Hadronic light-by-light contribution to muon $g - 2$: dispersive approach

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2 Lorentz structure of the HLbL tensor
3 Master formula for \((g - 2)_\mu\)
4 Mandelstam representation
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Magnetic moment

- relation of spin and magnetic moment of a lepton:

\[ \vec{\mu}_\ell = g_\ell \frac{e}{2m_\ell} \vec{s} \]

- Dirac’s prediction: \( g_e = 2 \)

- anomalous magnetic moment: \( a_\ell = (g_\ell - 2)/2 \)

- helped to establish QED and QFT as the framework for elementary particle physics

- today: probing not only QED but entire SM
\[(g - 2)_\mu: \text{comparison of theory and experiment}\]

![Graph showing comparison of theory and experiment](image)

- HMNT (06)
- JN (09)
- Davier et al, \(\tau\) (10)
- Davier et al, \(e^+e^-\) (10)
- JS (11)
- HLMNT (10)
- HLMNT (11)

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- Experiment
- BNL
- BNL (new from shift in \(\lambda\))

\[a_\mu \times 10^{10} - 11659000\]

\[\rightarrow\] Hagiwara et al. 2012
\((g - 2)_{\mu}\): theory vs. experiment

- discrepancy between SM and experiment \(\sim 3\sigma\)
- hint to new physics?
- new experiments (FNAL, J-PARC) aim at reducing the experimental error by a factor of 4
- theory error completely dominated by hadronic effects
- hadronic vacuum polarisation responsible for largest uncertainty, but will be systematically improved with better data input
Hadronic light-by-light (HLbL) scattering

- up to now only model calculations
- uncertainty estimate based rather on consensus than on a systematic method
- will dominate theory error in a few years
Model calculations of HLbL

Table 13
Summary of the most recent results for the various contributions to $a_{\mu}^{LbL;had} \times 10^{11}$. The last column is our estimate based on our new evaluation for the pseudoscalars and some of the other results.

<table>
<thead>
<tr>
<th>Contribution</th>
<th>BPP</th>
<th>HKS</th>
<th>KN</th>
<th>MV</th>
<th>BP</th>
<th>PdRV</th>
<th>N/JN</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0, \eta, \eta'$</td>
<td>85±13</td>
<td>82.7±6.4</td>
<td>83±12</td>
<td>114±10</td>
<td>–</td>
<td>114±13</td>
<td>99±16</td>
</tr>
<tr>
<td>$\pi, K$ loops</td>
<td>–19±13</td>
<td>–4.5±8.1</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–19±19</td>
<td>–19±13</td>
</tr>
<tr>
<td>$\pi, K$ loops + other subleading in $N_c$</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0±10</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>axial vectors</td>
<td>2.5±1.0</td>
<td>1.7±1.7</td>
<td>–</td>
<td>22±5</td>
<td>–</td>
<td>15±10</td>
<td>22±5</td>
</tr>
<tr>
<td>scalars</td>
<td>–6.8±2.0</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–7±7</td>
<td>–7±2</td>
</tr>
<tr>
<td>quark loops</td>
<td>21±3</td>
<td>9.7±11.1</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>2.3</td>
<td>21±3</td>
</tr>
<tr>
<td>total</td>
<td>83±32</td>
<td>89.6±15.4</td>
<td>80±40</td>
<td>136±25</td>
<td>110±40</td>
<td>105±26</td>
<td>116±39</td>
</tr>
</tbody>
</table>

→ Jegerlehner, Nyffeler 2009

- pseudoscalar pole contribution most important
- pion-loop second most important
- differences between models, large uncertainties
How to improve HLbL calculation?

- lattice QCD making progress
- dispersive approach
Dispersive approach to HLbL

- make use of fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- relate HLbL to experimentally accessible quantities
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The HLbL tensor: definitions

- hadronic four-point function:
  \[ \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int dxdydz e^{-i(q_1 x + q_2 y + q_3 z)} \langle 0 | T j_{em}^\mu(x) j_{em}^\nu(y) j_{em}^\lambda(z) j_{em}^\sigma(0) | 0 \rangle \]

- EM current:
  \[ j_{em}^\mu = \sum_{i=u,d,s} Q_i \bar{q}_i \gamma^\mu q_i \]

- Mandelstam variables:
  \[ s = (q_1 + q_2)^2, \quad t = (q_1 + q_3)^2, \quad u = (q_2 + q_3)^2 \]

- Ward identities: \[ \{ q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma \} \Pi_{\mu\nu\lambda\sigma} = 0 \]
Solution for the Lorentz decomposition, following a recipe by Bardeen, Tung (1968) and Tarrach (1975):

\[
\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)
\]

- Lorentz structures manifestly gauge invariant
- Crossing symmetry manifest: only 7 distinct structures, 47 follow from crossing
- Scalar functions \( \Pi_i \) free of kinematic singularities
  \( \Rightarrow \) ideal quantities for a dispersive treatment
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Master formula: contribution to $(g - 2)_\mu$

- from gauge invariance:
  \[
P_{\mu\nu\lambda\rho} = -q_4^\sigma \frac{\partial}{\partial q_4^\rho} P_{\mu\nu\lambda\sigma}
  \]

- for $(g - 2)_\mu$: afterwards take $q_4 \to 0$

- no kinematic singularities in scalar functions: perform these steps with the derived Lorentz decomposition

