Complex Langevin simulations of a finite density Matrix Model for QCD

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Work in collaboration with J. Glesaaen (Frankfurt U.), O. Philipsen (Frankfurt U.), J. Verbaarschot (Stony Brook U.)
Many approaches to attack the sign problem

- Conventional/Monte Carlo based methods
  - Reweighting
  - Taylor expansion
  - Imaginary $\mu$
  - Strong Coupling Expansion
  - Mean Field analyses

- Alternative methods
  - Stochastic Quantization-Complex Langevin
  - Lefschetz Thimble
  - Canonical ensembles
  - Dual variables
  - Density of States
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Stochastic quantization as an alternative

- consider the trivial "QFT" given by the partition function
  \[ Z = \int e^{-S(x)} \, dx \]
- in the real Langevin formulation
  \[ x(t + \delta t) = x(t) - \partial_x S(x(t)) \delta t + \delta \xi \]
- stochastic variable $\delta \xi$ with zero mean and variance given by $2\delta t$
- generalization to complex actions \cite{Parisi1983, Klauder1983}
  \[ x \rightarrow z = x + iy \]
  \[ z(t + \delta t) = z(t) - \partial_z S(z(t)) \delta t + \delta \xi \]
- one can study gauge theories with complex actions \cite{Aarts, James, Seiler, Sexty, Stamatescu, ...}
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Is this "the" solution to the sign problem?

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- Simulations are not guaranteed to converge to "the correct solution".
- Criteria of convergence not fulfilled in practical simulations.
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- mass dependence of the chiral condensate $\langle \bar{\eta} \eta \rangle = \partial_m \log Z$
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The Stephanov Model

\[ Z = \int D W e^{-n\Sigma^2 \text{Tr} W W^\dagger} \det^{N_f} (m \quad iW + \mu \\
\quad iW^\dagger + \mu \quad m) \]

Stephanov (1996)

- solve via bosonization
- \[ Z(m, \mu) = \int d\sigma d\sigma^* e^{-n\sigma^2} (\sigma\sigma^* + m(\sigma + \sigma^*) + m^2 - \mu^2)^n \]
- where \( \sigma \) is an \( N_f \times N_f \) matrix
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The phase transition

- in the thermodynamic limit evaluate $Z$ via a saddle point approximation
- there is a phase transition separating a phase with zero and non-zero baryon density
- In the chiral limit $\mu_c = 0.527$ for $\mu \in \mathbb{R}$
- $\mu_c = i$ for $\mu \in \mathbb{I}$
- we can compute $\Sigma(m, \mu)$ and $n_B(m, \mu)$ and compare it with the Complex Langevin simulation
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First attempts in the Osborn model Mollgaard and Splittorff (2013-2014), Nagata, Nishimura, Shimasaki (2015-2016)
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\[ W = a + ib \]

- compute the drift terms \( \partial S / \partial a_{ij} \) and \( \partial S / \partial b_{ij} \)
- complexify the dof \( a, b \in \mathbb{R} \rightarrow a, b \in \mathbb{C} \)
- \( a_{ij}(t + \delta t) = a_{ij}(t) - \partial_{a_{ij}} S(x(t)) \delta t + \delta \xi_{ij} \)
- \( b_{ij}(t + \delta t) = b_{ij}(t) - \partial_{b_{ij}} S(x(t)) \delta t + \delta \xi_{ij} \)
- \( \langle \xi_{ij} \rangle = 0 \) and \( \langle \xi_{ij}(t) \xi_{kl}(t') \rangle = 2 \delta t \delta(t - t') \delta_{ik} \delta_{jl} \)
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$m$-scan for $\mu = 0$

$\langle \eta^\dagger \eta \rangle$ for $\mu = 0$

$\langle \bar{\eta} \eta \rangle$ for $\mu = 0$
Numerical Validity - Matrix Size

\[ \langle \bar{\eta}\eta \rangle \text{ for } \mu = 1 \]

\[ \langle \bar{\eta}\eta \rangle / N \]

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\[ \langle \eta^\dagger\eta \rangle \text{ for } m = 0 \]

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Numerical Validity - Step Size

\[ \langle \eta \eta \rangle / N \]  
\[ \Delta t = 10^{-4} \]  
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\langle \eta^\dagger \eta \rangle / N 
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\langle \eta^\dagger \eta \rangle \text{ for } m = 0
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- Do the simulations converge?
- If yes to which theory?
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\]

\[
\langle \tilde{\eta} \eta \rangle \text{ for } m = 1
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Conclusions and outlook

- studied the Complex Langevin algorithm for an RMT model for QCD
- can compare with exact analytical results for all the range of parameters $(m, \mu)$
- compared to previous similar studies this model possess a phase transition to a phase with non-zero baryon density
- fails to converge to $QCD$ and it converges to $|QCD|$ standard ways to fix it → gauge cooling Seiler, Sexty and Stamatescu (2012), Nagata, Nishimura, Shimasaki (2015)
- work in progress...
- Thanks a lot for your attention!
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Stay Tuned!

for upcoming results . . .