Effective field theories for muonic hydrogen

Clara Peset

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Based on the work in collaboration with Antonio Pineda:
arxiv:1403.3408, 1406.4524, 1508.01948

Outline

Introduction

EFTs for bound states

Dynamics of NR systems

Effective Field Theories approach:

- model independent
- efficient
- systematic (power counting)
- unified framework to determine the nonperturbative effects

EFTs for bound states: NR systems

Non-relativistic systems fulfil the relation: $m_r \gg |\mathbf{p}| \gg E$

When bounded by QED, $\alpha \sim v$ is the only expansion parameter

Scales in bound state		Coulomb interaction
Hard scale: m_r	$\longrightarrow$	m_r
Soft scale: $\|\mathbf{p}\|$	$\longrightarrow$	$m_r\alpha$
Ultrasoft scale: E	$\longrightarrow$	$m_r\alpha^2$

Scales are well separated

We can integrate out the hard and soft scales to obtain **pNRQED**

It describes systems such as: positronium, muonium, hydrogen, muonic hydrogen, etc.

Muonic Hydrogen

The EFT for muonic hydrogen

Scales in bound state		Muonic hydrogen (Coulomb interaction)	
Hard scale: m_r	$\longrightarrow$	m_r	**well separated scales**
Soft scale: $\lvert\mathbf{p}\rvert$	$\longrightarrow$	$m_r\alpha$	
Ultrasoft scale: E	$\longrightarrow$	$m_r\alpha^2$	**pNRQED**

Scales in μH: $m_\rho \sim m_p,\ m_\mu \sim m_\pi \sim m_r,\ m_r\alpha \sim m_e$

small expansion parameters: $\frac{m_\pi}{m_p} \sim \frac{m_\mu}{m_p} \approx \frac{1}{9},\ \frac{m_e}{m_r} \sim \frac{m_r\alpha}{m_r} \sim \alpha \approx \frac{1}{137}$

Energy levels: $E(\mu p) = \frac{-m_r\alpha^2}{2n^2}(1 + c_1\alpha + c_2\alpha^2 + \dots)$

$c_1 \sim c_1\left[\frac{m_\mu\alpha}{m_e}\right]$ pure QED, and $c_n \sim \sum_{j=0}^{\infty} c_n^{(j)}\left(\frac{m_\pi}{m_p}\right)^j;\ c_n^{(j)} \sim c_n^{(j)}\left[\frac{m_r}{m_\mu}, \frac{m_\mu}{m_\pi}, \dots\right]$

pNRQED

- pNRQED is a theory for ultrasoft photons

$$\text{HBChPT} \overset{(m_{\pi/\mu},\Delta)}{\Longrightarrow} \text{NRQED} \overset{(m_\mu\alpha)}{\Longrightarrow} \text{pNRQED}\,.$$

- The pNRQED Lagrangian:

$$L_{\rm pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} S^\dagger(\mathbf{x},\mathbf{X},t)\Bigg\{ i\partial_0 - \frac{\mathbf{p}^2}{2m_r} + \frac{\mathbf{p}^4}{8m_\mu^3} + \frac{\mathbf{p}^4}{8m_p^3} - \frac{\mathbf{P}^2}{2M}$$
$$- V(\mathbf{x},\mathbf{p},\sigma_1,\sigma_2) + e\left(\frac{Z_\mu m_p + Z_p m_\mu}{m_p + m_\mu}\right)\mathbf{x}\cdot\mathbf{E}(\mathbf{X},t)\Bigg\} S(\mathbf{x},\mathbf{X},t) - \int d^3\mathbf{X}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

- The potential:

$$V(\mathbf{x},\mathbf{p},\sigma_1,\sigma_2) = V^{(0)}(r) + V^{(1)}(r) + V^{(2)}(r) + \cdots\,,$$

$$V^{(n)} \propto \frac{1}{m_\mu^n}, \quad V^{(n,r)} \propto \frac{1}{m_\mu^n}\alpha^r + \text{expansions in small parameters}$$

Muonic Hydrogen

The Lamb shift and the proton radius

Muonic Hydrogen: Lamb shift and the proton radius

Theoretical prediction for the Lamb shift: $2S_{1/2} \to 2P_{1/2}$

CP, A. Pineda

$$\Delta E_{LS} = 206.0243(30) - 5.2270(7)\frac{r_p^2}{\mathrm{fm}^2} + 0.0455(125) + \mathcal{O}\left(m_\mu \alpha^5 \frac{m_\mu^3}{m_\rho^3}, m_\mu \alpha^6\right) \mathrm{meV}$$

