and disconnected. Pole determined by nearest isoscalar \( \rightarrow \eta \).
Strangeness contribution to the spin of the nucleon

Ji decomposition: \[ \frac{1}{2} = \frac{1}{2} \Delta \Sigma + \sum_{q, \bar{q}} L_q + J_g \]

Longitudinal quark spins \[ \Delta \Sigma = \Delta u + \Delta \bar{u} + \Delta d + \Delta \bar{d} + \Delta s + \Delta \bar{s} + \cdots \]

C.f. from SIDIS [COMPASS,1001.4654] DSSV: [de Florian,0904.3821]

<table>
<thead>
<tr>
<th>((\Delta s + \Delta \bar{s})(5 \text{GeV}^2))</th>
<th>(-0.02 \pm 0.02 \pm 0.02)</th>
<th>(-0.10 \pm 0.02 \pm 0.02)</th>
</tr>
</thead>
</table>

Naive Extrap. combined with DSSV
Pion electromagnetic form factor

\[ \langle \pi(p_f) | V_\mu | \pi(p_i) \rangle = F_\pi(Q^2)(p_f + p_i)_\mu \]

[Huber,0809.3052]

Expt: JLab Hall C Fpi12, charged pion form factor to 6 GeV^2.
Pion electromagnetic form factor: Low $Q^2$

\[ F_\pi(Q^2) = Q^2 \ [\text{GeV}^2] \]

**HPQCD** $N_f = 2 + 1 + 1$:
- $m_\pi \sim 133$ MeV,
- $a = 0.088 - 0.15$ fm,
- $Lm_\pi = 3.3 - 3.9$.

**ETMC** Lattice 2016:
- $m_\pi \sim 131$ MeV,
- $a = 0.093$ fm,
- $Lm_\pi = 3$ and 4.

**PACS** Lattice 2016:
- $m_\pi = 145$ MeV,
- $a = 0.085$ fm,
- $Lm_\pi = 6$. 
Pion electromagnetic form factor: Low $Q^2$

\[ \langle r^2 \rangle_v \text{ [fm}^2\text{]} \]

\[ m^2 \pi \text{ [GeV}^2\text{]} \]

\[ \langle r^2 \rangle_\pi \text{ [fm}^2\text{]} \]

\[ N_f = 2 + 1 \]

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\[ N_f = 2 \]

\[ N_f = 2 \]

Owen
ETMC
Mainz
JLQCD
RBC/UKQCD
PACS
Expt

HPQCD (2015)
PACS-CS (Lattice 2016)
JLQCD (2015)
PACS-CS (2011)
RBC/UKQCD (2008)
Mainz (2013)
QCDSF (2006)
JLQCD/TWQCD (2009)
ETMC (2008)
Pion electromagnetic form factor: High $Q^2$

CSSM/UKQCD/QCDSF [Chambers,1511.07090]: Feynman-Hellmann approach.

$m_\pi = 470$ MeV, $a = 0.07$ fm.
Meson and baryon distribution amplitudes

Intuitive picture of a hadron in the infinite momentum frame: superposition of Fock states with different numbers of quarks and gluons

\[ |B\rangle = |qqq\rangle + |qqqg\rangle + |qqq\bar{q}q\rangle + \ldots, \]

\[ |M\rangle = |\bar{q}q\rangle + |\bar{q}qg\rangle + |\bar{q}q\bar{q}q\rangle + \ldots. \]

Within Fock state decomposition: (light cone) distribution amplitudes (DAs)

→ Hadron wave functions at small transverse distances of the constituents
→ describe the distribution of lightcone momentum.

In hard exclusive processes higher Fock states are power-suppressed → at high momentum transfer the valence contribution plays the most important role.

DAs complementary to parton distribution functions (PDFs)

PDFs: single-particle probabilities (or densities).

“Directly” extracted from fits to DIS and SIDIS data.

DAs: amplitudes (or wavefunctions).