- only 12 linear combinations of the scalar functions $\Pi_i$ contribute to $(g - 2)_\mu$
Master formula: contribution to \((g - 2)_\mu\)

\[
a^{\text{HLbL}}_\mu = e^6 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \left( \sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2) \right) \frac{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m^2_\mu][(p - q_2)^2 - m^2_\mu]}{q_1 q_2 (q_1 + q_2)^2}
\]

- \(\hat{T}_i\): known integration kernel functions
- five loop integrals can be performed with Gegenbauer polynomial techniques
- Wick rotation possible even in the presence of anomalous thresholds
Master formula for \((g - 2)_\mu\)

Master formula: contribution to \((g - 2)_\mu\)

\[
a^\text{HLbL}_\mu = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1 - \tau^2} Q_1^3 Q_2^3 \times \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau),
\]

- \(T_i\): known integration kernels
- \(\bar{\Pi}_i\): linear combinations of the scalar functions \(\Pi_i\)
- Euclidean momenta: \(Q_i^2 = -q_i^2\)
- \(Q_3^2 = Q_1^2 + Q_2^2 + 2Q_1 Q_2 \tau\)
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Analytic properties of scalar functions

- right- and left-hand cuts in each Mandelstam variable
- double-spectral regions (box topologies)
- anomalous thresholds for large photon virtualities
Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

\[
\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \Pi_{\mu\nu\lambda\sigma} + \ldots
\]
Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \Pi_{\mu\nu\lambda\sigma} + \ldots \]

one-pion intermediate state:
Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \Pi_{\mu\nu\lambda\sigma} + \ldots \]

two-pion intermediate state in both channels:
Mandelstam representation

• we limit ourselves to intermediate states of at most two pions

• writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \Pi_{\mu\nu\lambda\sigma}^{\bar{\Pi}} + \ldots \]

two-pion intermediate state in first channel:
Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \tilde{\Pi}_{\mu\nu\lambda\sigma} + \ldots$$

neglected so far: higher intermediate states
Pion pole

- input: doubly-virtual and singly-virtual pion transition form factors $\mathcal{F}_{\gamma^*\gamma^*\pi^0}$ and $\mathcal{F}_{\gamma^*\gamma\pi^0}$
- dispersive analysis of transition form factor:

→ Hoferichter et al., EPJC 74 (2014) 3180
Pion box

- simultaneous two-pion cuts in two channels
- Mandelstam representation explicitly constructed

\[ \Pi_i = \frac{1}{\pi^2} \int ds' dt' \frac{\rho_{st}^{st}(s', t')}{(s' - s)(t' - t)} + (t \leftrightarrow u) + (s \leftrightarrow u) \]

- \(q^2\)-dependence: pion vector form factors \(F_V^\pi(q_i^2)\) for each off-shell photon factor out
Pion box

- sQED loop projected on BTT basis fulfils the same Mandelstam representation
- only difference are factors of $F_V^\pi$
- $\Rightarrow$ box topologies are identical to FsQED:

\[ \equiv F_\pi^V (q_1^2) F_\pi^V (q_2^2) F_\pi^V (q_3^2) \]

- model-independent definition of pion loop
Pion box

Pion vector form factor in the space-like region:

\[ |F|^2 \]

\begin{align*}
\text{NA7 (1986)} & \quad \text{ETMC - quadratic fit} \\
\text{ETMC - logarithmic fit} & \quad \text{Volmer et al. (Fpi coll.) (2001)} \\
\text{Our fit} & \quad \text{VMD}
\end{align*}

Preliminary results:

\[ a_{\mu}^{\pi-\text{box}} = -15.9 \cdot 10^{-11}, \quad a_{\mu}^{\pi-\text{box}, \text{VMD}} = -16.4 \cdot 10^{-11} \]
Mandelstam representation

Pion-box saturation with photon virtualities

![Graph showing pion-box saturation in percentage vs. cutoff on the virtualities in GeV.](image)
Rescattering contribution

- neglect left-hand cut due to multi-particle intermediate states in crossed channel
- two-pion cut in only one channel
- unitarity relates it to the helicity amplitudes of the subprocess \( \gamma^* \gamma^{(*)} \rightarrow \pi\pi \)
- imaginary part expanded into partial waves
Rescattering contribution

- fixed-$s/t/u$ dispersion relations allow to reconstruct $\bar{\Pi}_{\mu\nu\lambda\sigma}$
- sum rules ensure cancellation of unphysical helicity amplitudes
- $S$- and $D$-waves: contain model-independent description of scalar and tensor contributions
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Conclusion and outlook

Summary

- our dispersive approach to HLbL scattering is based on fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- we take into account the lowest intermediate states: $\pi^0$-pole and $\pi\pi$-cuts
- relation to experimentally accessible (or again with data dispersively reconstructed) quantities
- a step towards a model-independent calculation of $\alpha_\mu$
Conclusion and outlook

A roadmap for HLbL

$e^+e^- \rightarrow e^+e^- \pi^0$

$e^+e^- \rightarrow \pi^0\gamma$

Pion transition form factor $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$

$e^+e^- \rightarrow 3\pi$

$\omega, \phi \rightarrow 3\pi$

$\omega, \phi \rightarrow \pi^0\gamma^*$

$\gamma\pi \rightarrow \pi\pi$

$\pi\pi \rightarrow \pi\pi$

Pion vector form factor $F^\pi_V$

$\pi\pi \rightarrow \pi\pi$

Partial waves for $\gamma^*\gamma^* \rightarrow \pi\pi$

$e^+e^- \rightarrow \pi\pi\gamma$

$e^+e^- \rightarrow e^+e^-\pi\pi$

$\gamma\pi \rightarrow \gamma\pi$

→ Flowchart by M. Hoferichter