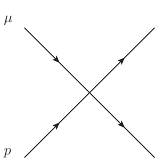
pure QED hadronic effects

Proton radius puzzle: CODATA vs μH values of r_p are $\sim 7\sigma$ away!

$$c_{D,\overline{\mathrm{MS}}}^{(p)}(\nu) \equiv Z_p + \frac{4}{3}\frac{Z_p^3 \alpha}{\pi} \ln\left(\frac{m_p^2}{\nu^2}\right) + \frac{4}{3} r_p^2 m_p^2 + \mathcal{O}(\alpha^2).$$

Hadronic effects are encoded in $\delta^{(3)}(r)$-potential terms

$$\text{EFT} \begin{cases} \mathrm{H}: \ \mathcal{O}(m_e \alpha^6) \\ \mu\mathrm{H}: \ \mathcal{O}(m_\mu \alpha^4) \end{cases}$$



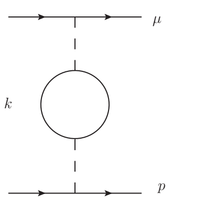
Pure QED corrections

Theoretical prediction for the Lamb shift

$$\Delta E_{LS} = \mathbf{206.0243(30)} - 5.2270(7)\frac{r_p^2}{\mathrm{fm}^2} + 0.0455(125) + \mathcal{O}\left(m_\mu \alpha^5 \frac{m_\mu^3}{m_\rho^3}, m_\mu \alpha^6\right) \mathrm{meV}$$

CP, A. Pineda

The LEADING CONTRIBUTION: electron vacuum polarization (VP)



$$V_{\rm VP} \sim V^{(0,1)}, \quad E_{\rm VP} \sim \mathcal{O}(m_\mu \alpha^3)$$

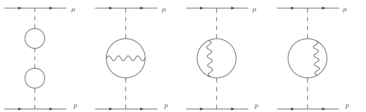
Pure QED corrections

Theoretical prediction for the Lamb shift

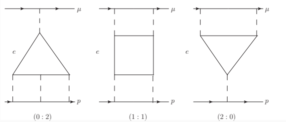
$$\Delta E_{LS} = \mathbf{206.0243(30)} - 5.2270(7)\frac{r_p^2}{\mathrm{fm}^2} + 0.0455(125) + \mathcal{O}\left(m_\mu \alpha^5 \frac{m_\mu^3}{m_\rho^3}, m_\mu \alpha^6\right)\,\mathrm{meV}$$

CP, A. Pineda

HIGHER ORDER CONTRIBUTIONS: 2 & 3 loop VP + light-by-light



$\mathcal{O}(m_r \alpha^4)$



$\mathcal{O}(m_r \alpha^5)$

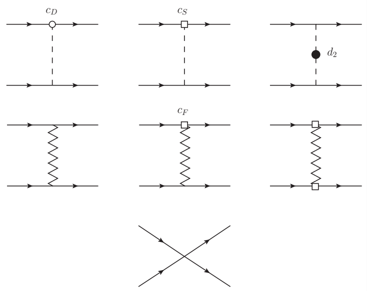
Pure QED corrections

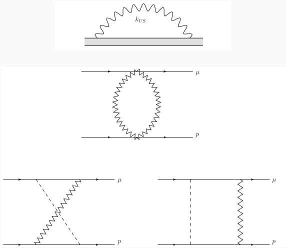
Theoretical prediction for the Lamb shift

$$\Delta E_{LS} = \mathbf{206.0243(30)} - 5.2270(7)\frac{r_p^2}{\mathrm{fm}^2} + 0.0455(125) + \mathcal{O}\left(m_\mu \alpha^5 \frac{m_\mu^3}{m_\rho^3}, m_\mu \alpha^6\right)\,\mathrm{meV}$$

CP, A. Pineda

RELATIVISTIC and ULTRASOFT corrections to $\mathcal{O}(m_r\alpha^5)$





Muonic Hydrogen Lamb shift: theoretical EFT prediction

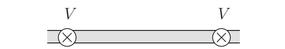
Pure QED corrections

Theoretical prediction for the Lamb shift

$$\Delta E_{LS} = \mathbf{206.0243(30)} - 5.2270(7)\frac{r_p^2}{\mathrm{fm}^2} + 0.0455(125) + \mathcal{O}\left(m_\mu\alpha^5\frac{m_\mu^3}{m_\rho^3}, m_\mu\alpha^6\right)\,\mathrm{meV}$$