Appear always in convolutions in expressions for hard exclusive processes. Difficult to extract from experiment without contamination from other hadronic uncertainties.
Meson and baryon distribution amplitudes

**DA$s$:** nonperturbative input for the theoretical description of hard exclusive processes.

**Collinear factorisation at large $Q^2$.**

**Pion:** $\gamma\gamma^* \rightarrow \pi$ form factor

\[
F_{\pi\gamma}(Q^2) = \frac{2f_\pi}{3} \int_0^1 dx \, T_{\gamma\pi}^H(x, Q^2) \phi_\pi(x, Q^2)
\]

[**Wu,1206.0466**]
Pion distribution amplitudes

DA’s: non-local matrix elements involving light-like quark separation,

\[
\langle 0| \bar{d}(-z)\gamma_\mu \gamma_5 [-z, z] u(z) |\pi^+ \rangle = i f_\pi p_\mu \int_{-1}^{1} d\xi e^{-i\xi p \cdot z} \phi_\pi (\xi, \mu),
\]

\( f_\pi \) is the pion decay constant, \( z \) is a light-like vector, \( \xi = 2x - 1 \) and \([-z, z]\) is a Wilson line connecting the \( u \) and \( \bar{d} \) fields.

Moments of DA’s: are related to matrix elements of leading twist local operators.

\[
\langle \xi^n \rangle = \int d\xi \xi^n \phi(\xi, \mu)
\]

Gegenbauer expansion:

\[
\phi(\xi, \mu) = \frac{3}{4} \left( 1 + \sum_{n=1}^{\infty} a_\pi^n(\mu) C_n^{3/2}(\xi) \right)
\]

[Collinear conformal symmetry, SL(2, \( \mathbb{R} \)), \( C_n^{3/2}(\xi) \) analogous to \( Y^{lm}(\theta, \phi) \) in O(3).]

\( \langle \xi^n \rangle \) and \( a_\pi^n \) related by simple algebraic expressions.
Meson and baryon distribution amplitudes

DAs are universal: involved in many processes

Pion:

Pion form factor to 6 GeV$^2$ \((\text{JLab Hall C})\)

Weak exclusive \(B\) and \(\Lambda_b\) decays \((\text{LHCb})\).

Nucleon:

Electric and magnetic nucleon form factor \(Q^2 \sim 14\ \text{GeV}^2\) \((\text{JLab, FAIR})\)

Electric neutron form factor \(Q^2 \sim 8\ \text{GeV}^2\) \((\text{JLab, FAIR})\)

Electroproduction of nucleon resonances at large \(Q^2 \sim 14\ \text{GeV}^2\) \((\text{JLab})\)
Pion distribution amplitude

RQCD: [Braun,1503.03656], $N_f = 2$, $m_\pi = 150 - 490$ MeV, $L m_\pi = 3.4 - 6.7$.

$$\phi_\pi(x) = 6x(1 - x) \left[ 1 + a_2^\pi(\mu) C_2^{3/2} (2x - 1) \right], \quad \xi = 2x - 1$$

RQCD [Braun,1503.03656]
RBC/UKQCD [Arthur,1011.5906]
QCDSF/UKQCD [Braun,hep-lat/0606012]

$\overline{a}_2^{\overline{\text{MS}}}(2 \text{ GeV}) = 0.136(15)(15)$,
$\overline{a}_2^{\overline{\text{MS}}}(2 \text{ GeV}) = 0.233(30)(60)$,
$\overline{a}_2^{\overline{\text{MS}}}(2 \text{ GeV}) = 0.211(114)$
Nucleon

RQCD: [Braun,1403.4189], $N_f = 2$, $m_\pi = 150 - 490$ MeV, $Lm_\pi = 3.4 - 6.7$.