CP, A. Pineda

PERTURBATION THEORY corrections to $\mathcal{O}(m_r\alpha^5)$



$$\delta E_{nlj}^{V\times V} = \langle\psi_{nlj}|V\frac{1}{(E_n-h)'}V|\psi_{nlj}\rangle$$
$$= \int \mathrm{d}\mathbf{r_2}\mathrm{d}\mathbf{r_1}\psi^*_{nlj}(\mathbf{r_2})V(\mathbf{r_2})G'_{nl}(\mathbf{r_1},\mathbf{r_2})V(\mathbf{r_1})\psi_{nlj}(\mathbf{r_1})$$

QED error estimates

Theoretical prediction for the Lamb shift

$$\Delta E_{LS} = \mathbf{206.0243(30)} - 5.2270(7)\frac{r_p^2}{\mathrm{fm}^2} + 0.0455(125) + \mathcal{O}(m_\mu \alpha^5 \frac{m_\mu^3}{m_\rho^3}, \mathbf{m_\mu \alpha^6})\,\mathrm{meV}$$

CP, A. Pineda

$\mathcal{O}(m_r\alpha^3)$	$V_{\mathrm{VP}}^{(0)}$	205.00737
$\mathcal{O}(m_r\alpha^4)$	$V_{\mathrm{VP}}^{(0)}$	1.50795
$\mathcal{O}(m_r\alpha^4)$	$V_{\mathrm{VP}}^{(0)}$	0.15090
$\mathcal{O}(m_r\alpha^5)$	$V_{\mathrm{VP}}^{(0)}$	0.00752
$\mathcal{O}(m_r\alpha^5)$	$V_{\mathrm{LbL}}^{(0)}$	$-0.00089(2)$
$\mathcal{O}(m_r\alpha^4 \times \frac{m_\mu^2}{m_p^2})$	$V^{(2,1)} + V^{(3,0)}$	0.05747
$\mathcal{O}(m_r\alpha^5)$	$V_{\mathrm{no-VP}}^{(2,2)}$ + ultrasoft	-0.71896
$\mathcal{O}(m_r\alpha^5)$	$V_{\mathrm{VP}}^{(2,2)} + V^{(2,1)} \times V_{\mathrm{VP}}^{(0,2)}$	0.01876
$\mathcal{O}(m_r\alpha^6 \times \ln(\frac{m_\mu}{m_e}))$	$V^{(2,3)}$; $c_D^{(\mu)}$	-0.00127
$\mathcal{O}(m_r\alpha^6 \times \ln\alpha)$	$V_{\mathrm{VP}}^{(2,3)}$; $c_D^{(\mu)}$	-0.00454

Hadronic effects

$$\delta\mathcal{L} = \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} - e\frac{c_D^{(p)}}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p + \frac{c_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu$$

- Hadronic contribution encoded in the δ-potential: $\delta V^{\text{had}} = \frac{D_d^{\text{had}}}{m_p^2}\delta^{(3)}(\mathbf{r})$
- Contributes to energy levels as: $\delta E_{nl}^{\text{had}} = \frac{D_d^{\text{had}}}{m_p^2}\frac{(m_r\alpha)^3}{\pi n^3}\delta_{l,0}$

Dependence on the matching coefficients:

$$D_d^{\text{had}} \equiv -c_3^{\text{had}} - 16\pi\alpha d_2^{\text{had}} + \frac{\pi\alpha}{2} c_D^{\text{had}}$$

two photon exchange

Hadronic effects

$$\delta\mathcal{L} = \frac{d_2}{m_p^2} F_{\mu\nu} D^2 F^{\mu\nu} - e\frac{c_D^{(p)}}{m_p^2} N_p^\dagger \nabla \cdot \mathbf{E} N_p + \frac{c_3}{m_p^2} N_p^\dagger N_p \mu^\dagger \mu$$

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hadronic vacuum polarization

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definition of the proton radius

Hadronic effects

Theoretical prediction for the Lamb shift

$$\Delta E_{LS} = 206.0243(30) - \mathbf{5.2270(7)}\frac{\mathbf{r_p^2}}{\mathbf{fm^2}} + 0.0455(125) + \mathcal{O}\left(\mathbf{m_\mu \alpha^5 \frac{m_\mu^3}{m_\rho^3}}, m_\mu\alpha^6\right)\,\mathrm{meV}$$

CP, A. Pineda

$\mathcal{O}(m_r\alpha^4 \times m_r^2 r_p^2)$	$V^{(2,1)}$; $c_D^{(p)}$; r_p^2	$-5.19745\,\frac{r_p^2}{\mathrm{fm}^2}$
$\mathcal{O}(m_r\alpha^5 \times m_r^2 r_p^2)$	$V^{(2,2)}_{\mathrm{VP}}$; $c_D^{(p)}$; r_p^2	$-0.02815\,\frac{r_p^2}{\mathrm{fm}^2}$
$\mathcal{O}(m_r\alpha^6 \ln\alpha \times m_r^2 r_p^2)$	$V^{(2,3)}$; $c_D^{(p)}$; r_p^2	$-0.00136\,\frac{r_p^2}{\mathrm{fm}^2}$
$\mathcal{O}(m_r\alpha^5 \times \frac{m_r^2}{m_\rho^2})$	$V^{(2)}_{\mathrm{VP_{had}}}$; d_2^{had}	0.0111(2)
$\mathcal{O}(m_r\alpha^5 \times \frac{m_r^2}{m_\rho^2}\frac{m_\mu}{m_\pi})$	$V^{(2)}$; c_3^{had}	0.0344(125)

$$D_d^{\mathrm{had}} \equiv -c_3^{\mathrm{had}} - 16\pi\alpha d_2^{\mathrm{had}} + \frac{\pi\alpha}{2} c_D^{\mathrm{had}}$$

definition of the proton radius

Hadronic effects

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CP, A. Pineda

$\mathcal{O}(m_r\alpha^4 \times m_r^2 r_p^2)$	$V^{(2,1)}; c_D^{(p)}; r_p^2$	$-5.19745\frac{r_p^2}{\mathrm{fm}^2}$
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$\mathcal{O}(m_r\alpha^6 \ln\alpha \times m_r^2 r_p^2)$	$V^{(2,3)}; c_D^{(p)}; r_p^2$	$-0.00136\frac{r_p^2}{\mathrm{fm}^2}$
$\mathcal{O}(m_r\alpha^5 \times \frac{m_r^2}{m_\rho^2})$	$V_{\mathrm{VP_{had}}}^{(2)}; d_2^{\mathrm{had}}$	0.0111(2)
$\mathcal{O}(m_r\alpha^5 \times \frac{m_r^2}{m_\rho^2}\frac{m_\mu}{m_\pi})$	$V^{(2)}; c_3^{\mathrm{had}}$	0.0344(125)

$$D_d^{\mathrm{had}} \equiv -c_3^{\mathrm{had}} - 16\pi\alpha d_2^{\mathrm{had}} + \frac{\pi\alpha}{2}c_D^{\mathrm{had}}$$

hadronic vacuum polarization
(obtained from DR) F. Jegerlehner

Hadronic effects

Theoretical prediction for the Lamb shift

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CP, A. Pineda

$\mathcal{O}(m_r\alpha^4 \times m_r^2 r_p^2)$	$V^{(2,1)}; c_D^{(p)}; r_p^2$	$-5.19745\frac{r_p^2}{\mathrm{fm}^2}$
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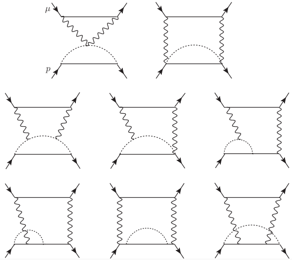
$\mathcal{O}(m_r\alpha^5 \times \frac{m_r^2}{m_\rho^2}\frac{m_\mu}{m_\pi})$	$V^{(2)}; c_3^{\mathrm{had}}$	0.0344(125)

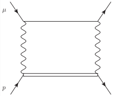
$$D_d^{\mathrm{had}} \equiv -c_3^{\mathrm{had}} - 16\pi\alpha d_2^{\mathrm{had}} + \frac{\pi\alpha}{2} c_D^{\mathrm{had}}$$

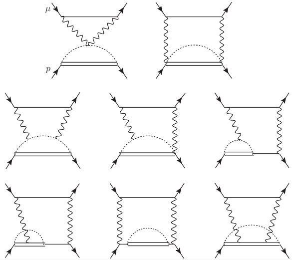
$$c_3^{\mathrm{had}} = c_3^{\mathrm{Born}} + c_3^{\mathrm{pol}} \sim \alpha^2 \frac{m_\mu}{m_\pi}\left(1 + \# \frac{m_\pi}{\Delta} + \ldots\right) + \mathcal{O}\left(\frac{\alpha^2 m_\mu}{\Lambda_{QCD}}\right)$$