Leading twist distribution amplitude:

$$
\Phi_3(x_i, \mu) = 120 f_N(\mu) x_1 x_2 x_3 \left\{ 1 + \phi_{10}(\mu)(x_1 - 2x_2 + x_3) \\
+ \phi_{11}(\mu)(x_1 - x_3) + \phi_{20}(\mu)[1 + 7(x_2 - 2x_1 x_3 - 2x_2^2)] \\
\phi_{21}(\mu)(1 - 4x_2)(x_1 - x_3) + \phi_{22}(\mu)[3 - 9x_2 + 8x_2^2 - 12x_1 x_3] + \ldots \right\}
$$

Momentum fractions in the proton:

$$
\langle x_1 \rangle = 0.372(7), \quad \langle x_2 \rangle = 0.314(3), \quad \langle x_3 \rangle = 0.314(7).
$$
Baryon octet

RQCD: [Bali,1512.02050], $N_f = 2 + 1$, $a = 0.086$ fm, $m_\pi = 220 - 420$ MeV.

Barycentric plots ($x_1 + x_2 + x_3 = 1$) showing deviation from asymptotic shape $\Phi^{as} \equiv 120x_1x_2x_3$.

$N$, $u \uparrow u \downarrow d \uparrow$  
$\Lambda$, $u \uparrow d \downarrow s \uparrow$  
$\Sigma$, $d \uparrow d \downarrow s \uparrow$  
$\Xi$, $s \uparrow s \downarrow u \uparrow$

$N$, $\Sigma$, $\Xi$: $q_1$ favoured over $q_2$, strange quarks carry more momentum.  
$\Lambda$: maximum of distribution shifted to strange quark.
Distribution amplitudes from a position space method


Example: pion

\[ T = \langle \pi^0(p) | \bar{q}(y/2) \Gamma_1 q(y/2) \bar{q}(-y/2) \Gamma_2 q(-y/2) | 0 \rangle \]

with \( \Gamma_1 = S = 1 \), \( \Gamma_2 = P = \gamma_5 \). To leading order and leading twist:

\[ T = \frac{y \cdot p}{\pi^2 y^4} f_\pi \int_0^1 dx \ e^{i(2x-1)p \cdot y/2} \phi_\pi(x, \mu) = \frac{y \cdot p}{\pi^2 y^4} \mathcal{F}(p \cdot y) \]

Evaluate: \( C^{(p)}_{\pi \text{SP}} \)

\[ \mathcal{F}(p \cdot y, t) \propto Z_S Z_P \frac{\pi^2 y^4}{y \cdot p} \frac{C^{(p)}_{\pi \text{SP}}(t)}{\sqrt{2E_\pi}} e^{E_\pi t/2} \]

Pion two-point function: \( C^{(p)}_{\pi}(t) \)

Local operator renormalisation factors: \( Z_S, Z_P \).
Achieving a good signal at high momenta

RQCD [Bali,1602.05525]:

Quark smearing that maintains small statistical errors and good overlaps with ground state wavefunctions.

Test case: $N_f = 2$, $m_\pi = 295$ MeV, $a = 0.07$ fm. Momenta up to 2.8 GeV achieved.
Pion distribution amplitude from a position space method


Test case: $N_f = 2$, $m_\pi = 295$ MeV, $a = 0.07$ fm.
Summary and Outlook

Benchmark quantities:

- First calculations with main systematics considered (continuum limit, finite $V$, physical point results, excited states, ...). More in progress.
- The same effort needs to be applied to $\langle x \rangle_{u-d}$.

Efficient methods for achieving statistical precision + $m_{\pi}^{\text{phys}}$ simulations.

Impact on:

- Form factor determinations at low $Q^2$, $F_\pi(Q^2)$, $G_{E,M}(Q^2)$, $G_A(Q^2)$, $\tilde{G}_P(Q^2)$.
- High $Q^2$ also being explored.

Calculations of individual quark contributions:

- Strangeness axial form factors, $\Delta s + \Delta \bar{s}$, also at the physical point.

Distribution amplitudes:

- Provides information complementary to PDFs.
- Lower moments computed for mesons and baryons.
- Feasibility study of direct determination.