The two photon exchange

The virtual forward Compton tensor







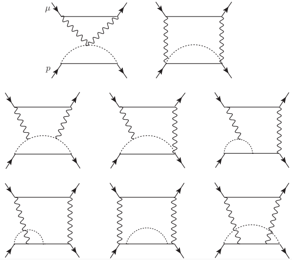
$$T^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) S_1(\rho, q^2) + \frac{1}{m_p^2}\left(p^\mu - \frac{m_p \rho}{q^2} q^\mu\right)\left(p^\nu - \frac{m_p \rho}{q^2} q^\nu\right) S_2(\rho, q^2)$$

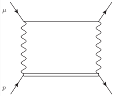
$$- \frac{i}{m_p}\,\epsilon^{\mu\nu\rho\sigma} q_\rho s_\sigma A_1(\rho, q^2) - \frac{i}{m_p^3}\,\epsilon^{\mu\nu\rho\sigma} q_\rho \big((m_p \rho) s_\sigma - (q \cdot s) p_\sigma\big) A_2(\rho, q^2)$$

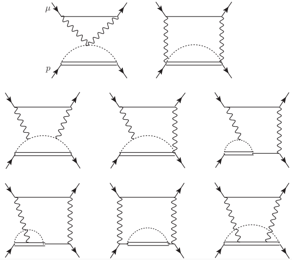
Muonic Hydrogen Lamb shift: theoretical EFT prediction

The two photon exchange

We have computed the TPE in HBET at NLO including the $\Delta(1232)$:







Total TPE energy shift

$$\delta E_L{}^{\mathrm{TPE}} = \delta E_L^{\mathrm{Born}} + \delta E_L^{\mathrm{pol}} = \underset{\text{(LO)}}{28.59(\pi)} + \underset{\text{(NLO)}}{5.86(\pi\&\Delta)} = 34.4(12.5)\mu\mathrm{eV}$$

CP, Pineda

chiral error: $\frac{m_\pi}{\Lambda_{\mathrm{QCD}}} \sim \frac{1}{3}$, Δ-error: $\frac{\Delta}{\Lambda_{\mathrm{QCD}}} \sim \frac{1}{2}$

Muonic Hydrogen Lamb shift: theoretical EFT prediction

c_3^{had}: Born contribution

$$c_{3,\text{Born}} = 4(4\pi\alpha)^2 m_p^2 m_\mu \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{1}{\mathbf{q}^6} G_E^{(0)} G_E^{(2)}(-\mathbf{q}^2) = \frac{\pi}{3}\alpha^2 m_p m_\pi \langle r^3 \rangle_{(2)}$$

- Zemach momenta

(fm)	$\langle r^3 \rangle$	$\langle r^4 \rangle$	$\langle r^5 \rangle$	$\langle r^6 \rangle$	$\langle r^7 \rangle$	$\langle r^3 \rangle_{(2)}$
π	0.4980	0.6877	1.619	5.203	20.92	0.9960
$\pi\&\Delta$	0.4071	0.6228	1.522	4.978	20.22	0.8142
Dipole	0.7706	1.083	1.775	3.325	7.006	2.023
Kelly	0.9838	1.621	3.209	7.440	19.69	2.526
Distler *et al*	1.16(4)	2.59(19)(04)	8.0(1.2)(1.0)	29.8(7.6)(12.6)	—	2.85(8)

important dependence on the fitted function & large difference with EFT

- Born energy shift

(μeV)	DR	Pachucki	Carlson	HBET	(π)	($\pi\&\Delta$)
δE_L^{Born}		23.2(1.0)	24.7(1.6)		10.1(5.1)	8.3(4.3)

less difference with the DR analysis was expected

c_3^{had}: polarizability contribution

$$c_3^{\rm pol} = -e^4 m_p m_{l_i} \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k_E^4 + 4m_\mu^2 k_{0,E}^2} \times \left\{ (3k_{0,E}^2 + \mathbf{k}^2) S_1^{\rm pol}(ik_{0,E}, -k_E^2) - \mathbf{k}^2 S_2^{\rm pol}(ik_{0,E}, -k_E^2) \right\} + \mathcal{O}(\alpha^3)$$

- Different results for the polarizability contribution

	DR+Model				BχPT(π)	HBET(π)	(π&Δ)
	Pachucki	Martynenko	Carlson et al	Gorchtein et al	Alarcon et al		CP, Pineda
$\delta E_L^{\rm pol}$ (μeV)	12(2)	11.5	7.4 (2.4)	15.3(5.6)	8.2 ($^{+1.2}_{-2.5}$)	18.5(9.3)	26.2(10.0)

-computed with the EFT is larger than from combination of DR & models

-The TPE contribution agrees better with DR results than Born and polarizability do separately

Muonic Hydrogen

The hyperfine splitting

EFT framework

NRQED: $\delta\mathcal{L} = -\frac{c_4^{pl_i}}{m_p^2} N_p^\dagger \sigma N_p l^\dagger \sigma l$ pNRQED: $\delta V = \frac{2c_4^{pl_i}}{m_p^2} (\mathbf{S_1} + \mathbf{S_2})^2 \delta^{(3)}(\mathbf{r})$

Hyperfine splitting: $\Delta E_{HFS} = \frac{4c_4^{pl_i}(\alpha m_r)^3}{\pi m_p^2}$

$$c_4^{pl_i} = c_{4,\mathrm{R}}^{pl_i} + c_{4,\text{point-like}}^{pl_i} + c_{4,\mathrm{Born}}^{pl_i} + c_{4,\mathrm{pol}}^{pl_i} + \mathcal{O}(\alpha^3)$$

- $c_{4,\text{point-like}}^{pl_i} = \left(1 - \frac{\kappa_p^2}{4}\right) \alpha^2 \ln \frac{m_{l_i}^2}{\nu^2}$
- 1S hydrogen HFS (Karshenboim '05): $\Delta E_{\mathrm{QED}} - \Delta E_{\mathrm{exp}} = -0.04633(1)\mathrm{MHz}$

For *ep* bound states

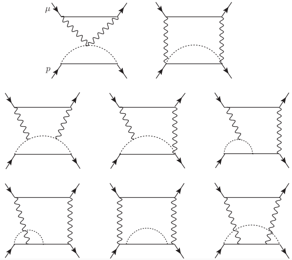
$$c_4^{pe} = -48.06(1)\alpha^2$$

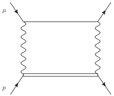
For μp bound states

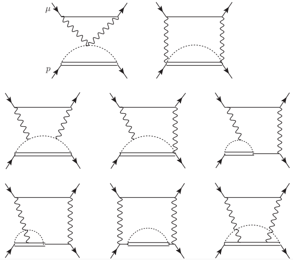
$$c_4^{p\mu} = c_4^{pe} + \left[c_{4,\text{point-like}}^{p\mu} - c_{4,\text{point-like}}^{pe}\right] + \left[c_{4,\mathrm{pol}}^{p\mu} - c_{4,\mathrm{pol}}^{pe}\right] + \mathcal{O}(\alpha^3, \alpha^2 \frac{m_\mu}{\Lambda_{\mathrm{QCD}}})$$

The two photon exchange

The virtual forward Compton tensor





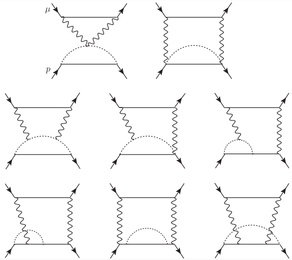


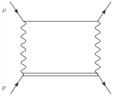
$$T^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) S_1(\rho, q^2) + \frac{1}{m_p^2}\left(p^\mu - \frac{m_p\rho}{q^2} q^\mu\right)\left(p^\nu - \frac{m_p\rho}{q^2} q^\nu\right) S_2(\rho, q^2)$$

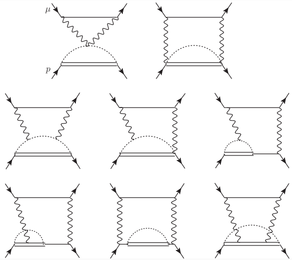
$$- \frac{i}{m_p}\,\epsilon^{\mu\nu\rho\sigma} q_\rho s_\sigma A_1(\rho, q^2) - \frac{i}{m_p^3}\,\epsilon^{\mu\nu\rho\sigma} q_\rho \big((m_p\rho)s_\sigma - (q\cdot s)p_\sigma\big) A_2(\rho, q^2)$$

The two photon exchange

We have computed the TPE in HBET at NLO including the $\Delta(1232)$:







Total polarizability contribution

$$c^{p\mu}_{4,\text{pol}} - c^{pe}_{4,\text{pol}} = [0.167(\pi) + 0.076(\pi\&\Delta)]\,\alpha^2 = 0.243(93)\alpha^2$$

(LO) (NLO)

$$c^{p\mu}_{4} = -45.7(1)\alpha^2$$

CP, Pineda preliminary

chiral error: $\frac{m_\pi}{\Lambda_{\text{QCD}}} \sim \frac{1}{3}$, Δ-error: $\frac{\Delta}{\Lambda_{\text{QCD}}} \sim \frac{1}{2}$

Final remarks

Final remarks

- we develope the **N^3LO potential** for pNRQED for **different masses**

Muonic hydrogen: Lamb shift

- We predict it theoretically in a model independent way in an *EFT framework*
- The main source of uncertainty is the TPE, which we compute using HBET
- The EFT Born & polarizability contributions are separately different from the DR+model results

- Proton radius:

μH value:	$r_p = 0.8413(15)\,\mathrm{fm}$ (CP, Pineda)	6.8σ away!
CODATA value:	$r_p = 0.8775(51)\,\mathrm{fm}$	

We give **model independent** significance to the proton radius puzzle

Final remarks

- we develope the **N^3LO potential** for pNRQED for **different masses**

Muonic hydrogen: hyperfine splitting

- Keep in mind the possibility of a large subleading $\mathcal{O}(\frac{m_\mu^2}{m_p^2})$ Born contribution.

Future perspectives

- New experimental results: see R. Hill plenary talk
  - new *ep*-scattering data: MAMI, JLAB, MUSE
  - experiments involving different muonic bound states: PSI
  - remeasure H energy levels: MPQ, Garching, Toronto
  - independent measurement of μH
- Compute the complete $\mathcal{O}(m\alpha^6)$ for future experimental determination of c_3 and c_4

Thank you!

Computing within the EFT

Summary of the EFT framework

- Matching HBET to NRQED: compute HBET diagrams
- Matching NRQED to pNRQED: compute potential or Wilson loops or HQET diagrams (equivalent)
- Observable: spectrum or decay

Corrections to the Green function:

$$G_s(E) = P_s \frac{1}{h_s^{(0)} - H_I - E} P_s = G_s^{(0)} + \delta G_s$$

- ultrasoft loops: $\mathbf{x} \cdot \mathbf{E}$ (Lamb shift)
- quantum mechanics perturbation theory

EFT definition of the proton radius

$$r_p^2 = 6\frac{dG_{p,E}(q^2)}{dq^2}\Big|_{q^2=0}$$

It is IR divergent!

NRQCD Lagrangian

$$\delta\mathcal{L} = -e\frac{c_D^{(p)}}{m_p^2} N_p^\dagger \nabla\cdot\mathbf{E} N_p$$

taking into account
$$\begin{cases} G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2}F_2(q^2) \\ c_D = Z_p + 2F_2(0) + 8F_1'(0) \end{cases}$$

General Expression

$$c^{(p)}_{D,\overline{\mathrm{MS}}}(\nu) \equiv Z_p + \frac{4}{3}\frac{Z_p^3\alpha}{\pi}\ln\left(\frac{m_p^2}{\nu^2}\right) + \frac{4}{3}r_p^2 m_p^2 + \mathcal{O}(\alpha^2).$$

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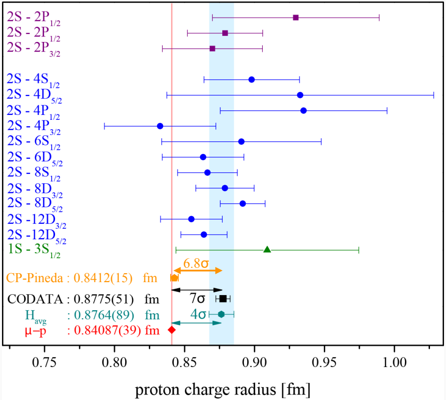
The proton radius in hydrogen spectroscopy

Experimental measurements



Theoretical prediction

Lamb shift:

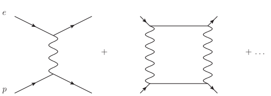
$$\Delta E_{LS} \sim r_p^2$$

in EFT counting:

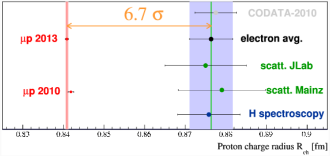
$$\Delta E_{LS} \sim \mathcal{O}\left(m_e \alpha^4 \frac{m_e^2}{m_p^2}\right) \sim \mathcal{O}(m_e \alpha^6)$$

The proton radius in *ep* scattering

The scattering at low q^2 can be related to r_p



$$\frac{d\sigma}{d\Omega} \sim \frac{d\sigma}{d\Omega}_{\text{point}} \left(1 - \frac{q^2}{4m^2}\right)\left(G_E^2(q^2) - \frac{q^2}{4m^2} G_M^2(q^2)\right)$$



- need to compute the two photon exchange (TPE)
- very sensitive to low q^2 data:
  extrapolation from $|\mathbf{q}| \geq 100$ MeV
  to $|\mathbf{q}| \sim m_e\alpha \sim 1$ MeV
- dependence on the fitting functions:
  normalization factors, full data set ...

Born and polarizability contributions

- Born contribution:

$$c_{3,\mathrm{Born}} = \frac{\pi}{3}\alpha^2 m_p m_\pi \langle r^3\rangle_{(2)}$$

$$\frac{\langle r^3\rangle_{(2)}}{\mathrm{fm}^3} = \frac{48}{\pi}\int \frac{d^3k}{4\pi}\frac{1}{\mathbf{k}^6}\left(G_E^2 - 1 + \frac{1}{3}\langle r^2\rangle \mathbf{k}^2\right)$$

- Polarizability contribution:

$$c_{3,\mathrm{sub}} = -e^4 m_p m_\mu \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^4}\frac{1}{k_E^4 + 4m_\mu^2 k_{0,E}^2}(3k_{0,E}^2 + \mathbf{k}^2)S_1(0,-k_E^2)$$

$$= -\frac{\alpha^2 m_p}{2m_\mu}\int_0^\infty \frac{dQ^2}{Q^2}\left\{1 + \left(1 - \frac{Q^2}{2m_{l_i}^2}\right)\left(\sqrt{\frac{4m_{l_i}^2}{Q^2} + 1} - 1\right)\right\} S_1(0,-Q^2),$$

$$c_{3,\mathrm{inel}} = -e^4 m_p m_\mu \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^4}\frac{1}{k_E^4 + 4m_\mu^2 k_{0,E}^2}$$

$$\times \left\{(3k_{0,E}^2 + \mathbf{k}^2)(S_1(ik_{0,E},-k_E^2) - S_1(0,-k_E^2)) - \mathbf{k}^2 S_2(ik_{0,E},-k_E^2)\right\}.$$

Logarithmic functions

$E_n^C = -\frac{C_F^2 \alpha_s^2 m_r}{2n^2}$

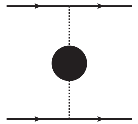
$L_\nu = \ln\left(\frac{n\nu}{2m_r C_F \alpha_s}\right) + S_1(n+l)$ $L_{US} = \ln\left(\frac{C_F \alpha_s n}{2}\right) + S_1(n+l)$

$L_H = \ln\left(\frac{n}{C_F \alpha_s}\right) + S_1(n+l)$

The Bethe logarithm is defined as

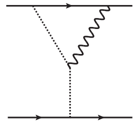
$$L_n^E = \frac{1}{(C_F\alpha_s)^2 E_n^C} \int_0^\infty \frac{d^3k}{(2\pi)^3} |\langle \mathbf{r} \rangle_{\mathbf{k}n}|^2 \left(E_n^C - \frac{k^2}{2m_r}\right)^3 \ln \frac{E_1^C}{E_n^C - \frac{k^2}{2m_r}}$$

Energy dependence in the Coulomb gauge



$$= -\frac{ig_B^4}{3} \frac{k_0^2 \mathbf{k}^{2\epsilon-4} \csc(\pi\epsilon)}{2^{4\epsilon+4}\pi^{\epsilon+\frac{1}{2}}\Gamma\left(\epsilon+\frac{5}{2}\right)} \Big[3C_F T_F n_f \epsilon(1+\epsilon)$$

$$-\frac{C_A C_F}{4}\left((\epsilon+1)(\epsilon(56\epsilon+121)+60)\right) - \frac{5\Gamma\left(\epsilon+\frac{3}{2}\right)^2}{\sqrt{\pi}\Gamma\left(2\epsilon+\frac{5}{2}\right)} 4^{\epsilon+1}(\epsilon+1)(2\epsilon+3)(4\epsilon+3)\Big]$$



$$= -\frac{ig_B^4}{3m_i} C_A C_F \frac{\mathbf{k}^{2\epsilon-4}(\epsilon+1)(2\epsilon+1)\Gamma(1-\epsilon)\Gamma(2\epsilon)}{4^{2\epsilon+1}\pi^{\epsilon+\frac{3}{2}}\Gamma\left(2\epsilon+\frac{3}{2}\right)}$$

$$\left[E_i\left(\mathbf{k}^2 - (\mathbf{p}'^2-\mathbf{p}^2)^2\right) + E_i'\left(\mathbf{k}^2 + (\mathbf{p}'^2-\mathbf{p}^2)^2\right)\right]$$

The proton radius puzzle

What is going on?

- Are experimental measurements wrong?
- Are theoretical predictions mistaken?
- Is there a more exotic explanation?

we need to accommodate this within our known theory